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A MOST BASIC TRIAD OF PARABOLAS ASSOCIATED WITH A TRIANGLE

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ABSTRACT. For a given triangle we study the triad of parabolas each with one vertex as focus and its opposite side as directrix. Among the 6 angle bisectors and the perpendicular bisectors of the 3 sides, we show that each of these parabolas is tangent to 4 of the angle bisectors and 2 perpendicular bisectors. The 18 points of tangency fall on another set of 6 lines each joining the antipode (on the circumcircle) of a vertex to the two remaining vertices. Some perspectivity of triangles will also be exhibited. The 6 points of tangency on the external bisectors lie on a conic with a simple barycentric equation with respect to the excentral triangle.

1. INTRODUCTION

In this paper we study a triad of parabolas most naturally associated with a triangle. Given a triangle ABC, we consider the parabolas $\mathscr{P}_a$ with focus A and directrix BC, $\mathscr{P}_b$ with focus B and directrix CA, and $\mathscr{P}_c$ with focus C and directrix AB, and prove some interesting results about these parabolas. In §**??**, we redefine these parabolas in terms of their tangents, and locate the points of tangency. Specifically we show that each of $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$ is tangent to 4 angle bisectors and 2 perpendicular bisectors of ABC, and establish some perspectivity of triangles with vertices among the points of tangency. In the final section, we show that the 6 points of tangency with the external bisectors lie on a conic.

Notations. For triangle ABC, we denote respectively by

a, b, c	the sides BC, CA, AB,
a′, b′, c′	the perpendicular bisectors of a, b, c,
$\mathcal{L}_a$, $\mathcal{L}_b$, $\mathcal{L}_c$	the bisectors of angles A, B, C,
$\mathcal{L}_a^\perp$, $\mathcal{L}_b^\perp$, $\mathcal{L}_c^\perp$	the external bisectors of angles A, B, C;
$\mathcal{L}^\perp(P)$	the perpendicular to the line $\mathcal{L}$ at (or from) the point P.

Apart from basic elementary results on parabolas we shall also make use of homogeneous barycentric coordinates. A basic reference is [**?**]. Let a, b, c denote the lengths of the sides BC, CA, AB respectively. We shall make use of Conway's notations

$$S_A := \frac{b^2+c^2-a^2}{2}, \quad S_B := \frac{c^2+a^2-b^2}{2}, \quad S_C := \frac{a^2+b^2-c^2}{2}. \tag{1}$$

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The use of these symbols greatly simplifies expressions for coordinates involving only even powers of a, b, c. These satisfy basic relations like

$$S_B + S_C = a^2, \qquad S_C + S_A = b^2, \qquad S_A + S_B = c^2; \tag{2}$$

$$S_{BC} + S_{CA} + S_{AB} = S^2, \tag{3}$$

where $S_{AB} := S_A S_B$ etc and S stands for *twice* the area of triangle ABC (see [**?**, §3.4.1]. In fact, $S_A = S \cot A$ etc. For examples, the homogeneous barycentric coordinates of the circumcenter O are $(a^2 S_A : b^2 S_B : c^2 S_C)$, the orthocenter and the symmedian point are the points $H = (S_{BC} : S_{CA} : S_{AB})$ and $K = (S_B + S_C : S_C + S_A : S_A + S_B)$ respectively. For later use we record the coordinates of the antipodes A', B', C' of A, B, C on the circumcircle:

$$\begin{aligned} A' &= (-S_{BC} : b^2 S_B : c^2 S_C), \\ B' &= (a^2 S_A : -S_{CA} : c^2 S_C), \\ C' &= (a^2 S_A : b^2 S_B : -S_{AB}). \end{aligned} \tag{4}$$

Instead of using homogeneous linear equations to represent lines, we shall use line coordinates. Thus, the sidelines are $\mathsf{a} = [1 : 0 : 0]$, $\mathsf{b} = [0 : 1 : 0]$, $\mathsf{c} = [0 : 0 : 1]$, and the Brocard axis is

$$OK = [b^2c^2(b^2 - c^2) : c^2a^2(c^2 - a^2) : a^2b^2(a^2 - b^2)]. \tag{5}$$

2. Three pairs of points on the perpendicular bisectors

Consider the following points on the perpendicular bisectors of the sides of triangle ABC.

	$\mathscr{P}_a$	$\mathscr{P}_b$	$\mathscr{P}_c$
a'		$X_b := \mathsf{a}' \cap \mathsf{b}^\perp(C)$	$X_c := \mathsf{a}' \cap \mathsf{c}^\perp(B)$
b'	$Y_a := \mathsf{b}' \cap \mathsf{a}^\perp(C)$		$Y_c := \mathsf{b}' \cap \mathsf{c}^\perp(A)$
c'	$Z_a := \mathsf{c}' \cap \mathsf{a}^\perp(B)$	$Z_b := \mathsf{c}' \cap \mathsf{b}^\perp(A)$	

Proposition 1. *The points Y_a, Z_a are on the parabola $\mathscr{P}_a$; similarly, Z_b, X_b are on $\mathscr{P}_b$, and X_c, Y_c are on $\mathscr{P}_c$.*

Proof. Since $Y_aC = Y_aA$ and $Y_aC \perp \mathsf{a}$, Y_a is a point on the parabola $\mathscr{P}_a$; so is Z_a. □

Proposition 2. *The points X_b and X_c are inverse in the circumcircle; so are Y_c and Y_a, Z_a and Z_b.*

Proof. Note that in Figure **??** triangles OCX_b and ABC are oppositely similar, since the directed angles (see [**?**, §§16–19])

$$\begin{aligned} (OC, OX_b) &= \frac{1}{2}(OC, OB) = (AC, AB) = -(AB, AC), \\ (X_bO, X_bC) &= (X_bO, BC) + (BC, X_bC) = \frac{\pi}{2} + (BC, X_bC) \\ &= (X_bC, AC) + (BC, X_bC) = (BC, AC) = -(CA, CB). \end{aligned}$$

By the law of sines,

$$\frac{OX_b}{OC} = \frac{AC}{AB} = \frac{b}{c} \implies OX_b = \frac{b}{c} \cdot R$$

where R is the circumradius of triangle ABC. The distances from O to these six points are summarized below.

