SOME FIXED POINT THEOREMS IN THE PROJECTIVE TENSOR PRODUCT OF 2-BANACH SPACES

DIPANKAR DAS, NILAKSHI GOSWAMI, AND VISHNU NARAYAN MISHRA

ABSTRACT. Let $X$ and $Y$ be two normed spaces which are also 2-Banach spaces. For the projective tensor product $X \otimes_\gamma Y$, we have defined a 2-norm for which $X \otimes_\gamma Y$ is a 2-Banach space. Considering the closed and bounded subspace $D_X, D_Y$ and $D_{X \otimes_\gamma Y}$ of $X, Y$ and $X \otimes_\gamma Y$ respectively, we take two pairs of mappings $T_1, S_1 : D_{X \otimes_\gamma Y} \to D_X$ and $T_2, S_2 : D_{X \otimes_\gamma Y} \to D_Y$ satisfying some specific characteristics. Using these two pairs we define a pair of self mappings $T$ and $S$ on $D_{X \otimes_\gamma Y}$. Some fixed point theorems for $T$ and $S$ on $D_{X \otimes_\gamma Y}$ are derived here with suitable example. Moreover, taking $X \otimes_\gamma Y$ as a 2-Banach algebra, we establish another fixed point theorem for a self mapping $\hat{T}$ on $D_{X \otimes_\gamma Y}$ with a proper example.

1. INTRODUCTION

In 1963, Gähler [7] introduced the notion of 2-metric spaces and and their topological structures. The concept of 2-normed linear spaces can be found in Gähler’s paper [8] in 1965. Gähler investigated a lot of results in this space and also proved that if $(X, \|\|)$ is a linear normed space then a 2-norm can be defined on $X$.

In 1969, White [19] introduced the concept of 2-Banach space and gave some important definitions and examples. Since then a lot of applications of such spaces can be found in various aspects.

Many authors viz., Iseki [10], Khan and Khan [11], Rhoads [15], Hadžić [9] etc. developed the applications of 2-metric spaces and 2-Banach spaces in the field of fixed point theory.

In this paper, we consider the projective tensor product space as a 2-Banach space and derive some fixed point theorems here.

PRELIMINARIES

Definition 1.1 [8] Let $X$ be a real linear space of dimension greater than 1 and let $\|\cdot\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(i) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent,
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha$ being real, $x, y \in X$,
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$, for all, $x, y, z \in X$

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Then $\|.,.\|$ is called a 2-norm on $X$ and $(X,\|.,.\|)$ is called a linear 2-normed space.


A 2-Banach algebra $(X,\|.,.\|)$ is a real algebra (of dimension greater than 2) which is a 2-Banach space (with respect to 2-norm topology) and in addition, the following condition holds:

$$\|a, bc\| \leq M\|a, b\|\|a, c\|, \ M > 0 \ \forall a, b, c \in X.$$

**Definition 1.2** A sequence $\{x_n\}$ in a 2-normed space $(X,\|.,.\|)$ is said to be a Cauchy sequence if $\lim_{n,m \to \infty} \|x_n - x_m, a\| = 0$ for all $a \in X$.

**Definition 1.3** A sequence $\{x_n\}$ in a 2-normed space $X$ is called a convergent sequence if there is an $x$ in $X$ such that $\lim_{n \to \infty} \|x_n - x, a\| = 0$ for all $a \in X$.

**Definition 1.4** A 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space.

**Definition 1.5** Let $X$ and $Y$ be two linear 2-normed spaces. An operator $T : X \to Y$ is said to be continuous at $x \in X$ if for every sequence $\{x_n\}$ in $X$, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ in $Y$ as $n \to \infty$.

**Example** [19] Let $P_n$ denotes the set of all real polynomials of degree $\leq n$, on the interval $[0,1]$. Clearly, $P_n$ is a linear vector space over the reals with respect to usual addition and scalar multiplication. Let $\{x_0, x_1, ..., x_{2n}\}$ be distinct fixed points in $[0,1]$ and $g, h \in P_n$. Define the following 2-norm on $P_n$:

$$\|g, h\| = \begin{cases} \sum_{i=0}^{2n} |g(x_i)h(x_i)|, & \text{if } g \text{ and } h \text{ are linearly independent} \\ 0, & \text{if } g \text{ and } h \text{ are linearly dependent} \end{cases}$$

then $(P_n,\|.,.\|)$ is a 2-Banach space.

**Algebraic tensor product:** [6] Let $X, Y$ be normed spaces over $F$ with dual spaces $X^*$ and $Y^*$ respectively. Given $x \in X, y \in Y$. Let $x \otimes y$ be the element of $BL(X^*, Y^*; F)$ (which is the set of all bounded bilinear forms from $X^* \times Y^*$ to $F$), defined by

$$x \otimes y(f, g) = f(x)g(y), \ (f \in X^*, g \in Y^*)$$

The algebraic tensor product of $X$ and $Y$, $X \otimes Y$ is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $BL(X^*, Y^*; F)$.

**Projective tensor product:** [6] Given normed spaces $X$ and $Y$, the projective tensor norm $\gamma$ on $X \otimes Y$ is defined by

$$\|u\|_\gamma = \inf \{\sum_i \|x_i\||y_i| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of $u$.

**Lemma:** [6] $X \otimes_\gamma Y$ can be represented as a linear subspace of $BL(X^*, Y^*; F)$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$, where $\sum_i \|x_i\||y_i| < \infty$. Moreover, $\|u\|_\gamma = \inf \{\sum_i \|x_i\||y_i|\}$ over all such representations of $u$.

Let $X$ and $Y$ be both normed as well as 2-normed spaces. For the projective tensor product $X \otimes_\gamma Y$ we define another 2-norm as follows:

for $u = \sum_i x_i \otimes y_i$, $v = \sum_j p_j \otimes q_j$ and $w = \sum_k g_k \otimes h_k$
\[
\|u, v\| = \begin{cases}
\frac{1}{2} \sum_{i,j} \left(\|x_i, p_j\| \|y_i, q_j\| + \|x_i, q_j\| \|y_i, p_j\|\right), & \text{if } u \text{ and } v \text{ are linearly independent} \\
0, & \text{if } u \text{ and } v \text{ are linearly dependent}
\end{cases}
\]

(i) If \(u\) and \(v\) are linearly dependent then by definition, \(\|u, v\| = 0\).

