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ANOTHER PURELY SYNTHETIC PROOF OF DAO'S THEOREM ON SIX CIRCUMCENTERS

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ABSTRACT. Using the notion of directed angle, I will introduce a purely synthetic proof of Dao's theorem that is different from Telv Cohl's proof in [1].

In 2014, Dao Thanh Oai proposed an interesting theorem on six circumcenters associated with six points that are concyclic (see [2]). Dao's theorem is stated accurately as follow.

Theorem 1. *Let be given six points A, B, C, D, E and F belonging to one and the same circle. Let us denote $X = FA \cap BC$, $Y = AB \cap CD$, $Z = BC \cap DE$, $U = CD \cap EF$, $V = DE \cap FA$ and $W = EF \cap AB$. Let O_1, O_2, O_3, O_4, O_5 and O_6 be the circumcenters of triangles XAB, YBC, ZCD, UDE, VEF and WFA respectively. Then, either O_1O_4, O_2O_5 and O_3O_6 are concurrent or O_1O_4, O_2O_5 and O_3O_6 are pairwise parallel or coincident.*

In [2], different from the above accurate assertion, Dao Thanh Oai inaccurately asserted that "O_1O_4, O_2O_5 and O_3O_6 are concurrent".

Using complex numbers, Nikolaos Dergiades introduced the first proof of theorem 1 (see [3]). Immediately after that, Telv Cohl gave a purely synthetic of theorem 1 (see [1]). However, Telv Cohl's proof did not cover all the possible cases.

In this article, using the concepts of directional angle between two lines and directional angle between two vectors, we will introduce a purely synthetic proof of theorem 1 that covers all the possible cases.

The definitions of directional angles between two lines and directional angles between two vectors can be found in [5].

To facilitate the readers, we would like to present a few results related to these two concepts that will be used in this article.

i) For three lines a, b, c, we have $(a, b) \equiv (a, c) + (c, b) \pmod{\pi}$.

ii) For three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, we have $(\overrightarrow{a}, \overrightarrow{b}) \equiv (\overrightarrow{a}, \overrightarrow{c}) + (\overrightarrow{c}, \overrightarrow{b}) \pmod{2\pi}$.

iii) If points A, B, C, D are not linear, then A, B, C, D belong to the same circle if and only if $(AC, AD) \equiv (BC\, BD) \pmod{\pi}$.

iv) If $OA = OB = R$, then point M belongs to the circle of center O and with radius R if and only if $(MA, MB) \equiv \frac{1}{2}(\overrightarrow{OA}, \overrightarrow{OB}) \pmod{\pi}$.

v) Triangles $ABC, A'B'C'$ are similar in the same direction if and only if one of the following conditions occurs:

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• $\dfrac{AB}{AC} = \dfrac{A'B'}{A'C'}$; $(\overrightarrow{AB}, \overrightarrow{AC}) \equiv (\overrightarrow{A'B'}, \overrightarrow{A'C'}) \pmod{2\pi}$.
• $(AB, AC) \equiv (A'B', A'C') \pmod{\pi}$; $(BC, BA) \equiv (B'C', B'A') \pmod{\pi}$.

To prove theorem 1, we need three lemmas.

Lemma 2. *Let be given six points A, B, C, A', B' and C' belonging to one and the same circle. Let us denote $X = AC \cap BC'$ and $X' = A'C' \cap B'C$. Let (O) and (O') be the circumcircles of triangles XAB and $X'A'B'$ respectively. Let A_0 and B_0 be the second intersections of AA' and BB' with (O) respectively. Then,*

1) Triangles XA_0B_0 and $X'A'B'$ are images of each other through a homothety or a translation.

2) AA', BB' and OO' are either concurrent or pairwise parallel.