	$OX_b = \frac{b}{c} \cdot R$	$OX_c = \frac{c}{b} \cdot R$
$OY_a = \frac{a}{c} \cdot R$		$OY_c = \frac{c}{a} \cdot R$
$OZ_a = \frac{a}{b} \cdot R$	$OZ_b = \frac{b}{a} \cdot R$	

From these, it is clear that X_b and X_c are inverse in the circumcircle; so are Y_c and Y_a, Z_a and Z_b. □

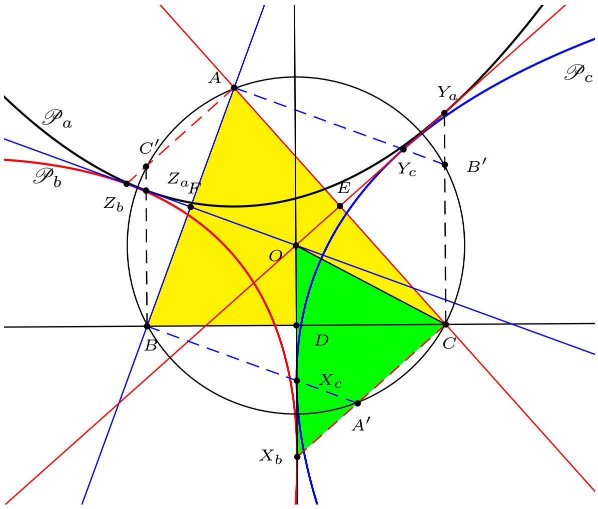

FIGURE 1.

Proposition 3. *The homogeneous barycentric coordinates of* X_b, X_c, Y_c, Y_a, Z_a, Z_b *on* a′, b′, c′ *are as follows.*

a′	$X_b = (-a^2S_C : a^2b^2 : S^2 + S_{BC})$	$X_c = (-a^2S_B : S^2 + S_{BC} : c^2a^2)$
b′	$Y_c = (S^2 + S_{CA} : -b^2S_A : b^2c^2)$	$Y_a = (a^2b^2 : -b^2S_C : S^2 + S_{CA})$
c′	$Z_a = (c^2a^2 : S^2 + S_{AB} : -c^2S_B)$	$Z_b = (S^2 + S_{AB} : b^2c^2 : -c^2S_A)$

Proof. These follow from two sets of line coordinates.

(1) The perpendiculars from the vertices to the sidelines:

$\mathrm{a}^\perp(B) = [S_B : 0 : S_B + S_C]$	$\mathrm{a}^\perp(C) = [S_C : S_B + S_C : 0]$
$\mathrm{b}^\perp(A) = [0 : S_A : S_C + S_A]$	$\mathrm{b}^\perp(C) = [S_C + S_A : S_C : 0]$
$\mathrm{c}^\perp(A) = [0 : S_A + S_B : S_A]$	$\mathrm{c}^\perp(B) = [S_A + S_B : 0 : S_B]$

(6)

(2) The perpendicular bisectors of the sides:

$$\begin{aligned}\mathrm{a}' &= [S_B - S_C : -(S_B + S_C) : S_B + S_C] = [b^2 - c^2 : \ a^2 : \ -a^2],\\ \mathrm{b}' &= [S_C + S_A : S_C - S_A : -(S_C + S_A)] = [-b^2 : \ c^2 - a^2 : \ b^2],\\ \mathrm{c}' &= [-(S_A + S_B) : S_A + S_B : S_A - S_B] = [c^2 : \ -c^2 : \ a^2 - b^2].\end{aligned} \tag{7}$$

For these, it is enough to verify that $[S_B - S_C : \ -(S_B + S_C) : \ S_B + S_C]$ contains the midpoint $(0 : 1 : 1)$ of BC and the circumcenter

$$O = (S_A(S_B + S_C) : S_B(S_C + S_A) : S_C(S_A + S_B)).$$

The first is clear. For the circumcenter, it follows from

$$\begin{aligned} & (S_B - S_C)S_A(S_B + S_C) - (S_B + S_C)S_B(S_C + S_A) + (S_B + S_C)S_C(S_A + S_B) \\ = & \ (S_B + S_C)(S_A(S_B - S_C) - S_B(S_C + S_A) + S_C(S_A + S_B)) \\ = & \ 0. \end{aligned}$$

From the coordinates given in (**??**) and (**??**), we compute

$$\begin{aligned} X_b = & [S_B - S_C : \ -(S_B + S_C) : \ S_B + S_C] \cap [S_C + S_A : \ S_C : \ 0] \\ = & \ (-S_C(S_B + S_C) : \ (S_B + S_C)(S_C + S_A) \\ & \qquad : \ (S_B - S_C)S_C + (S_B + S_C)(S_C + S_A)) \\ = & \ (-S_C(S_B + S_C) : \ (S_B + S_C)(S_C + S_A) : \ 2S_{BC} + S_{AB} + S_{CA}) \\ = & \ (-S_C(S_B + S_C) : \ (S_B + S_C)(S_C + S_A) : \ S^2 + S_{BC}) \\ = & \ (-a^2S_C : a^2b^2 : S^2 + S_{BC}). \end{aligned}$$