If \(\|u, v\| = 0\) then \(|x_i, p_j| \|y_i, q_j| = 0\) and \(|y_i, q_j| \|x_i, p_j| = 0\) \(\forall i, j\).

If \(\|x_i, p_j| \neq 0\) then \(|y_i| = 0\) or \(|q_j| = 0\) \(\Rightarrow y_i = 0\) or \(q_j = 0\) \(\forall i, j\).

(Clearly then \(x_i \neq 0\) and \(p_j \neq 0\) otherwise \(\|x_i, p_j| = 0\).)

\(\Rightarrow u = 0\) or \(v = 0\) which shows that \(u\) and \(v\) are linearly dependent.

Similarly, if \(\|y_i, q_j| 
eq 0\) then \(u = 0\) or \(v = 0\) and thus \(u\) and \(v\) are linearly dependent.

Now, let \(|x_i, p_j| = 0\) and \(|y_i, q_j| = 0\) \(\forall i, j\).

\(\Rightarrow x_i\) and \(p_j\) are linearly dependent and \(y_i\) and \(q_j\) are linearly dependent \(\forall i, j\)(being 2-norms).

This also implies that \(x_i, x_j\'\)s are linearly dependent \(\forall i, j\), and \(y_i, y_j\'\)s are linearly dependent \(\forall i, j\).

Therefore, \(u\) can be expressed as: \(u = \alpha(x_1 \otimes y_1)\), for some scalar \(\alpha\).

Similarly, \(p_i, p_j\'\)s are linearly dependent \(\forall i, j\), and \(q_i, q_j\'\)s are linearly dependent \(\forall i, j\).

So, we can take \(v = \beta(p_1 \otimes q_1)\), for some scalar \(\beta\). But \(x_1, p_1\) and \(y_1, q_1\) are linearly dependent. Hence, \(u\) and \(v\) are linearly dependent.

(ii) \(\|u, v\| = \|v, u\|\)

(iii) \(\|u, \alpha v\| = |\alpha| \|u, v\|\), \(\alpha\) is a scalar

(iv) \(\|u, v + w\| \leq \|u, v\| + \|u, w\|\) (all these properties follow by definition.)

(Moreover, if \(X\) and \(Y\) are two 2-Banach spaces, then \(X \otimes_\gamma Y\) is also a 2-Banach space for the above 2-norm.)

Let \(D_X, D_Y\) and \(D_{X \otimes_\gamma Y}\) denote closed and bounded subsets of \(X, Y\) and \(X \otimes_\gamma Y\) respectively.

Let \(T_1, S_1 : D_{X \otimes_\gamma Y} \to D_X\) and \(T_2, S_2 : D_{X \otimes_\gamma Y} \to D_Y\) be such that for any \(u, v \in D_{X \otimes_\gamma Y}\) and \(a \otimes b \in D_{X \otimes_\gamma Y}\) with \(\|a\| \cdot \|b\| \geq 1\) and positive \(k, k'\),

(A) \(\|T_1(u) - T_1(v)\| \leq \frac{1}{M_2 N_1}(k \|u, v, a \otimes b\| - \psi(k \|u, v, a \otimes b\|))\)

(B) \(\|T_2(u) - T_2(v)\| \leq \frac{1}{M_1 N_2}(k' \|u, v, a \otimes b\| - \psi(k' \|u, v, a \otimes b\|))\)

(C) \(\|T_1(u) - T_1(v), a\| \leq \frac{1}{N_1^2}(k \|u, v, a \otimes b\| - \psi(k \|u, v, a \otimes b\|))\)

(D) \(\|T_2(u) - T_2(v), b\| \leq \frac{1}{N_2^2}(k \|u, v, a \otimes b\| - \psi(k \|u, v, a \otimes b\|))\)

where,

(a) \(\psi : [0, \infty) \to [0, \infty)\) is continuous and non-decreasing. \(\psi(0) = 0\),

(b) \(\max\|T_1u, \|S_1u\| \leq N_1\) and \(\max\|T_2u, \|S_2u\| \leq N_2\) (where \(u \in D_{X \otimes_\gamma Y}\). (Here \(D_X, D_Y\) and \(D_{X \otimes_\gamma Y}\) are bounded in norm by \(N_0\).

(c) \(\max\|T_1u, \|S_1u, a\| \leq M_1\) and \(\max\|T_2u, \|S_2u, b\| \leq M_2\) (where \(u \in D_{X \otimes_\gamma Y}\). (Here \(D_X, D_Y\) and \(D_{X \otimes_\gamma Y}\) are bounded in 2-norm by \(M_0\) and \(M_1\) and \(M_2\) respectively.)
From the pairs \((T_1, S_1)\) and \((T_2, S_2)\), we define self mappings \(T\) and \(S\) on \(D_{X\otimes Y}\) as earlier such that \(Tu = T_1u \otimes T_2u\) and \(Su = S_1u \otimes S_2u\) \(\forall u \in D_{X\otimes Y}\).

**Main Results**

**Theorem 2.1** For normed spaces \(X\) and \(Y\) which are also 2-Banach spaces, the mapping \(T\) derived by the pair of mappings \((T_1, T_2)\) satisfying (A), (B), (C) and (D) has a unique fixed point in \(D_{X\otimes Y}\) if

(i) \(p, q \in D_{X\otimes Y}\) with \(\|p\| > \|q\| \Rightarrow \|x, p\| > \|x, q\| \forall x \in D_{X\otimes Y}\) and

(ii) \(k + k' \leq 1\)

