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# A SIMPLE PROOF OF DAO'S THEOREM ON SIX CIRCUMCENTERS

NGUYEN TIEN DUNG

ABSTRACT. In this article, we give a simple proof of Dao's theorem on six circumcenters.

## 1. INTRODUCTION

Dao Thanh Oai [3] proposed the problem of proving the following remarkable theorem:

**Theorem 1** (Dao). *Let six points $A, B, C, D, E, F$ lie on a circle and let $X = FA \cap BC$, $Y = AB \cap CD$, $Z = BC \cap DE$, $U = CD \cap EF$, $V = DE \cap FA$, $W = EF \cap AB$. Denote by $O_1$, $O_2$, $O_3$, $O_4$, $O_5$, $O_6$ the cicumcenters of six triangles $XAB$, $YBC$, $ZCD$, $UDE$, $VEF$, $WFA$ respectively. Then, three lines $O_1O_4$, $O_2O_5$, $O_3O_6$ are concurrent.*

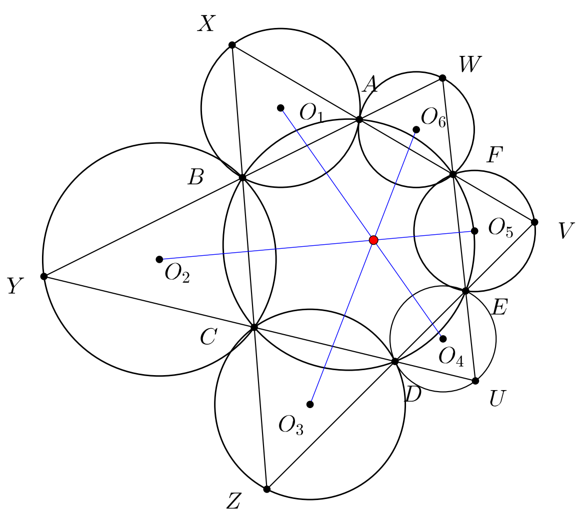


FIGURE 1

In 2014, Nikolaos Dergiades [6] published an elegant proof using complex numbers of Theorem 1. After that, Telv Cohl presented a purely synthetic proof of Theorem 1 (see [2]).

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Noted in the Figure 1, when the circle passing through six points $A, B, C, D, E, F$ is a Tucker circle of the triangle $WYU$, the concurrence of three lines $O_1O_4$, $O_2O_5$, $O_3O_6$ lies on a line passing through the triangle centers $X(6)$, $X(24)$, $X(54)$, $X(973)$, $X(2917)$, $X(3567)$ of the triangle $WYU$ (see [5] and [7]). When the hexagon $ABCDEF$ degenerates into the triangle $WYU$, the concurrence, which is the Kosnita point $X(54)$ of the triangle $WYU$, is also the point $X(3649)$ of the triangle $ZVX$ (see Figure 2). Some problems around Dao's theorem on six circumcenters can be found in [4] and [8]. In this article, we introduce another simple proof of Theorem 1.

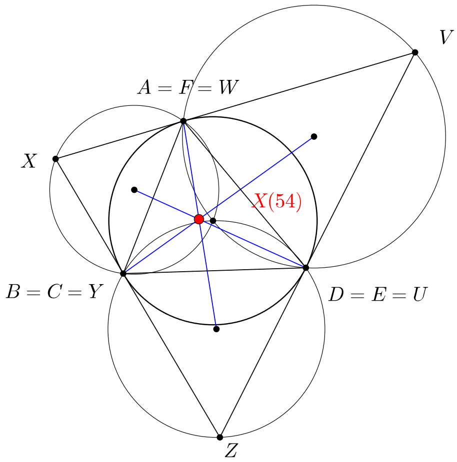


FIGURE 2

## 2. PROOF OF DAO'S THEOREM ON SIX CIRCUMCENTERS

We directly use two following lemmas to prove Theorem 1:

**Lemma 2.** *Let six points $A, B, C, A', B', C'$ lie on a circle and let $D = CA' \cap AB$, $E = AB \cap B'C'$, $F = B'C' \cap CA'$, $A_a = BB' \cap CC'$, $B_b = CC' \cap AA'$, $C_c = AA' \cap BB'$. Denote by $(O_1), (O_2), (O_3)$ the circumcircles of three triangles $DBC, EC'A, FA'B'$ respectively. Then, $A_aO_1, B_bO_2, C_cO_3$ are concurrent.*

*Proof.* Let $R_i$ be the radius of the circle $(O_i)$ and let $d_{ia}, d_{ib}, d_{ic}$ be the distances from $O_i$ to $AA', BB', CC'$ for $i = 1, 2, 3$ respectively. Let the circle $(O_2)$ intersect $CC'$ again at $Y$ and let the circle $(O_3)$ intersect $BB'$ again at $Z$ (see Figure 3). Since $\angle EYC' = \angle EAC' = \angle BCC'$, $EY \parallel BC$. Similarly, we have $EY \parallel FZ \parallel BC$. Since $\angle YO_2C' = 2\angle YEC' = 2\angle ZFB' = \angle ZO_3B'$, two isosceles triangles $YO_2C'$ and $ZO_3B'$ are similar. It follows that $\frac{d_{2c}}{d_{3b}} = \frac{R_2}{R_3}$. Similarly, we have $\frac{d_{3a}}{d_{1c}} = \frac{R_3}{R_1}$ and $\frac{d_{1b}}{d_{2a}} = \frac{R_1}{R_2}$. Let $M = A_aO_1 \cap B_bC_c$, $N = B_bO_2 \cap C_cA_a$, $P = C_cO_3 \cap A_aB_b$. Denote by $[ABC]$ the area of the triangle $ABC$, then we have:

$$\frac{MB_b}{MC_c} \cdot \frac{NC_c}{NA_a} \cdot \frac{PA_a}{PB_b} = \frac{[O_1A_aB_b]}{[O_1A_aC_c]} \cdot \frac{[O_2B_bC_c]}{[O_2B_bA_a]} \cdot \frac{[O_3C_cA_a]}{[O_3C_cB_b]}$$