The others can be computed similarly. □

Proposition 4. *The lines BX_c and CX_b intersect at the antipode A' of A on the circumcircle.*

Proof. The intersection of BX_c and CX_b is the point

$$\begin{aligned} & (0:1:0) \cdot (-a^2S_B : S^2 + S_{BC} : c^2a^2) \cap (0:0:1) \cdot (-a^2S_C : a^2b^2 : S^2 + S_{BC}) \\ = & [c^2a^2 : 0 : a^2S_B] \cap [-a^2b^2 : -a^2S_C : 0] \\ = & [c^2 : 0 : S_B] \cap [b^2 : S_C : 0] \\ = & (-S_{BC} : b^2S_B : c^2S_C). \end{aligned}$$

This is A' given in (**??**)). Similarly, $CY_a \cap AY_c = B'$ and $AZ_b \cap BZ_a = C'$. □

We shall show in Proposition **??** below that b$'$ and c$'$ are tangent to $\mathscr{P}_a$; similarly for the other two parabolas.

3. The parabolas $\mathscr{P}_a$ etc each defined by five tangents

Lemma 5. (a) *The reflection of the focus of a parabola in a tangent is a point on the directrix.*

(b) *The circumcircle of the triangle bounded by three tangents to a parabola passes through the focus of the parabola.*

Theorem 6 ([**?**]). *The parabola tangent to the $\mathcal{L}_b$, $\mathcal{L}_c$, $\mathcal{L}_b^\perp$ and $\mathcal{L}_c^\perp$ has focus at vertex A and directrix the line* a. *It is the parabola $\mathscr{P}_a$.*

Proof. Let I, I_b, I_c be the incenter and the B-, C-excenters of triangle ABC. The bisectors $\mathcal{L}_b$, $\mathcal{L}_b^\perp$ and $\mathcal{L}_c$ bound the triangle IBI_c. The bisectors $\mathcal{L}_c$, $\mathcal{L}_c^\perp$, and $\mathcal{L}_b$ bound the triangle ICI_b. Apart from I, the circumcircles of these triangles intersect at A. This is the focus of the parabola by Lemma **??**(b).

The reflections of A in the $\mathcal{L}_b$ and $\mathcal{L}_c$ are points on a. By Lemma **??**(a), the line a is the directrix. This shows that the parabola tangent to the four bisectors of angles B and C is the parabola $\mathscr{P}_a$. □

The dual conic of $\mathscr{P}_a$ is a line conic $\mathscr{P}_a^*$ containing $\mathcal{L}_b$, $\mathcal{L}_b^\perp$, $\mathcal{L}_c$, $\mathcal{L}_c^\perp$, which have line coordinates

$$[c : \ 0 : \ -a], \quad [c : \ 0 : \ a], \quad [-b : \ a : \ 0], \quad [b : \ a : \ 0].$$

Since $\mathscr{P}_a$ is a parabola, it is tangent to the line of infinity. Therefore, $\mathscr{P}_a^*$ also contains $[1 : 1 : 1]$. From these five line coordinates we find the barycentric equation of $\mathscr{P}_a^*$:

$$a^2x^2 - b^2y^2 + (b^2 + c^2 - a^2)yz - c^2z^2 = 0, \tag{8}$$

Proposition 7. *The perpendicular bisectors* b' *and* c' *are tangent to the parabola* $\mathscr{P}_a$.

Proof. Rewriting (**??**) as

$$a^2(x^2 - yz) - (y - z)(b^2y - c^2z) = 0,$$

we have, upon substitution of the line coordinates of $\mathrm{b}' = [-b^2 : c^2 - a^2 : b^2]$ given in (**??**),

$$\begin{aligned} & a^2(b^4 - (c^2 - a^2)b^2) - (c^2 - a^2 - b^2)(b^2(c^2 - a^2) - c^2b^2) \\ = \ & a^2b^2(b^2 - c^2 + a^2) + a^2b^2(c^2 - a^2 - b^2) \\ = \ & 0. \end{aligned}$$

Therefore the dual conic $\mathscr{P}_a^*$ contains b'. A similar calculation shows that it also contains c'. These two perpendicular bisectors are tangent to the parabola $\mathscr{P}_a$. □

Corollary 8. *The parabola* $\mathscr{P}_a$ *is tangent to* b' *and* c' *at* Y_a *and* Z_a *respectively.*

Figure **??** shows the 6 points of tangency of the parabola $\mathscr{P}_a$ with b', c', $\mathcal{L}_b$, $\mathcal{L}_b^\perp$, $\mathcal{L}_c$, $\mathcal{L}_c^\perp$. The counterparts of Theorem **??**, Proposition **??**, and Corollary **??** hold for the parabolas $\mathscr{P}_b$ and $\mathscr{P}_c$.

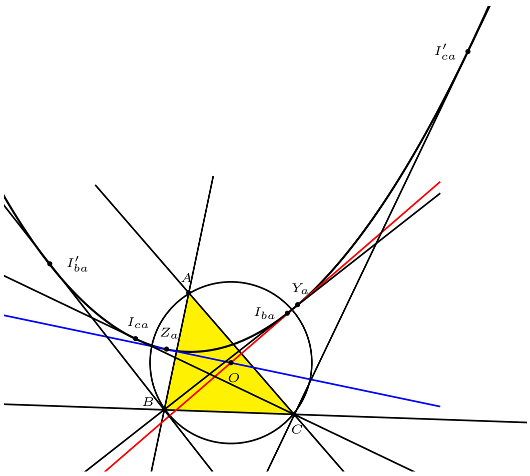

FIGURE 2.