**Proof.** For \(u, v \in D_{X\otimes Y}\) and an arbitrary \(a \otimes b \in D_{X\otimes Y}\) with \(\|a\|\|b\| \geq 1\), we have,

\[
\|Tu - Tv, a \otimes b\| = \|T_1u \otimes T_2u - T_1v \otimes T_2v, a \otimes b\|
\leq \|(T_1u - T_1v) \otimes T_2u, a \otimes b\| + \|T_1v \otimes (T_2u - T_2v), a \otimes b\|
= \frac{1}{2}\|T_1u - T_1v\|\|T_2u\|\|b\| + \|T_2u, b\|\|T_1u - T_1v\|\|a\|
+ \|T_2u - T_2v, b\|\|T_1v\|\|a\| + \|T_1v, a\|\|T_2u - T_2v\|\|b\|
\leq \frac{1}{2}\|\frac{1}{N_2}\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)\|N_2N_2
+ \frac{1}{M_2N_1}\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)\|M_2N_1
+ \|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)\|M_1N_1
+ \|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)\|M_1N_2\]

\[
= (k + k')\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|) - \psi(k\|u - v, a \otimes b\|)
\leq \|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|) + \psi(k\|u - v, a \otimes b\|)\] (for \(k + k' \leq 1\))

Let \(x_0 \in D_{X\otimes Y}\) be fixed. We take \(x_{n+1} = Tx_n, n = 0, 1, 2, \ldots \) Now,

\[
\|x_{n+1} - x_n, a \otimes b\| = \|Tx_n - Tx_{n-1}, a \otimes b\|
\leq \|x_n - x_{n-1}, a \otimes b\| - \psi(k\|x_n - x_{n-1}, a \otimes b\|)
- \psi(k\|x_n - x_{n-1}, a \otimes b\|)
\leq \|x_n - x_{n-1}, a \otimes b\|
\]

Hence \(\{\|x_{n+1} - x_n, a \otimes b\|\}\) is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say \(r\).

Taking \(n \to \infty\), we get

\[
r \leq r - (\psi(kr) + \psi(k\|r\|), \text{ (by continuity of } \psi)\)
\Rightarrow \psi(kr) + \psi(k\|r\|) \leq 0,
\]

this is possible only when \(r = 0\). So,

\[
\lim_{n \to \infty} \|x_{n+1} - x_n, a \otimes b\| = 0.
\]
Let \( p \otimes q \in D_{X \otimes Y} \) be such that \( \|p \otimes q\| < 1 \). If
\[
\lim_{n \to \infty} (x_{n+1} - x_n) \neq k(p \otimes q) \quad \text{for any } k,
\]
then
\[
\lim_{n \to \infty} \|x_{n+1} - x_n, p \otimes q\| \neq 0.
\]
Again \( \|a \otimes b\| \geq 1 \) and \( \|p \otimes q\| < 1 \). Therefore,
\[
\|a \otimes b\| > \|p \otimes q\| \Rightarrow \|u, a \otimes b\| > \|u, p \otimes q\| \quad \forall u \in D_{X \otimes Y} \quad \text{(by condition (i))}.
\]
So,
\[
\|x_{n+1} - x_n, a \otimes b\| > \|x_{n+1} - x_n, p \otimes q\| \quad n = 0, 1, 2, ...
\]
Taking limit as \( n \to \infty \), we get,
\[
\lim_{n \to \infty} \|x_{n+1} - x_n, a \otimes b\| \geq \lim_{n \to \infty} \|x_{n+1} - x_n, p \otimes q\| \neq 0, \quad \text{a contradiction.}
\]
So, \( \lim_{n \to \infty} (x_{n+1} - x_n) = k(p \otimes q) \) for some \( k \).

Thus, \( \lim_{n \to \infty} \|x_{n+1} - x_n, p \otimes q\| = 0 \) for any \( p \otimes q \in D_{X \otimes Y} \) with \( \|p \otimes q\| < 1 \).

So, \( \lim_{n \to \infty} \|x_{n+1} - x_n, u\| = 0 \forall u \in D_{X \otimes Y} \). Now, for any integer \( p > 0 \),
\[
\|x_n - x_{n+p}, u\| \leq \|x_n - x_{n+1}, u\| + \|x_{n+1} - x_{n+2}, u\| + \ldots + \|x_{n+p-1} - x_{n+p}, u\|
\]
\[
\rightarrow 0 \quad \text{as } n \to \infty
\]
showing that \( \{x_n\} \) is a Cauchy sequence in \( D_{X \otimes Y} \). Let it converge to some \( z \in D_{X \otimes Y} \). Now,
\[
\|z - Tz, u\| \leq \|z - x_{n+1}, u\| + \|x_{n+1} - Tz, u\|
\]
\[
= \|z - x_{n+1}, u\| + \|Tx_n - Tz, u\|
\]
\[
\leq \|z - x_{n+1}, u\| + \|x_n - z, u\| - [\psi(k\|x_n - z, u\|)] + [\psi(k\|x_n - z, u\|)]
\]
\[
\rightarrow 0 \quad \text{as } n \to \infty
\]
Hence, \( \|z - Tz, u\| = 0 \Rightarrow z = Tz \).

To show the uniqueness:
Let \( z_1 \) and \( z_2 \) be two distinct fixed points for \( T \) in \( D_{X \otimes Y} \). Now,
\[
\|z_1 - z_2, u\| = \|Tz_1 - Tz_2, u\|
\]
\[
\leq \|z_1 - z_2, u\| - [\psi(k\|z_1 - z_2, u\|)] + [\psi(k\|z_1 - z_2, u\|)]
\]
\[
\Rightarrow [\psi(k\|z_1 - z_2, u\|)] + [\psi(k\|z_1 - z_2, u\|)] \leq 0,
\]
possible only for \( z_1 = z_2 \) (since \( u \) is arbitrary).

Thus, \( T \) has a unique fixed point in \( D_{X \otimes Y} \). \( \square \)