Proof. It is easy to see that

$$(A_0B_0, A'B') \equiv (A_0B_0, A_0A) + (A'A, A'B') \equiv (BB_0, BA) + (BA, BB') \pmod{\pi}$$
$$\equiv (BB_0, BB') \equiv 0 \pmod{\pi}.$$

Therefore, $A_0B_0 \parallel A'B'$. Thus,

$$(XA_0, X'A') \equiv (XA_0, B_0A_0) + (B'A', C'A') \equiv (XB, B_0B) + (B'B, C'B) \pmod{\pi}$$
$$\equiv (XB, C'B) \equiv 0 \pmod{\pi}.$$

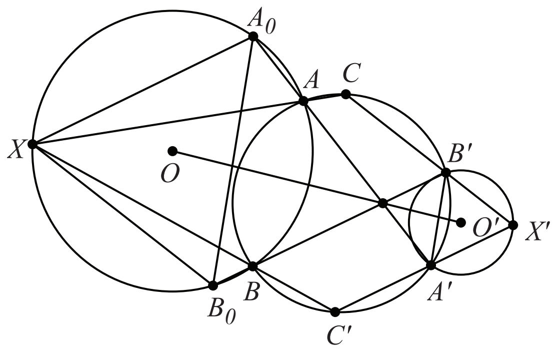
Therefore, $XA_0 \parallel X'A'$. Likewise, $XB_0 \parallel X'B'$.

Thus, triangles XA_0B_0 and $X'A'B'$ have correspondingly parallel sides.

This means that:

1) Triangles XA_0B_0 and $X'A'B'$ are images of each other through a homothety or a translation.

2) AA', BB' and OO' are either concurrent or pairwise parallel.



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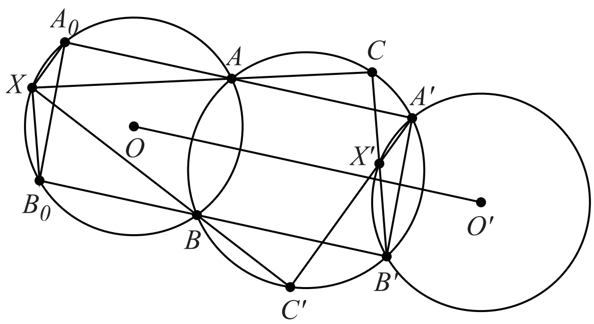


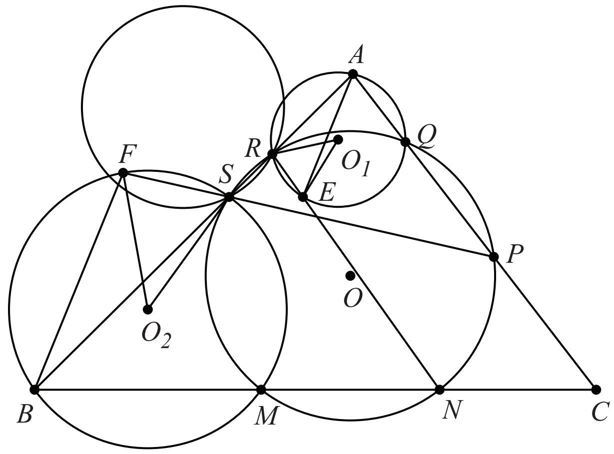
f.2

Lemma 2 is more detail and accurate statement than Lemma 1 of Telv Cohl (see [1]). □

Lemma 3. *Let be given a triangle ABC and a circle (O) not passing through A, B and C and cutting BC, CA and AB at pairs of points (M, N), (P, Q) and (R, S) respectively. Let (O_1) and (O_2) be circumcircles of triangles AQR and BMS respectively. Let NR and PS cut (O_1) and (O_2) again at E and F respectively. Then, triangles O_1RE and O_2SF are similar in the same direction.*

Proof. It is easy to see that (f.3)

$$\frac{1}{2}(\overrightarrow{O_1R},\overrightarrow{O_1E})-\frac{1}{2}(\overrightarrow{O_2S},\overrightarrow{O_2F}) \equiv (AR,AE)-(BS,BF)\equiv(BF,AE)\,(\text{mod }\pi)$$
$$\equiv (BF,BM)+(NM,NR)+(RE,AE)\equiv(SF,SM)+(QM,QR)+(RQ,AQ)\,(\text{mod }\pi)$$
$$\equiv (SP,SM)+(QM,PQ)\equiv(QP,QM)+(QM,PQ)\equiv(QP,PQ)\equiv 0\,(\text{mod }\pi).$$



f.3

Therefore, $(\overrightarrow{O_1R},\overrightarrow{O_1E})\equiv(\overrightarrow{O_2S},\overrightarrow{O_2F})\,(\text{mod}2\pi)$.