Altogether, the triad of parabolas $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$, together with 6 angle bisectors and the 3 perpendicular bisectors, accounts for 18 points of tangency shown in Table 1 below.

Table 1. Points of tangency of the triad of parabolas

	$\mathcal{L}_a$	$\mathcal{L}_b$	$\mathcal{L}_c$	$\mathcal{L}_a^\perp$	$\mathcal{L}_b^\perp$	$\mathcal{L}_c^\perp$	a'	b'	c'
$\mathscr{P}_a$		I_{ba}	I_{ca}		I'_{ba}	I'_{ca}		Y_a	Z_a
$\mathscr{P}_b$	I_{ab}		I_{cb}	I'_{ab}		I'_{cb}	X_b		Z_b
$\mathscr{P}_c$	I_{ac}	I_{bc}		I'_{ac}	I'_{bc}		X_c	Y_c	

4. 18 POINTS OF TANGENCY ON 6 LINES

In this section we find another interesting arrangement of these 18 points of tangency in Table 1.

From the equation of $\mathscr{P}_a^*$, we find that of $\mathscr{P}_a$. The matrix of $\mathscr{P}_a^*$ is

$$M_a^* := \begin{bmatrix} a^2 & 0 & 0 \\ 0 & -b^2 & S_A \\ 0 & S_A & -c^2 \end{bmatrix}.$$

This has adjoint matrix

$$M_a = \begin{pmatrix} S^2 & 0 & 0 \\ 0 & -c^2a^2 & -a^2S_A \\ 0 & -a^2S_A & -a^2b^2 \end{pmatrix}$$

representing the parabola $\mathscr{P}_a$. The polar of B with respect to $\mathscr{P}_a$ is the line

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} M_a = -a^2[0 \quad c^2 \quad S_A]. \tag{9}$$

It contains the points of tangency of $\mathscr{P}_a$ with $\mathcal{L}_b$ and $\mathcal{L}_b^\perp$, namely, the points I_{ba} and I'_{ba}. We can simply find I_{ba} as $[c : 0 : -a] \cap [0 : c^2 : S_A]$ and I'_{ba} as $[c : 0 : a] \cap [0 : c^2 : S_A]$; similarly for the remaining points of tangency of the three parabolas with the angle bisectors. We summarize the coordinates in the proposition below.

Proposition 9. *The coordinates of the points of tangency of $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$ with the angle bisectors are as follows.*

	$\mathscr{P}_a$	$\mathscr{P}_b$	$\mathscr{P}_c$
$\mathcal{L}_a$		$I_{ab} = (-S_B : bc : c^2)$	$I_{ac} = (-S_C : b^2 : bc)$
$\mathcal{L}_b$	$I_{ba} = (ca : -S_A : c^2)$		$I_{bc} = (a^2 : -S_C : ca)$
$\mathcal{L}_c$	$I_{ca} = (ab : b^2 : -S_A)$	$I_{cb} = (a^2 : ab : -S_B)$	
$\mathcal{L}_a^\perp$		$I'_{ab} = (S_B : bc : -c^2)$	$I'_{ac} = (S_C : -b^2 : bc)$
$\mathcal{L}_b^\perp$	$I'_{ba} = (ca : S_A : -c^2)$		$I'_{bc} = (-a^2 : S_C : ca)$
$\mathcal{L}_c^\perp$	$I'_{ca} = (ab : -b^2 : S_A)$	$I'_{cb} = (-a^2 : ab : S_B)$	

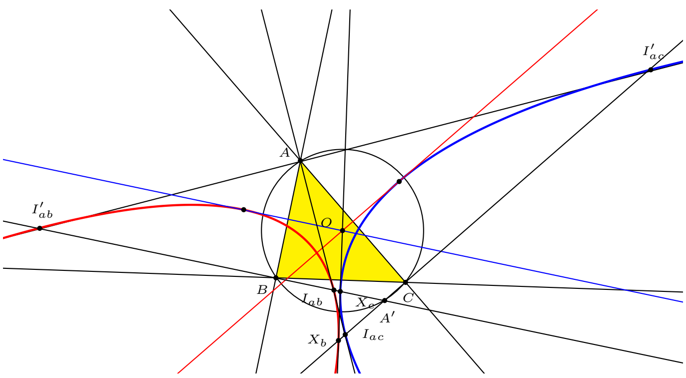

FIGURE 3.

Note that the polar of B given in (**??**) is the same as $\mathrm{c}^\perp(A)$ given (**??**). As such, it contains the antipode B' of B on the circumcircle. It also contains the vertex A and the point of tangency Y_c of $\mathscr{P}_c$ with b'. In other words, the line AB' contains the three points of tangency I_{ba}, I'_{ba}, and Y_c.

Theorem 10. *The* 18 *points of tangency of the parabolas* $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$ *with the* 6 *angle bisectors and the* 3 *perpendicular bisectors fall on* 6 *lines, each joining the antipode* (*on the circumcircle*) *of a vertex of triangle* ABC *to the remaining two vertices.*

AB'	I_{ba}	I'_{ba}	Y_c
AC'	I_{ca}	I'_{ca}	Z_b
BC'	I_{cb}	I'_{cb}	Z_a
BA'	I_{ab}	I'_{ab}	X_c
CA'	I_{ac}	I'_{ac}	X_b
CB'	I_{bc}	I'_{bc}	Y_a

Figure **??** shows the two lines BA' and CA' containing I_{ab}, I'_{ab}, X_c, and I_{ac}, I'_{ac}, X_b respectively.