**Example 2.2** We define the following 2-norms (in the sense of White) on the normed spaces \( l^1 \) and \( K \):
\[
\|., .\| : K \times K \to \mathbb{R}^+ \cup \{0\} \quad \text{such that}
\]
\[
\|x, b\| = \begin{cases} \|x\| \|b\|, & \text{if } x \text{ and } b \text{ are linearly independent} \\ \|x\|_1 \|b\|_1, & \text{if } x \text{ and } b \text{ are linearly dependent} \\ \end{cases}
\]
and \( \|., .\| : l^1 \times l^1 \to \mathbb{R}^+ \cup \{0\} \) such that
\[
\|\{x_n\}, \{b_n\}\| = \begin{cases} \|\{x_n\}\| \|\{b_n\}\|, & \text{if } \{x_n\}_n \text{ and } \{b_n\}_n \text{ are linearly independent} \\ 0, & \text{if } \{x_n\}_n \text{ and } \{b_n\}_n \text{ are linearly dependent} \\ \end{cases}
\]
We define \( T \) and \( T \). Using these norms we can now define a 2-norm in the projective tensor product \( l^1 \otimes \gamma \mathbf{K} \), i.e., \( l^1(\mathbf{K}) \) as:

for \( u = \sum_i x_i \otimes y_i, v = \sum_j p_j \otimes q_j \in l^1 \otimes \gamma \mathbf{K} \)

\[
\|u, v\| = \frac{1}{2} \sum_{i,j} (\|x_i, p_j\| \|y_i\| \|q_j\| + \|y_i, q_j\| \|x_i\| \|p_j\|) = \frac{1}{2} \sum_{i,j} (\|x_i\| \|p_j\| \|y_i\| \|q_j\| + \|x_i\| \|p_j\| \|y_i\| \|q_j\|) = \|x_i\| \|p_j\| \|y_i\| \|q_j\| \text{ (for linearly independent } \{x_i\} \text{ and } \{p_j\}) \text{ and } \{y_i\} \text{ and } \{q_j\})
\]

Let \( D_1, D_\mathbf{K} \) and \( D_1 \otimes \gamma \mathbf{K} \) be the closed subsets of the normed spaces \( l^1, \mathbf{K} \) and \( l^1 \otimes \gamma \mathbf{K} \), bounded by the constants \( K, K \) and \( K^2 \) respectively (\( K \geq 1 \)).

We define \( T_1 : D_1 \otimes \gamma \mathbf{K} \rightarrow D_1 \) by

\[
T_1(\sum_i a_i \otimes x_i) = \frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}, \text{ where } a_i = \{a_{i_n}\}
\]

and \( T_2 : D_1 \otimes \gamma \mathbf{K} \rightarrow D_\mathbf{K} \) by \( T_2(\sum_i a_i \otimes x_i) = \frac{1}{4} \sum_i ||a_i|| \|x_i\| \).

For arbitrary \( b_j = \{b_{j_n}\} \in D_1, b \in D_\mathbf{K} \) with \( ||b|| \|b\| \geq 1 \)

\[
\|T_1(\sum_i a_i \otimes x_i), b_j\| = \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}, b]\| = \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}\| \|b_j\| \text{ (for the 2-norm)} \leq \frac{1}{2K^5} \sum_i ||a_i|| \|x_i\| ||b_j||
\]

So, taking the projective tensor norm in \( l^1 \otimes \gamma \mathbf{K} \) i.e., \( l^1(\mathbf{K}) \), we get,

\[
\|T_1(\sum_i a_i \otimes x_i), b\| \leq \frac{1}{2K^5} \|\sum_i a_i \otimes x_i\| \|b\| \leq \frac{1}{2K^5} K^3 = \frac{1}{2K^2}(= M_1)
\]

and

\[
\|T_1(\sum_i a_i \otimes x_i)\| = \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}\| \leq \frac{1}{2K^5} \sum_i ||a_i|| \|x_i\|
\]

So, taking the projective tensor norm, in \( l^1 \otimes \gamma \mathbf{K} \), we get,

\[
\|T_1(\sum_i a_i \otimes x_i)\| \leq \frac{1}{2K^5} \|\sum_i a_i \otimes x_i\| \leq \frac{1}{2K^5} K^2 = \frac{1}{2K^3}(= N_1)
\]
Similarly,

\[ \| T_2(\sum_i a_i \otimes x_i), b \| = \frac{1}{4} \sum_i \| a_i \| \| x_i \| \| b \| \]

\[ = \frac{1}{4} \sum_i \| a_i \| \| x_i \| \| b \| \] (for the 2-norm)

\[ \leq \frac{1}{4} \sum_i \| a_i \| \| x_i \| \| b \| \]

So, taking the projective tensor norm,

\[ \| T_2(\sum_i a_i \otimes x_i), b \| \leq \frac{1}{4} \sum_i \| a_i \otimes x_i \| \| b \| \leq \frac{K^3}{4} (= M_2) \]

and

\[ \| T_2(\sum_i a_i \otimes x_i) \| \leq \frac{1}{4} \sum_i \| a_i \otimes x_i \| \leq \frac{K^2}{4} (= N_2) \]

For \( u = \sum_i a_i \otimes x_i \) and \( v = \sum_i d_i \otimes y_i \) in \( D_{l_1} \otimes_{\gamma_K} K \), we have,

\[ \| T_1 u - T_1 v, b_j \| = \| \frac{1}{2K^3} \sum_i \{ a_i x_i \} - \frac{1}{2K^3} \sum_i \{ d_i y_i \} \| b_j \| \]

\[ \leq \| \frac{1}{2K^3} \sum_i a_i \otimes x_i - \frac{1}{2K^3} \sum_i d_i \otimes y_i \| \| b_j \| \| b_j \| \| b \| \] (\( \because \| b_j \| \| b \| \geq 1 \))

\[ = \frac{1}{2K^3} \| u - v, b_j \otimes b \| \]

\[ = \frac{1}{4} \| u - v, b_j \otimes b \| \]

\[ = \frac{1}{4} \| u - v, b_j \otimes b \| \]

\[ = \frac{1}{N_2} \left[ \frac{1}{2} \| u - v, b_j \otimes b \| - \psi \left( \frac{1}{2} \| u - v, b_j \otimes b \| \right) \right], \]

where \( \psi(t) = \frac{t}{2}, k = \frac{1}{2} \)
and

$$\|T_1 u - T_1 v\| = \|\frac{1}{2K^3} \sum_i \{a_i x_i\} - \frac{1}{2K^3} \sum_i \{d_i y_i\}\|
\leq \frac{1}{4} \|u - v\| \leq \frac{1}{4} \|u - v\|||b_j||b|
\leq \frac{1}{2K^3} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| - \frac{1}{2} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| \right] \right]
\leq \frac{1}{2K^3} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| - \frac{1}{2} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| \right] \right] (\text{for the 2-norm})
= \frac{1}{M_2 N_1} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| - \psi \left( \frac{1}{2} \|u - v, b_j \otimes b\| \right) \right],
\text{where } \psi(t) = \frac{t}{2}, k = \frac{1}{2}