Combined with $\dfrac{O_1R}{O_1E}=1=\dfrac{O_2S}{O_2F}$, we can deduce that triangles O_1RE and O_2SF are similar in the same direction.

Comment.

- If O_1 belongs to RN, then O_2 belongs to PS.
- If (O_1) touches RN, then (O_2) touches PS.

The author of lemma 3 is Nguyen Tien Dung, a student of Hanoi University of Foreign Trade.

□

Lemma 4. *Let be given two lines Δ and Δ' and a non-zero number k. Then, the set of points M such that $\dfrac{\overline{MH}}{\overline{MH'}}=k$ is a line, where $H$and H' are the perpendicular projections of M on Δ and Δ' respectively.*

Note that

- Notation $\overline{MH}$ refers to the signed length of vector $\overrightarrow{MH}$.
- If we mention signed length $\overline{MH}$, then line MH is already directed by a unit vector.
- Parallel lines are always directed by the same unit vector.

Return to the proof of theorem 1.

There are two cases.

Case 1. AD, BE and CF are pairwise non-parallel.

There are five sub-cases to consider.

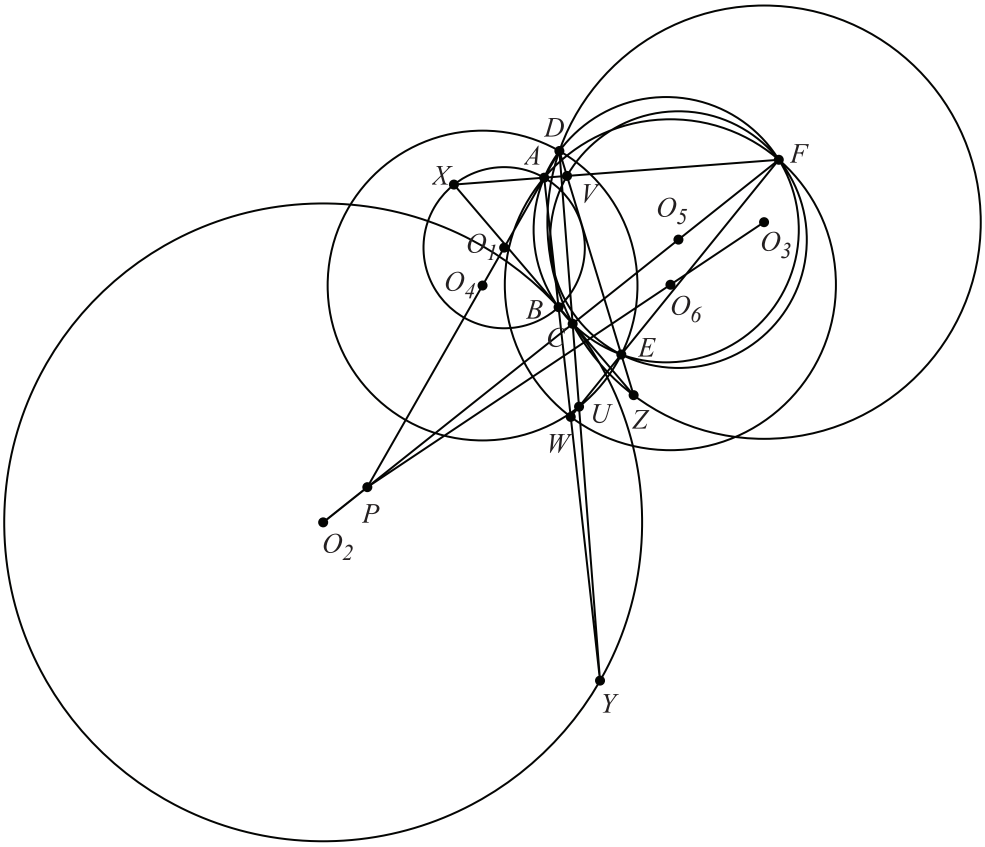
Sub-case 1.1. AD, BE and CF are concurrent.

Suppose that AD, BE and CF are concurrent at S.

By Lemma 1, groups of three lines (AD, BE, O_1O_4), (BE, CF, O_2O_5) and (CF, AD, O_3O_6) are concurrent at S simultaneously.

Thus, O_1O_4, O_2O_5 and O_3O_6 are concurrent at S.