5. Some perspectivities

5.1. Triangles with vertices among the tangency points on the perpendicular bisectors.

Proposition 11. *The triangle* $X_1Y_1Z_1$ *bounded by the lines* Y_aZ_a, Z_bX_b, X_cY_c *is perspective with* ABC. *The perspectrix is the Lemoine axis, and the perspector is the Kiepert perspector* $K\left(\frac{\pi}{2} - \omega\right)$, *where* ω *is the Brocard angle of triangle* ABC.

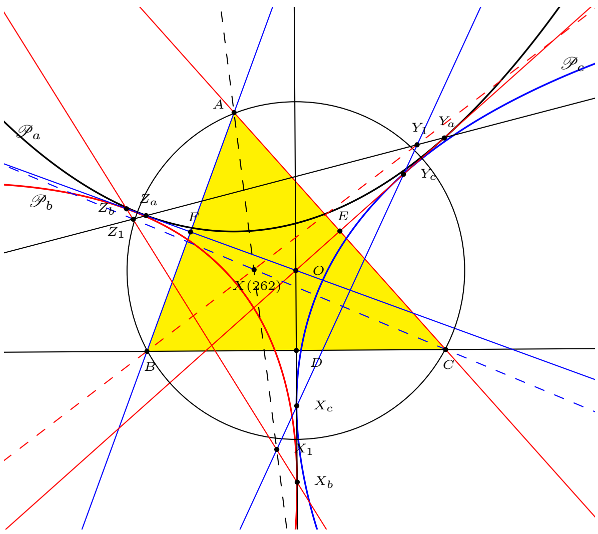

FIGURE 4.

Proof. The lines Y_aZ_a, Z_bX_b, X_cY_c are the polars of O with respect to the parabolas $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$ respectively. They have line coordinates

$$[-S_A : c^2 : b^2], \quad [c^2 : -S_B : a^2], \quad [b^2 : a^2 : -S_C].$$

These lines intersect a, b, c respectively at

$$(0 : b^2 : -c^2), \qquad (-a^2 : 0 : c^2), \qquad (a^2 : -b^2 : 0),$$

which are collinear on the Lemoine axis $\left[\frac{1}{a^2} : \frac{1}{b^2} : \frac{1}{c^2}\right]$. This shows that the triangle bounded by the lines is perspective with ABC with the Lemoine axis as perspectrix. The vertices of the triangle are

$$\begin{aligned}
X_1 = &\ [S_A + S_B : -S_B : S_B + S_C] \cap [S_C + S_A : S_B + S_C : -S_C] \\
= &\ (-(S_{BB} + S_{BC} + S_{CC}) : S^2 + (S_A + S_B + S_C)S_C : S^2 + (S_A + S_B + S_C)S_B), \\
Y_1 = &\ (S^2 + (S_A + S_B + S_C)S_C : -(S_{CC} + S_{CA} + S_{AA}) : S^2 + (S_A + S_B + S_C)S_A), \\
Z_1 = &\ (S^2 + (S_A + S_B + S_C)S_B : S^2 + (S_A + S_B + S_C)S_A : -(S_{AA} + S_{AB} + S_{BB})).
\end{aligned}$$

From these, the perspector with ABC is the point

$$\left(\frac{1}{S^2 + (S_A + S_B + S_C)S_A} : \frac{1}{S^2 + (S_A + S_B + S_C)S_B} : \frac{1}{S^2 + (S_A + S_B + S_C)S_C}\right).$$

Since $S_A + S_B + S_C = S_\omega$ for the Brocard angle ω of triangle ABC, $\frac{S^2}{S_A+S_B+S_C} = \frac{S^2}{S_\omega} = S\tan\omega$, the perspector is the same as

$$\left(\frac{1}{S_A + S\tan\omega} : \frac{1}{S_B + S\tan\omega} : \frac{1}{S_C + S\tan\omega}\right),$$

the Kiepert perspector $K\left(\frac{\pi}{2} - \omega\right)$. □

Proposition 12 ([**?**]). *The triangle $X_1Y_1Z_1$ bounded by the Y_aZ_a, Z_aX_a, X_aY_a is perspective to the triangle of the vertices of $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$. The perspector is the point*

$$((S_A + S_B + S_C)^2(S_{BB} + 4S_{BC} + S_{CC}) - (S_{BB} + S_{BC} + S_{CC})^2 : \cdots : \cdots),$$

and the perspectrix is the line

$$\frac{x}{S_{CA} + S_{AB} - S_{BC}} + \frac{y}{S_{AB} + S_{BC} - S_{CA}} + \frac{z}{S_{BC} + S_{CA} - S_{AB}} = 0.$$

Remarks. (1) Proposition **??** appears in [**?**] under the entry $X(262)$.

(2) The line $Y_aZ_a = [-S_A : c^2 : b^2]$ is also the directrix of the A-Artzt parabola, which is tangent to b and c at C and B. It has focus $(2S_A : b^2 : c^2)$, the second intersection of AK with the Brocard circle (of diameter OK). Similar results hold for the B- and C-Artzt parabolas. Therefore, the directrices of the A-, B-, C-Artzt parabolas bound a triangle with perspective with ABC at $X(262)$.

(3) The perspector in Proposition **??** is the triangle center $X(9748)$ in [**?**].