Again,

$$\|T_2 u - T_2 v, b\| = \|\frac{1}{4} \sum_i \|a_i\| |x_i| - \frac{1}{4} \sum_i \|d_i\| |y_i|, b\|
\leq \frac{1}{4} \left| \sum_i \|a_i\||x_i| - \sum_i \|d_i\||y_i| \right| ||b|| ||b_j|| ||b||$$

Taking the projective tensor norm,

$$\|T_2 u - T_2 v, b\| \leq \frac{K}{4} \|u - v\| ||b|| ||b_j||
\leq \frac{K}{4} \|u - v\|||b_j|| ||b|
= \frac{K}{4} \|u - v, b_j \otimes b\| (\text{for the 2-norm})
\Rightarrow \|T_2 u - T_2 v, b\| \leq 4K^6 \left( \frac{1}{4} \|u - v, b_j \otimes b\| - \frac{1}{2} \left[ \frac{1}{4} \|u - v, b_j \otimes b\| \right] \right) (\because K \geq 1)
= \frac{1}{4} \|u - v, b_j \otimes b\| - \psi \left( \frac{1}{4} \|u - v, b_j \otimes b\| \right)
\leq \frac{1}{N_2^2} \left[ \frac{1}{2} \|u - v, b_j \otimes b\| - \psi \left( \frac{1}{2} \|u - v, b_j \otimes b\| \right) \right],
\text{where } \psi(t) = \frac{t}{2}, k' = \frac{1}{4}$$
Also,
\[ \|T_2u - T_2v\| = \|\frac{1}{4} \sum_i |a_i||x_i| - \frac{1}{4} \sum_i |d_i||y_i|\]
\[ \leq \frac{1}{4} \left( \sum_i |a_i||x_i| - \sum_i |d_i||y_i| \right) \|b_j\| |b| \quad (\because |b_j||b| \geq 1) \]

Taking the projective tensor norm ,
\[ \|T_2u - T_2v\| \leq \frac{1}{4} \|u - v\| |b_j||b| \]
\[ = \frac{1}{4} \|u - v, b_j \otimes b\| \leq \|u - v, b_j \otimes b\| \]
\[ = \frac{1}{4} \|u - v, b_j \otimes b\| - \frac{3}{4} \left( \frac{1}{4} \|u - v, b_j \otimes b\| \right) \]
\[ = \frac{K^2}{4} \frac{1}{2K^2} M_1 M_2 , \text{ where } \psi(t) = \frac{t}{2}, k' = \frac{1}{4} \]

Therefore, \((T_1, T_2)\) satisfies the conditions (A), (B), (C), (D). Also,
\[ k + k' = \frac{1}{2} + \frac{1}{4} < 1 \]

Moreover, for \(\sum_i p_i \otimes x_i, \sum_i q_i \otimes y_i\) and \(\sum_j b_j \otimes c_j \in D^n \otimes K\),
\[ \|\sum_i p_i \otimes x_i\| > \|\sum_i q_i \otimes y_i\| \Rightarrow \sum_i \|p_i||x_i| > \sum_i \|q_i||y_i| \]
\[ \Rightarrow \sum_j \|b_j||c_j| \sum_i \|p_i||x_i| > \sum_j \|b_j||c_j| \sum_i \|q_i||y_i| \]
\[ \Rightarrow \sum_j \|b_j||p_i||c_j||x_i| > \sum_j \|b_j||p_i||c_j||y_i| \]
\[ \Rightarrow \|\sum_j b_j \otimes c_j, \sum_i p_i \otimes x_i\| > \|\sum_j b_j \otimes c_j, \sum_i q_i \otimes y_i| \]

So, by Theorem 2.1, the mapping \(T : D^n \otimes K \rightarrow D^n \otimes K\) defined by
\[ T(u) = T_1(u) \otimes T_2(u) \]

has a unique fixed point in \(D^n \otimes K\).

**Corollary 2.3** If in Theorem 2.1 \(T_1\) or \(T_2\) satisfies one of the following conditions:
\[ \|T_1(u) - T_1(v), a\| \leq \frac{k}{M_2} \|u - v, a \otimes b\|, \quad \|b\| \geq 1 \]
\[ \|T_2(u) - T_2(v), b\| \leq \frac{k'}{M_1} \|u - v, a \otimes b\|, \quad \|a\| \geq 1 \]
then $T$ has a unique fixed point if $k + k' \leq 1$.

**Corollary 2.4** If $T_1$ and $T_2$ satisfy both the above conditions, then $T$ has a unique fixed point if $k + k' < 1$.

Now, we discuss common fixed points for the pair of mappings $(T, S)$ derived by the pairs $(T_1, S_1)$ and $(T_2, S_2)$. For this, we replace the conditions (A), (B), (C) and (D) of the Theorem 2.1 by the following:

(E) $\|T_1(u) - S_1(v)\| \leq \frac{1}{M_2 N_1} (k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|))$

(F) $\|T_2(u) - S_2(v)\| \leq \frac{1}{M_1 N_2} (k'\|u - v, a \otimes b\| - \psi(k'\|u - v, a \otimes b\|))$

(G) $\|T_1(u) - S_1(v), a\| \leq \frac{1}{N_2} (k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|))$

(H) $\|T_2(u) - S_2(v), b\| \leq \frac{1}{N_1} (k'\|u - v, a \otimes b\| - \psi(k'\|u - v, a \otimes b\|))$

**Theorem 2.5** The mappings $T$ and $S$ derived by $(T_1, S_1)$ and $(T_2, S_2)$ satisfying (E), (F), (G) and (H) have a common unique fixed point in $D_{X \otimes Y}$, if $p, q \in D_{X \otimes Y}$

(i) $\|p\| > \|q\| \Rightarrow \|x, p\| > \|x, q\| (\forall x \in D_{X \otimes Y})$ and

(ii) $k + k' \leq 1$.