Sub-case 1.2. O_1O_4 either coincides with AD or coincides with BE (f.4).



f.4

Without loss of generality, suppose that O_1O_4 coincides with AD.

According to the first comment after lemma 3, O_2O_5 coincides with CF.

On the other hand, according to part 2 of lemma 2, AD, CF and O_3O_6 are concurrent.

Thus, O_1O_4, O_2O_5 and O_3O_6 are concurrent.

Sub-case 1.3. O_2O_5 either coincides with BE or coincides with CF.

Similar to sub-case 1.2, O_1O_4, O_2O_5 and O_3O_6 are concurrent.

Sub-case 1.4. O_3O_6 either coincides with CF or coincides with AD.

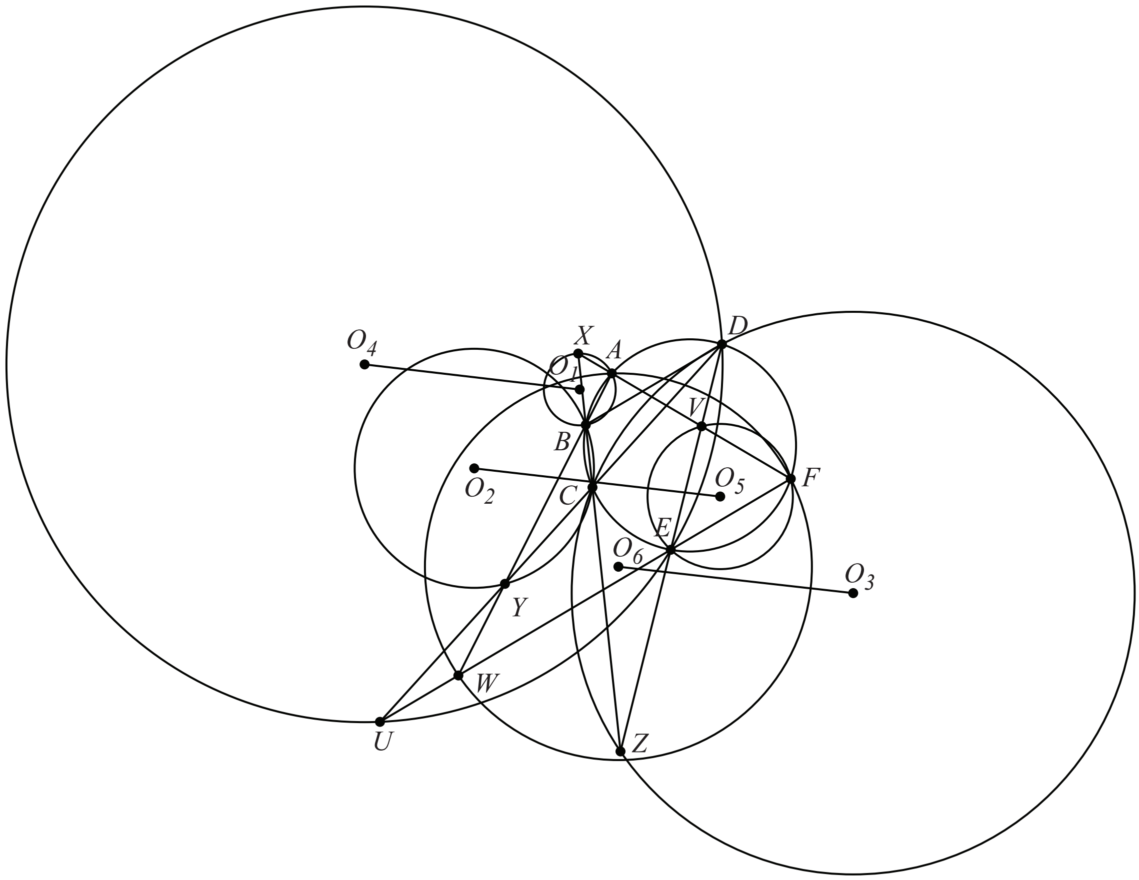
Similar to sub-case 1.2, O_1O_4, O_2O_5 and O_3O_6 are concurrent.

Sub-case 1.5. O_1O_4 does not coincide with AD and BE; O_2O_5 does not coincide with BE and CF; O_3O_6 does not coincide with CF and AD.

There are two sub-sub-cases to consider.