Proposition 13. *The points $X_2 := Y_cZ_a \cap Y_aZ_b$, $Y_2 := Z_aX_b \cap Z_bX_c$, $Z_2 := X_bY_c \cap X_cY_a$ are collinear on the Brocard axis, and the lines AX_2, BY_2, CZ_2 are concurrent at the Tarry point on the circumcircle.*

Proof. The triangles $X_bY_cZ_a$ and $X_cY_aZ_b$ are clearly perspective at the circumcenter O. They are also axis-perspective, i.e. the three intersections $X_2 = Y_cZ_a \cap Y_aZ_b$, $Y_2 = Z_aX_b \cap Z_bX_c$, $Z_2 = X_bY_c \cap X_cY_a$ are collinear. Making use of the coordinates given in Proposition **??**, we have

$$\begin{aligned} X_2 = &\ (a^2(S_A^3 + a^2(2S_{AA} - S_{BC}) - S_A(S_{BB} + S_{BC} + S_{CC})) : b^2c^2(S_{AB} - S_{CC}) \\ &\ : b^2c^2(S_{CA} - S_{BB})), \\ Y_2 = &\ (c^2a^2(S_{AB} - S_{CC}) : b^2(S_B^3 + b^2(2S_{BB} - S_{CA}) - S_B(S_{CC} + S_{CA} + S_{AA})) \\ &\ : c^2a^2(S_{BC} - S_{AA})), \\ Z_2 = &\ (a^2b^2(S_{CA} - S_{BB}) : a^2b^2(S_{BC} - S_{AA}) \\ &\ : c^2(S_C^3 + c^2(2S_{CC} - S_{AB}) - S_C(S_{AA} + S_{AB} + S_{BB}))). \end{aligned}$$

Substituting the coordinates of X_2 into the equation (**??**) of the Brocard axis, we have, after ignoring a common factor $a^2b^2c^2$,

$$\begin{aligned} &(S_B - S_C)(S_A^3 + 2S_{AA}(S_B + S_C) - S_A(S_{BB} + S_{BC} + S_{CC}) - S_{BC}(S_B + S_C)) \\ &\quad + (S_C - S_A)(S_A + S_B)(S_{AB} - S_{CC}) + (S_C + S_A)(S_A - S_B)(S_{CA} - S_{BB}) \\ = &\ S_A^3(S_B - S_C) + 2S_{AA}(S_{BB} - S_{CC}) - S_A(S_B^3 - S_C^3) - S_{BC}(S_{BB} - S_{CC}) \\ &\quad - S_A^3 S_B - S_{AA}(S_{BB} - S_{BC} - S_{CC}) + S_A S_C(S_{BB} + S_{BC} - S_{CC}) - S_B S_C^3 \\ &\quad + S_A^3 S_C - S_{AA}(S_{BB} + S_{BC} - S_{CC}) + S_A S_B(S_{BB} - S_{BC} - S_{CC}) + S_B^3 S_C \\ = &\ 0. \end{aligned}$$

This shows that X_2 lies on the Brocard axis. Similar calculations show that Y_2 and Z_2 also lies on the same line.

From the coordinates of X_2, Y_2, Z_2, the perspectivity of ABC and $X_2Y_2Z_2$ is

$$\left(\frac{1}{S_{BC} - S_{AA}} : \frac{1}{S_{CA} - S_{BB}} : \frac{1}{S_{AB} - S_{CC}}\right).$$

This is the Tarry point $X(98)$ on the circumcircle (see Figure **??**). □

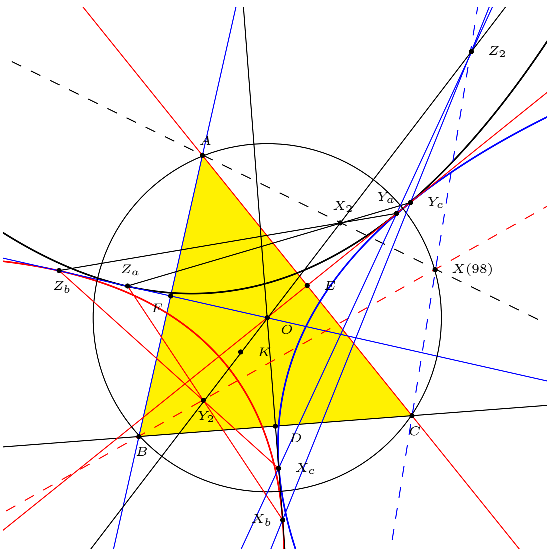

FIGURE 5.

5.2. Triangles with vertices among the tangency points on the angle bisectors.

Proposition 14 (R. Hudson). *The triangle bounded by the lines $I_{ba}I_{ca}$, $I_{cb}I_{ab}$, $I_{ac}I_{bc}$ is perspective with ABC, the perspectrix is the trilinear polar of the incenter, and the perspector the triangle center $X(1000)$.*

Proof. The lines $I_{ba}I_{ca}$, $I_{cb}I_{ab}$, $I_{ac}I_{bc}$ are the polars of the incenter I with respect to the parabolas $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$. They have line coordinates $[-(c+a-b)(a+b-c) : 2ca : 2ab]$, $[2ab : -(a+b-c)(b+c-a) : 2bc]$, $[2ca : 2bc : -(b+c-a)(c+a-b)]$, and intersect a, b, c respectively at $(0 : b : -c)$, $(-a : 0 : c)$, $(a : -b : 0$. Therefore, they bound a triangle perspective with ABC, the perspectrix being the line $\left[\frac{1}{a} : \frac{1}{b} : \frac{1}{c}\right]$, the trilinear polar of the incenter.