**Proof.** For $u, v \in D_{X \otimes Y}$, and $a \otimes b \in D_{X \otimes Y}$ with $\|a \otimes b\| \geq 1$

$$\|T u - S v, a \otimes b\| = \|T_1 u \otimes T_2 u - S_1 v \otimes S_2 v, a \otimes b\|$$

$$\leq \frac{1}{2} \|T_1 u - S_1 v\| \|T_2 u\| \|b\| + \|T_2 u\| \|T_1 u - S_1 v\| \|a\|$$

$$+ \|T_2 u - S_2 v\| \|S_1 v\| \|a\| + \|S_1 v\| \|T_2 u - S_2 v\| \|b\|$$

$$\leq \frac{1}{2} \frac{1}{N_2} [k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)].N_2 N_2$$

$$+ \frac{1}{M_2 N_1} [k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)]M_2 N_1$$

$$+ \frac{1}{N_1} [k'\|u - v, a \otimes b\| - \psi(k'\|u - v, a \otimes b\|)].N_1 N_1$$

$$+ \frac{1}{M_1 N_2} [k'\|u - v, a \otimes b\| - \psi(k'\|u - v, a \otimes b\|)]M_1 N_2]$$

$$\leq (k + k')\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|) - \psi(k'\|u - v, a \otimes b\|)$$

$$\leq \|u - v, a \otimes b\| - (\psi(k\|u - v, a \otimes b\|) + \psi(k'\|u - v, a \otimes b\|))$$
Let \( x_0 \in D_X \otimes Y \) be fixed. We take \( x_1 = Sx_0, x_2 = Tx_1, x_3 = Sx_2, x_4 = Tx_3, \ldots \), \( x_{2n+1} = Sx_{2n}, x_{2n+2} = Tx_{2n+1} \). Now,

\[
\|x_{2n+2} - x_{2n+1}, a \otimes b\| = \|Tx_{2n+1} - Sx_{2n}, a \otimes b\| \\
\leq \|x_{2n+1} - x_{2n}, a \otimes b\| - \psi(k\|x_{2n+1} - x_{2n}, a \otimes b\|) \\
\leq \|x_{2n+1} - x_{2n}, a \otimes b\| \\
= \|x_{2n+1} - x_{2n}, a \otimes b\| - \psi(k\|x_{2n+1} - x_{2n}, a \otimes b\|)
\]

Similarly, \( \|x_{2n+1} - x_{2n}, a \otimes b\| \leq \|x_{2n} - x_{2n-1}, a \otimes b\|, n = 0, 1, 2, \ldots \) (1)

Hence \( \{\|x_{n+1} - x_n, a \otimes b\|\} \) is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say \( r \).

Taking \( n \to \infty \) in (1), we get \( r = 0 \). Now, as in Theorem 2.1, we can show that

\[
\lim_{n \to \infty} \|x_{n+1} - x_n, u\| = 0 \forall u \in D_X \otimes Y
\]

Now, it can be easily shown that \( \{x_n\} \) is a Cauchy sequence in \( D_X \otimes Y \). Let it converge to some \( z \in D_X \otimes Y \). Now, proceeding as in Theorem 2.1, we can show that \( z = Sz \) and also \( z = Tz \).

Uniqueness can also be shown in a similar manner.

So, \( z \) is the common unique fixed point for \( T \) and \( S \) in \( D_X \otimes Y \). \( \square \)

Next we consider algebraic tensor product space as a 2-Banach algebra.

Let \( X \) and \( Y \) be two unital Banach algebras which are also 2-Banach algebras. We show that \( X \otimes Y \) with projective tensor norm is a 2-normed algebra (with the 2-norm already defined). For \( u = \sum_{i=1}^{n} x_i \otimes y_i, v = \sum_{j=1}^{m} p_j \otimes q_j \) and \( w = \sum_{r=1}^{k} e_r \otimes f_r \) in \( X \otimes Y \) we have,

\[
\|uw, v\| = \left\| \sum_{i=1}^{\min(n, k)} x_i e_r \otimes y_i f_r \sum_{j=1}^{m} p_j \otimes q_j \right\| \\
= 1/2 \sum_{i, r, j=1}^{\min(n, k)} \left( \|x_i e_r\| \|y_i f_r\| \|q_j\| \right) \\
\leq 1/2 \sum_{i, r, j=1}^{\min(n, k)} \left( K\|x_i\| \|p_j\| + K'\|y_i\| \|q_j\| \right)
\]

(for some constant \( K \) and \( K' \))
Let \( e = \min_{j=1,2,\ldots,m}(\|p_j\|, \|q_j\|), \) \( p_j \neq 0, q_j \neq 0. \) We choose \( N(\geq 1) \) be such that \( \max(K, K') \leq Ne. \) Now,

\[
\|uw, v\| \leq \frac{Ne}{2} \min(m,n,k) \sum_{i,j=1}^{m,n,k} \left( \|x_i, p_j\| \|e_i, p_j\| \|y_i, f_r\| \|q_j\| + \|y_i, q_j\| \|f_r, q_j\| \|x_i, e_r\| \|p_j\| \right) \\
= \frac{N}{2} \min(m,n,k) \sum_{i,j=1}^{m,n,k} \left[ \ |\|x_i, p_j\| \|y_i, q_j\| (\|e_i, p_j\| \|y_i, f_r\| \|q_j\|) + \right. \\
+ \left. \ |\|y_i, q_j\| \|x_i, e_r\| \|p_j\| \right] \\
\leq \frac{N}{2} \min(m,n,k) \sum_{i,j=1}^{m,n,k} \left( \|x_i, p_j\| \|y_i, q_j\| |e_i, p_j| \|y_i, f_r\| \|q_j\| + \|y_i, q_j\| \|x_i, e_r\| \|p_j\| + \right. \\
+ \left. \|y_i, q_j\| \|x_i, e_r\| \|p_j\| + \|y_i, q_j\| \|x_i, e_r\| \|p_j\| + \|y_i, q_j\| \|x_i, e_r\| \|p_j\| + \|y_i, q_j\| \|x_i, e_r\| \|p_j\| \right) \\
= M \sum_{i,j=1}^{m,n,k} \frac{1}{2} \left( \|x_i, p_j\| \|y_i, q_j\| \|x_i, e_r\| \|p_j\| \right) \frac{1}{2} \left( \|e_i, p_j\| \|f_r, q_j\| |e_i, p_j| \|f_r, q_j\| \right) \\
= M \|u, v\| \|w, v\|, \text{ where } M = 2N
\]

showing that \( X \otimes Y \) is a 2-normed algebra. Taking the completion with respect to the 2-norm, we can make it a 2-Banach algebra.