Sub-sub-case 1.5.1. O_1O_4, O_2O_5 and O_3O_6 are pairwise parallel or concurrent (f.5).

Theorem 1 is proved.

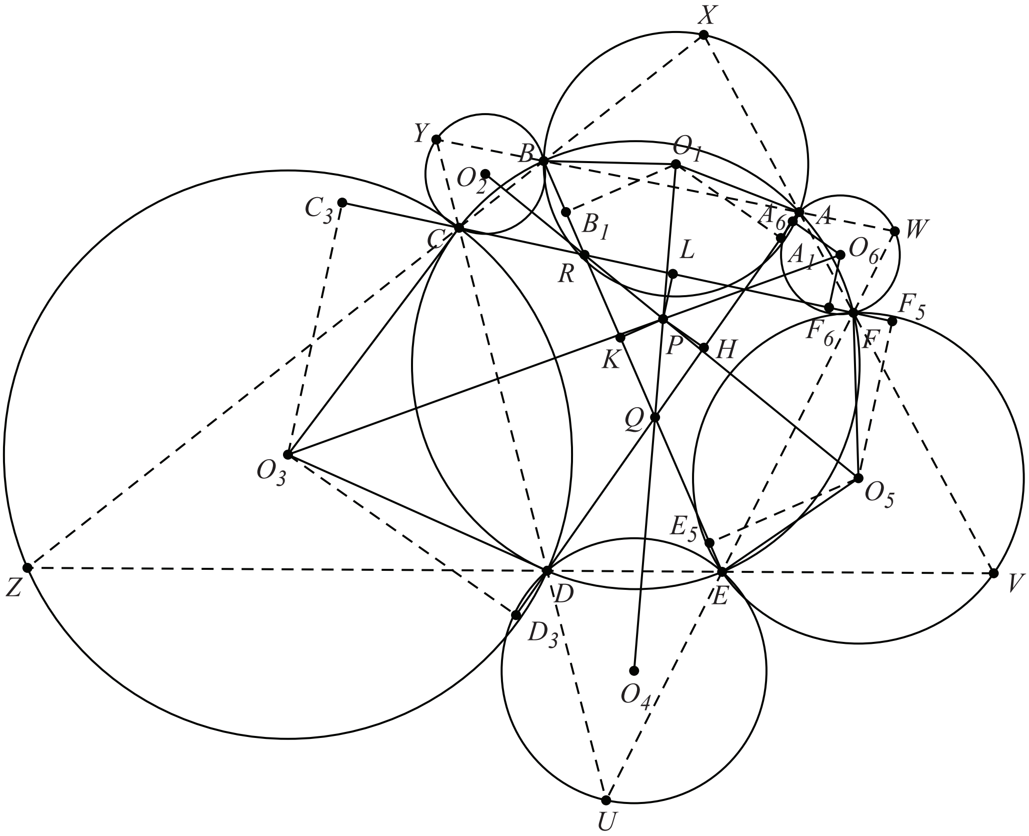


f.5

Sub-sub-cases 1.5.2. O_1O_4, O_2O_5 and O_3O_6 are not pairwise parallel or coincident.

Without loss of generality, suppose that O_1O_4 and O_2O_5 are not either parallel or coincident.

Let P be the intersection of O_1O_4 and O_2O_5. Let H, K and L be the perpendicular projections of P on AD, BE and CF respectively. Let $\overrightarrow{\alpha}$, $\overrightarrow{\beta}$ and $\overrightarrow{\gamma}$ be the unit vectors directing lines perpendicular to AD, lines perpendicular to BE and lines perpendicular to CF respectively. Let A_1 and B_1 be the perpendicular projections of O_1 on AD and BE respectively. Let E_5 and F_5 be the perpendicular projections of O_5 on BE and CF respectively. Let C_3 and D_3 be the perpendicular projections of O_3 on CF and AD respectively. Let F_6 and A_6 be the perpendicular projections of O_6 on CF and AD respectively (f.6).



f.6

Since AD, BE and CF are pairwise non-parallel, without loss of generality, suppose that BE cuts AD and CF. Let Q and R be the intersections of BE with AD and CF respectively. Since AD, BE and CF are not concurrent, $Q \neq R$.