The vertices of the triangle are

$$\begin{aligned} X_3 = \ & (-a^4 + 2a^3(b+c) - 2a(b-c)^2(b+c) + (b^2-c^2)^2 \\ & : \ 2bc(a^2+b^2-c^2-4ab) : \ 2bc(c^2+a^2-b^2-4ca)), \\ Y_3 = \ & (2ca(a^2+b^2-c^2-4ab) \\ & : -b^4 + 2b^3(c+a) - 2b(c-a)^2(c+a) + (c^2-a^2)^2 \\ & : \ 2ca(b^2+c^2-a^2-4bc)), \\ Z_3 = \ & (2ab(c^2+a^2-b^2-4ca) : \ 2ab(b^2+c^2-a^2-4bc) \\ & : \ -c^4 + 2c^3(a+b) - 2c(a-b)^2(a+b) + (a^2-b^2)^2). \end{aligned}$$

The perspector with triangle ABC is the triangle center

$$\left(\frac{1}{b^2+c^2-a^2-4bc} : \frac{1}{c^2+a^2-b^2-4ca} : \frac{1}{a^2+b^2-c^2-4ab}\right).$$

This is the triangle center $X(1000)$ in [**?**]. □

Proposition 15. *The triangle bounded by the lines $I'_{ba}I'_{ca}$, $I'_{cb}I'_{ab}$, $I'_{ac}I'_{bc}$ is perspective with ABC, the perspectrix being the trilinear polar of the incenter, and the perspector the orthocenter H.*

Proof. The lines $I'_{ba}I'_{ca}$, $I'_{cb}I'_{ab}$, $I'_{ac}I'_{bc}$ are the polars of I_a, I_b, I_c with respect to the parabolas $\mathscr{P}_a$, $\mathscr{P}_b$, $\mathscr{P}_c$ respectively. They have line coordinates $[(c+a-b)(a+b-c) : 2ca : 2ab]$, $[2bc : (a+b-c)(b+c-a) : 2ab]$, $[2bc : 2ca : (b+c-a)(c+a-b)]$. These lines intersect a, b, c respectively at $(0 : b : -c)$, $(-a : 0 : c)$, $(a : -b : 0)$. Therefore, they bound a triangle perspective with ABC, the perspectrix being the trilinear polar of the incenter.

The vertices of the triangle are

$$\begin{aligned} X_4 = \ & (a^4 - 2a^3(b+c) + 2a(b-c)^2(b+c) - (b^2-c^2)^2 : \ 2bc(a^2+b^2-c^2) \\ & : \ 2bc(c^2+a^2-b^2)), \\ Y_4 = \ & (2ca(a^2+b^2-c^2) : \ b^4 - 2b^3(c+a) + 2b(c-a)^2(c+a) - (c^2-a^2)^2 \\ & : \ 2ca(b^2+c^2-a^2)), \\ Z_4 = \ & (2ab(c^2+a^2-b^2) : \ 2ab(b^2+c^2-a^2) \\ & : \ c^4 - 2c^3(a+b) + 2c(a-b)^2(a+b) - (a^2-b^2)^2). \end{aligned}$$

The perspector with triangle ABC is $\left(\frac{1}{b^2+c^2-a^2} : \frac{1}{c^2+a^2-b^2} : \frac{1}{a^2+b^2-c^2}\right)$, the orthocenter. □

Proposition 16. *Let* $X_5 = I_bI'_{ba} \cap I_cI'_{ca}$, $Y_5 = I_cI'_{cb} \cap I_aI'_{ab}$, *and* $Z_5 = I_aI'_{ac} \cap I_bI'_{bc}$. *The triangle* $X_5Y_5Z_5$ *is perspective with* ABC *at the incenter, and the perspectrix is the trilinear polar of the Nagel point.*

Proof. In homogeneous coordinates, the points are

$$\begin{aligned}
X_5 &= I_bI'_{ba} \cap I_cI'_{ca}\\
&= (-a(b^2+6bc+c^2-a^2) : b(c+a-b)(a+b-c) : c(c+a-b)(a+b-c)),\\
Y_5 &= (a(a+b-c)(b+c-a) : -b(c^2+6ca+a^2-b^2) : c(a+b-c)(b+c-a)),\\
Z_5 &= (a(b+c-a)(c+a-b) : b(b+c-a)(c+a-b) : -c(a^2+6ab+b^2-c^2)).
\end{aligned}$$

From these, it is clear that triangles ABC and ABC are perspective at the incenter.

The lines Y_5Z_5, Z_5X_5, X_5Y_5 are

$$[(a+b+c)(a(b+c)-(b-c)^2) : a(a+b-c)(b+c-a) : a(b+c-a)(c+a-b)],$$
$$[b(c+a-b)(a+b-c) : (a+b+c)(b(c+a)-(c-a)^2) : b(b+c-a)(c+a-b)],$$
$$[c(c+a-b)(a+b-c) : c(a+b-c)(b+c-a) : (a+b+c)(c(a+b)-(a-b)^2)].$$

They intersect a, b, c respectively at

$$(0 : -(c+a-b) : a+b-c),$$
$$(b+c-a : 0 : -(a+b-c)),$$
$$(-(b+c-a) : c+a-b : 0),$$

which are collinear on $\frac{x}{b+c-a} + \frac{y}{c+a-b} + \frac{z}{a+b-c} = 0$, the trilinear polar of the Nagel point. This is the perspectrix of the triangles ABC and $X_5Y_5Z_5$. □