Let \( (T_1, T_2) \) be a pair of mappings where \( T_1 : D_{X \otimes Y} \to D_{X \otimes Y} \) and \( T_2 : D_{X \otimes Y} \to D_{X \otimes Y} \) are such that for any \( u, v, a, b \in D_{X \otimes Y} \cup \{e\} \) with \( \|a\|, \|b\| \geq 1, \)

(I) \( \|T_1(u) - T_1(v), a\| \leq \frac{1}{\sqrt{MM_2}} \sqrt{k\|u - v, ab\| - \psi(k\|u - v, ab\|)} \)

(J) \( \|T_2(u) - T_2(v), b\| \leq \frac{1}{\sqrt{MM_2}} \sqrt{k'\|u - v, ab\| - \psi(k'\|u - v, ab\|)}, \)

(for constants \( M_1, M_2 > 0 \) and \( M \) is defined as above)

where,

(a) \( \psi : [0, \infty) \to [0, \infty) \) is continuous and non-decreasing, \( \psi(0) = 0 \)

(b) \( \|T_1u, a\| \leq \frac{M_1}{M} \) and \( \|T_2u, b\| \leq \frac{M_2}{M} \) [Here \( D_{X \otimes Y} \) is a closed and bounded subspace of \( X \otimes Y \).

From the pair \( (T_1, T_2) \) we define a self mapping \( \hat{T} : D_{X \otimes Y} \to D_{X \otimes Y} \) such that \( \hat{T}u = T_1uT_2u \) for \( u \in D_{X \otimes Y}. \)

**Theorem 2.6** From the pair of mappings \( (T_1, T_2) \) satisfying (I) and (J) the mapping \( \hat{T} \) defined above has a unique fixed point in \( D_{X \otimes Y} \) if

(i) \( \|p, q \in D_{X \otimes Y}, \|p\| > \|q\| \Rightarrow \|u, p\| > \|u, q\| \forall u \in D_{X \otimes Y}, \)

(ii) \( k + k' \leq 1 \)
Proof. For $u, v, a, b \in DX\otimes Y$ with $\|a\| \geq 1, \|b\| \geq 1$

$$\|\hat{T}u - \hat{T}v, ab\| = \|T_1uT_2u - T_1vT_2v, ab\|$$
$$\leq \|T_1u - T_1v\|T_2u, ab\| + \|T_2v(T_2u - T_2v), ab\|$$
$$\leq M\|T_1u - T_1v, ab\|T_2u, ab\| + M\|T_1v, ab\|\|T_2u - T_2v, ab\|$$
$$\leq M^3\|T_1u - T_1v, a\|\|T_1u - T_1v, b\|T_2u, a\|\|T_2u, b\|$$
$$+ M^3\|T_1v, a\|\|T_1v, b\|\|T_2u - T_2v, a\|\|T_2u - T_2v, b\|$$
$$\leq M^3 \left[ \frac{1}{\sqrt{MM_2}} \sqrt{k\|u - v, ab\| - \psi(k\|u - v, ab\|)} \left( \frac{M_2}{M} \right) \right]^2$$
$$+ M^3 \left[ \frac{1}{\sqrt{MM_1}} \sqrt{k'\|u - v, ab\| - \psi(k'\|u - v, ab\|)} \left( \frac{M_1}{M} \right) \right]^2$$
$$= (k + k')\|u - v, ab\| - \psi(k\|u - v, ab\|) - \psi(k'\|u - v, ab\|)$$
$$\leq \|u - v, ab\| - (\psi(k\|u - v, ab\|) + \psi(k'\|u - v, ab\|))$$

Let $x_0 \in DX\otimes Y$ be fixed. We take $x_{n+1} = \hat{T}x_n$, $n = 0, 1, 2, \ldots$ Now, for arbitrary $a \in DX\otimes Y$ with $\|a\| \geq 1$,

$$\|x_{n+1} - x_n, a\| = \|x_{n+1} - x_n, ae\|$$
$$\leq \|x_n - x_{n-1}, ae\| - \psi(k\|x_n - x_{n-1}, ae\|)$$
$$- \psi(k'\|x_n - x_{n-1}, ae\|)$$
$$\leq \|x_n - x_{n-1}, a\|$$

Hence $\{\|x_{n+1} - x_n, a\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say $r$.

Taking $n \to \infty$, we get

$$r \leq r - (\psi(kr) + \psi(k'r)) \quad \text{(by continuity of } \psi)$$

$$\Rightarrow \psi(kr) + \psi(k'r) \leq 0,$$

this is possible only when $r = 0$. So, $\lim_{n \to \infty} \|x_{n+1} - x_n\| \to 0$. Now, let $p \in DX\otimes Y$ be such that $\|p\| < \|a\|$. By condition (i), $\|x_{n+1} - x_n, p\| < \|x_{n+1} - x_n, a\|$, $n = 0, 1, 2, \ldots$ Taking limit as $n \to \infty$ on both sides, we get,

$$\lim_{n \to \infty} \|x_{n+1} - x_n, p\| = 0.$$

Hence, $\lim_{n \to \infty} \|x_{n+1} - x_n, u\| = 0 \forall u \in DX\otimes Y$ So, $\{x_n\}$ is a Cauchy sequence in $DX\otimes Y$. $DX\otimes Y$ being closed, this sequence converges to some $z \in DX\otimes Y$. Now, as in Theorem 2.1, it can be shown that

$$\|z - \hat{T}z, u\| \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $z = \hat{T}z$. Uniqueness can be shown in a similar manner. Thus, $\hat{T}$ has a unique fixed point in $DX\otimes Y$. $\square$
Example 2.7 Let $D_{l^1 \otimes \gamma K}$ be the closed and bounded (in 2-norm by some constant $K$) subspace of the 2-Banach algebra $l^1 \otimes \gamma K$ with the 2-norm defined earlier. We define the self mappings $T_1$ and $T_2$ on $D_{l^1 \otimes \gamma K}$ by

$T_1(\sum_i a_i \otimes x_i) = \frac{1}{2d} \sum_i a_i \otimes x_i$ and

$T_2(\sum_i a_i \otimes x_i) = \frac{1}{4d} \sum_i a_i \otimes x_i$, where $a_i = \{a_{ik}\} \in l^1$, and $d$ is a positive constant such that $d \geq \sqrt{2K}$, and $d \geq 2$.