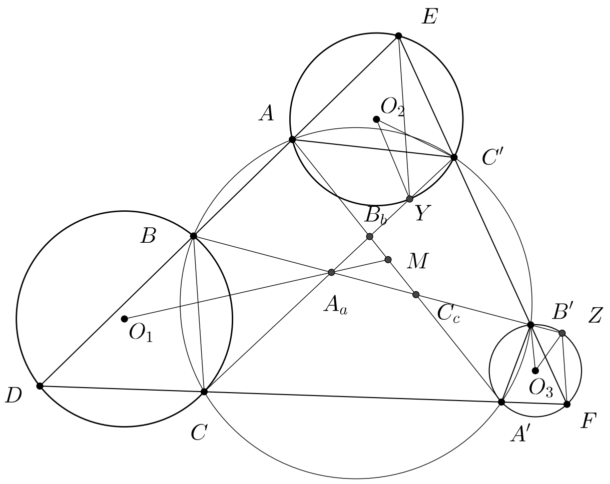


FIGURE 3

$$= \frac{[O_2B_bC_c]}{[O_3B_bC_c]} \cdot \frac{[O_3C_cA_a]}{[O_1C_cA_a]} \cdot \frac{[O_1A_aB_b]}{[O_2A_aB_b]} = \frac{d_{2a}}{d_{3a}} \cdot \frac{d_{3b}}{d_{1b}} \cdot \frac{d_{1c}}{d_{2c}}$$

$$= \frac{d_{2a}}{d_{1b}} \cdot \frac{d_{3b}}{d_{2c}} \cdot \frac{d_{1c}}{d_{3a}} = \frac{R_2}{R_1} \cdot \frac{R_3}{R_2} \cdot \frac{R_1}{R_3} = 1$$

By the converse of Ceva's theorem for the triangle $A_aB_bC_c$, we conclude that the lines $A_aO_1, B_bO_2, C_cO_3$ are concurrent. □

**Lemma 3** (Cohl [2])**.** *Let six points $A, B, C, A', B', C'$ lie on a circle and let $X = AB \cap A'C, X' = A'B' \cap AC'$. Denote by $(O)$ and $(O')$ the circumcircles of two triangles $XBC$ and $X'B'C'$. Then, $OO', BB', CC'$ are concurrent.*

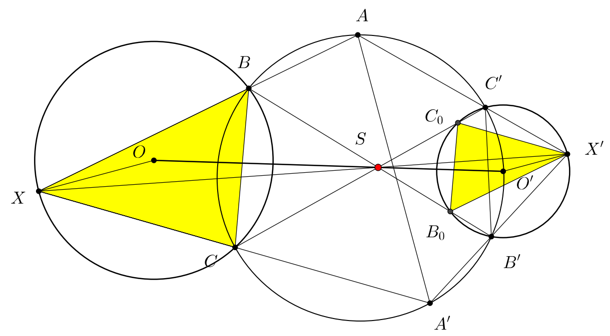


FIGURE 4

*Proof.* We have $\angle BXO = 90^0 - \angle XCB = 90^0 - \angle XAA'$, so $XO \perp AA'$. Similarly, we have $XO \parallel X'O' \perp AA'$. Let $B_0$ and $C_0$ be two points on the circle $(O')$ such that $X'B_0 \parallel XB$ and $X'C_0 \parallel XC$ (see Figure 4). We can easily deduce that $XX', OO', BB_0, CC_0$ are concurrent at the homothetic center $S$ of two circles $(O)$ and $(O')$. Since $\angle B_0B'C' = \angle B_0X'C' = 180^0 - \angle BAC' = \angle BB'C'$, $B, B'$ and $B_0$ are collinear. Similarly, $C, C'$ and $C_0$ are collinear. Hence, $OO', BB', CC'$ are concurrent. □

*Return to the proof of Theorem 1.* Let $M = BE \cap AD, N = AD \cap CF, P = CF \cap BE$ (see Figure 5). According to Lemma 2, $O_1M, O_3N, O_5P$ are concurrent. Applying Lemma 3, we have $O_1, M, O_4$ collinear, $O_3, N, O_6$ collinear and $O_5, P, O_2$ collinear. Therefore, $O_1O_4, O_2O_5, O_3O_6$ are concurrent. The proof of Theorem 1 is complete.

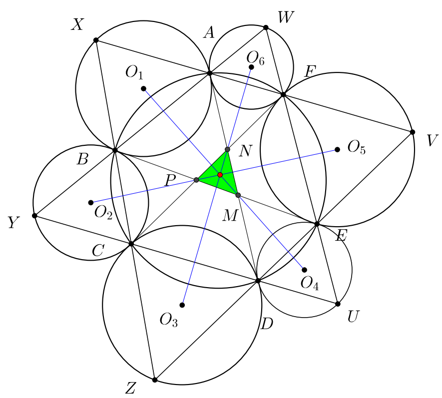


FIGURE 5

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NO 24, 3 PLANE, PHUC LOI WARD, LONG BIEN DISTRICT, HA NOI, VIET NAM.
*E-mail address*: tiendung12121993@gmail.com