6. A CONIC THROUGH THE POINTS OF TANGENCY WITH THE EXTERNAL ANGLE BISECTORS

Consider the excentral triangle $I_aI_bI_c$. From the coordinates of I'_{ab} given in Proposition **??** with coordinate sum $\frac{1}{2}(c+a-b)(a+b-c)$, and the expression

$$\begin{aligned}
&2a(c^2+a^2-b^2, 2bc, -2c^2)\\
= &-(a+b-c)(b+c-a)(a,-b,c) + (a+b+c)(c+a-b)(a,b,-c),
\end{aligned}$$

we have

$$I'_{ab} = \frac{-(b+c-a)I_b + (a+b+c)I_c}{2a}.$$

From this we have the homogeneous barycentric coordinates of I'_{ab} with respect to the excentral triangle $I_aI_bI_c$; similarly for the other five points of tangency on the external angle bisectors.

Proposition 17. *With respect to the* **excentral triangle**,

$$\begin{aligned}
I'_{ab} &= (0 : a+b+c : -(b+c-a)), & I'_{ac} &= (0 : -(b+c-a) : a+b+c),\\
I'_{bc} &= (-(c+a-b) : 0 : a+b+c), & I'_{ba} &= (a+b+c : 0 : -(c+a-b)),\\
I'_{ca} &= (a+b+c : -(a+b-c) : 0), & I'_{cb} &= (-(a+b-c) : a+b+c : 0).
\end{aligned}$$

Proposition 18. *The six points are on a conic $\mathscr{C}$ with barycentric equation*

$$(a+b+c)(x^2+y^2+z^2)+\sum_{\text{cyclic}}\frac{2(a^2+(b+c)^2)}{b+c-a}yz=0 \tag{10}$$

relative to the excentral triangle $I_aI_bI_c$.

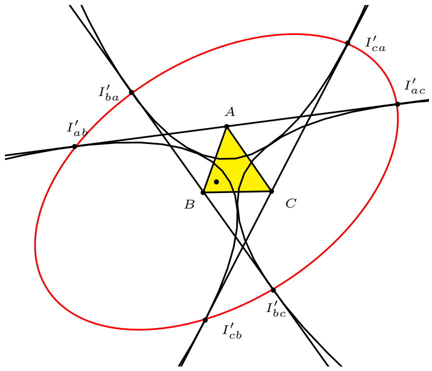

FIGURE 6.

Proof. Substituting $x=0$ into (**??**) we obtain

$$\begin{aligned}
0 =& \ (a+b+c)(y^2+z^2)+\frac{2(a^2+(b+c)^2)}{b+c-a}yz\\
=& \ \frac{1}{b+c-a}((a+b+c)(b+c-a)(y^2+z^2)+2(a^2+(b+c)^2)yz)\\
=& \ \frac{((b+c-a)y+(a+b+c)z)((a+b+c)y+(b+c-a)z)}{b+c-a}.
\end{aligned}$$

Therefore the conic intersects the line I_bI_c at $(0:a+b+c:-(b+c-a))$ and $(0:-(b+c-a):a+b+c)$. These are the points I'_{ab} and I'_{ac}. Similarly, the points I'_{bc}, I'_{ba}, I'_{ca}, I'_{cb} are also on the same conic. □

The center of a conic with given barycentric equation can be computed using the formula given in [**?**, §10.7.2]. Applying this to the conic $\mathscr{C}$ we have

Proposition 19. *Relative to the excentral triangle, the conic $\mathscr{C}$ through the six points I'_{ab}, I'_{ac}, I'_{bc}, I'_{ba}, I'_{ca}, I'_{cb} has center with homogeneous barycentric coordinates*

$$\begin{aligned}
&(a^2(c+a-b)(a+b-c)(a^4-2abc(b+c-a)-(b^2-c^2)^2)\\
:&\ b^2(a+b-c)(b+c-a)(b^4-2abc(c+a-b)-(c^2-a^2)^2)\\
:&\ c^2(b+c-a)(c+a-b)(c^4-2abc(a+b-c)-(a^2-b^2)^2)).
\end{aligned}$$

Remark. With reference to ABC, the conic $\mathscr{C}$ has barycentric equation

$$f_ayz+f_bzx+f_cxy+4abc(x+y+z)g(x,y,z)=0,$$

where

$$f_a = a^6 - a^4(b-c)^2 - a^2(b-c)^2(b^2+10bc+c^2) + 8abc(b-c)^2(b+c) + (b-c)^4(b+c)^2,$$

$$g(x,y,z) = \sum_{\text{cyclic}} bc(a+b-c)(a-b+c)x,$$

and f_b, f_c are obtained from f_a by cyclic rotations of a, b, c. The center of the conic has homogeneous barycentric coordinates

$$\begin{aligned}(a(a^{10} &- a^8(b^2-6bc+c^2) - 6a^7bc(b+c) - 2a^6(b^4+3b^3c-12b^2c^2+3bc^3+c^4)\\ &+ 6a^5bc(b-c)^2(b+c) + 2a^4(b-c)^2(b^4-b^3c-8b^2c^2-bc^3+c^4)\\ &+ 2a^3bc(b-c)^2(b+c)(3b+c)(b+3c) + a^2(b-c)^4(b+c)^2(b^2+8bc+c^2)\\ &- 2abc(b-c)^2(b+c)^3(3b^2-2bc+3c^2) - (b^2-c^2)^4(b^2+c^2))\\ &: \cdots : \cdots).\end{aligned}$$

This has ETC-search number $26.8848761379\cdots$.

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