If $P = Q$, then according to part 2 of lemma 2,

$$\begin{aligned} P &= Q = (O_1O_4 \cap O_2O_5) \cap (AD \cap BE) = O_2O_5 \cap BE \\ &= O_2O_5 \cap BE \cap CF = BE \cap CF = R. \end{aligned}$$

This means that AD, BE and CF are concurrent, contradiction. Thus, $P \neq Q$. Likewise, $P \neq R$.

Therefore, if $\overline{PH} = 0$, then O_1O_4 coincides with AD, contradiction. Thus, $\overline{PH} \neq 0$. Likewise, $\overline{PK} \neq 0$ and $\overline{PL} \neq 0$.

According to part 2 of lemma 2, AD, BE and O_1O_4 are concurrent at Q. Therefore, noting that P belongs to O_1O_4, $P \neq Q$, $PH \parallel O_1A_1$ and $PK \parallel O_1B_1$, we can deduce that $\dfrac{\overline{PH}}{\overline{PK}} = \dfrac{\overline{O_1A_1}}{\overline{O_1B_1}}$. Likewise, $\dfrac{\overline{PK}}{\overline{PL}} = \dfrac{\overline{O_5E_5}}{\overline{O_5F_5}}$.

Thus,

$$\frac{\overline{PH}}{\overline{PL}} = \frac{\overline{O_1A_1}}{\overline{O_1B_1}} \cdot \frac{\overline{O_5E_5}}{\overline{O_5F_5}} \tag{1}$$

According to lemma 3, triangles O_1AA_1 and O_5EE_5 are similar in the same direction to triangles O_5FF_5 and O_3DD_3 respectively. Therefore, without loss of generality, suppose that $(\overrightarrow{\alpha}, \overrightarrow{O_1A}) \equiv (\overrightarrow{\gamma}, \overrightarrow{O_5F}) \pmod{2\pi}$and $(\overrightarrow{\beta}, \overrightarrow{O_5E}) \equiv (\overrightarrow{\alpha}, \overrightarrow{O_3D}) \pmod{2\pi}$.

Thus,

$$\begin{aligned}(\overrightarrow{\gamma}, \overrightarrow{O_3C}) - (\overrightarrow{\beta}, \overrightarrow{O_1B}) &\equiv (\overrightarrow{\gamma}, \overrightarrow{O_5F}) + (\overrightarrow{O_5F}, \overrightarrow{O_3C}) - (\overrightarrow{\beta}, \overrightarrow{O_5E}) - (\overrightarrow{O_5E}, \overrightarrow{O_1B}) \pmod{2\pi}\\ &\equiv (\overrightarrow{\alpha}, \overrightarrow{O_1A}) + (\overrightarrow{O_5F}, \overrightarrow{O_3C}) - (\overrightarrow{\alpha}, \overrightarrow{O_3D}) - (\overrightarrow{O_5E}, \overrightarrow{O_1B}) \pmod{2\pi}\\ &\equiv (\overrightarrow{O_3D}, \overrightarrow{O_1A}) + (\overrightarrow{O_5F}, \overrightarrow{O_3C}) + (\overrightarrow{O_1B}, \overrightarrow{O_5E}) \pmod{2\pi}\\ &\equiv (\overrightarrow{O_3D}, \overrightarrow{O_3C}) + (\overrightarrow{O_5F}, \overrightarrow{O_1A}) + (\overrightarrow{O_1B}, \overrightarrow{O_5E}) \pmod{2\pi}\\ &\equiv (\overrightarrow{O_3D}, \overrightarrow{O_3C}) + (\overrightarrow{O_5F}, \overrightarrow{O_5E}) + (\overrightarrow{O_1B}, \overrightarrow{O_1A}) \pmod{2\pi}\\ &\equiv 2(ZD, ZC) + 2(VF, VE) + 2(XB, XA) \pmod{2\pi}\\ &\equiv 2(ZV, ZX) + 2(VX, VZ) + 2(XZ, XV) \pmod{2\pi}\\ &\equiv 2\left((ZV, ZX) + (VX, VZ) + (XZ, XV)\right) \equiv 0 \pmod{2\pi}.\end{aligned}$$

This means that $(\overrightarrow{\gamma}, \overrightarrow{O_3C}) \equiv (\overrightarrow{\beta}, \overrightarrow{O_1B}) \pmod{2\pi}$.

Thus, we have

$$\begin{aligned}&\frac{\overline{O_1A_1}}{\overline{O_1B_1}} \cdot \frac{\overline{O_5E_5}}{\overline{O_5F_5}} \cdot \frac{\overline{O_3C_3}}{\overline{O_3D_3}} = \frac{\overrightarrow{\alpha} \cdot \overrightarrow{O_1A}}{\overrightarrow{\beta} \cdot \overrightarrow{O_1B}} \cdot \frac{\overrightarrow{\beta} \cdot \overrightarrow{O_5E}}{\overrightarrow{\gamma} \cdot \overrightarrow{O_5F}} \cdot \frac{\overrightarrow{\gamma} \cdot \overrightarrow{O_3C}}{\overrightarrow{\alpha} \cdot \overrightarrow{O_3D}}\\ &= \frac{1 \cdot O_1A \cdot \cos(\overrightarrow{\alpha}, \overrightarrow{O_1A})}{1 \cdot O_1B \cdot \cos(\overrightarrow{\beta}, \overrightarrow{O_1B})} \cdot \frac{1 \cdot O_5E \cdot \cos(\overrightarrow{\beta}, \overrightarrow{O_5E})}{1 \cdot O_5F \cdot \cos(\overrightarrow{\gamma}, \overrightarrow{O_5F})} \cdot \frac{1 \cdot O_3C \cdot \cos(\overrightarrow{\gamma}, \overrightarrow{O_3C})}{1 \cdot O_3D \cdot \cos(\overrightarrow{\alpha}, \overrightarrow{O_3D})}\\ &= \frac{\cos(\overrightarrow{\alpha}, \overrightarrow{O_1A})}{\cos(\overrightarrow{\gamma}, \overrightarrow{O_5F})} \cdot \frac{\cos(\overrightarrow{\beta}, \overrightarrow{O_5E})}{\cos(\overrightarrow{\alpha}, \overrightarrow{O_3D})} \cdot \frac{\cos(\overrightarrow{\gamma}, \overrightarrow{O_3C})}{\cos(\overrightarrow{\beta}, \overrightarrow{O_1B})} = 1. \qquad (2)\end{aligned}$$

From (1) and (2), we can deduce that

$$\frac{\overline{PL}}{\overline{PH}} = \frac{\overline{O_3C_3}}{\overline{O_3D_3}} \qquad (3)$$

According to part 1 of lemma 2,

$$\frac{\overline{O_3C_3}}{\overline{O_3D_3}} = \frac{\overline{O_6F_6}}{\overline{O_6A_6}}. \qquad (4)$$

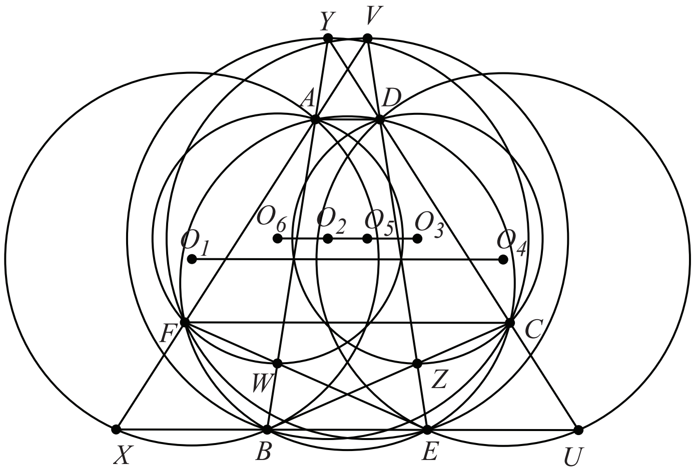
From (3) and (4), by Lemma 4, P belongs to O_3O_6.

Thus, O_1O_4, O_2O_5 and O_3O_6 are concurrent.

Case 2. AD, BE, CF are pairwise parallel (f.7).

According to part 2 of Lemma 2, $O_1O_4 \parallel AD \parallel BE$, $O_2O_5 \parallel BE \parallel CF$ and $O_3O_6 \parallel CF \parallel AD$.

Thus, O_1O_4, O_2O_5, O_3O_6 are pairwise parallel or coincident.



f.7

Remark. On Dao's theorem's configuration, there are still other interesting results (see [5]).

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