For $p = \sum_k b_k \otimes y_k$, $q = \sum_j c_j \otimes y_j' \in l^1 \otimes \gamma K$ with $\|p\| \geq 1$, $\|q\| \geq 1$,

$$\|T_1(\sum_i a_i \otimes x_i), \sum_k b_k \otimes y_k\| = \frac{1}{2d} \sum_i \|a_i \otimes x_i\| \sum_k \|b_k \otimes y_k\|$$

$$= \frac{1}{2d} \sum_{i,k} (\|a_i \| \|b_k \| \|x_i\| \|y_k\| + \|x_i\| \|y_k\| \|a_i\| \|b_k\|)$$

$$= \frac{1}{2d} \sum_{i,k} \|a_i\| \|b_k\| \|x_i\| \|y_k\| \|a_i\| \|b_k\| (\text{for the 2-norms in } l^1 \text{ and } \gamma K)$$

$$\leq \frac{1}{2d} \sum_{i,k} \|a_i\| \|b_k\| \|x_i\| \|y_k\| ((\sum_j \|c_j\| \|y_j'\|) \quad (\because \|q\| \geq 1)$$

$$= \frac{1}{2d} \sum_{i,k,j} \|a_i\| \|b_k\| \|c_j\| \|x_i\| \|y_k\| \|y_j'\|$$

$$= \frac{1}{2d} \sum_{i,k,j} \|a_i\| \|b_kc_j\| \|x_i\| \|y_ky_j'\|$$

$$= \frac{1}{2d} \sum_{i} \|a_i\| \|x_i\| \sum_{k,j} b_kc_j \otimes y_ky_j'$$

$$= \frac{1}{2d} \|\sum_{i} a_i \otimes x_i \sum_{k,j} b_kc_j \otimes y_ky_j'\|$$

$$= \frac{1}{2d} \|\sum_{i} a_i \otimes x_i, pq\|$$

$$\leq \frac{K}{2d} \leq \frac{1}{2d} d^2 (K \leq \frac{d^2}{2})$$

$$\leq \frac{1}{M} \left(= \frac{M_1}{M}, M = \frac{2}{d}, M_1 = \frac{1}{2} \right)$$

Similarly,

$$\|T_2(\sum_i a_i \otimes x_i), \sum_k b_k \otimes y_k\| \leq \frac{1}{4d} \|\sum_{i} a_i \otimes x_i \sum_{k,j} b_kc_j \otimes y_ky_j'\|$$

$$= \frac{1}{4d} \|\sum_{i} a_i \otimes x_i, pq\| \leq \frac{K}{4d}$$

$$= \frac{1}{M} \left(= \frac{M_2}{M} \right)$$
For arbitrary \(u, v \in D_{l_1} \otimes K\), we have,

\[
\|T_1 u - T_1 v, p\| = \left\| \frac{1}{2d} u - \frac{1}{2d} v, \sum_k b_k \otimes y_k \right\| \leq \frac{1}{4d} \|u - v, pq\| \frac{1}{2}
\]

\[
= \sqrt{\frac{1}{4d} (u - v, pq) \left\| \frac{1}{4d} (u - v, pq) \right\|^2}
\]

\[
\leq \sqrt{\frac{1}{4d} (u - v, pq) \left( \frac{1}{4d} \|u, pq\| + \frac{1}{4d} \|v, pq\| \right)^2}
\]

\[
\leq \sqrt{\frac{1}{4d} (u - v, pq) \sqrt{\frac{K}{2d}} \frac{\|u - v, pq\|^2}{\sqrt{d}}}
\]

\[
\leq \sqrt{\frac{1}{4d} (u - v, pq) \frac{1}{\sqrt{d}} \sqrt{\frac{1}{2}}}
\]

\[
\leq \sqrt{\frac{2}{d} \frac{\sqrt{\frac{1}{4} \|u - v, pq\|}}{\sqrt{2d}}} \quad , \quad M = \frac{1}{d} = \frac{2}{d}
\]

\[
\leq 4 \sqrt{\frac{1}{4} \|u - v, pq\|} \quad , \quad M = \frac{1}{d} = \frac{2}{d}
\]

\[
\leq \sqrt{\frac{1}{4} \|u - v, pq\|} \quad , \quad M = \frac{1}{d} = \frac{2}{d}
\]

\[
\|T_1 u - T_1 v, p\| \leq \frac{1}{\sqrt{MM_2}} \left\| \frac{1}{2} \|u - v, pq\| - \psi \left( \frac{1}{2} \|u - v, pq\| \right) \right\| , \quad \text{where} \quad \psi(t) = \frac{t}{2}, k = \frac{1}{2}
\]

Similarly,

\[
\|T_2 u - T_2 v, p\| \leq \frac{1}{\sqrt{MM_1}} \left\| \frac{1}{2} \|u - v, pq\| - \psi \left( \frac{1}{2} \|u - v, pq\| \right) \right\| , \quad \text{where} \quad \psi(t) = \frac{t}{2}, k' = \frac{1}{2}
\]
Therefore, $T_1$ and $T_2$ satisfy the conditions (I) and (J) respectively. Also

$$k + k' = \frac{1}{2} + \frac{1}{2} = 1$$

Now, proceeding as in Example 2.2, we can show that for $p, q \in l^1 \otimes K$

$$\|p\| > \|q\| \Rightarrow \|u, p\| > \|u, q\| \forall u \in D_{l^1 \otimes K}$$

So, the mapping $\hat{T} : D_{l^1 \otimes K} \to D_{l^1 \otimes K}$ defined by $\hat{T}(u) = T_1 u T_2 u$ has a unique fixed point in $D_{l^1 \otimes K}$.

2. CONCLUDING REMARKS:

In 1973 [2], Gähler, et al. introduced the concept of 2-inner product spaces. For a linear space $X$ of dimension greater than 1 let $(., | .)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

(i) $(x, x | z) \geq 0$; $(x, x | z) = 0$ if and only if $x$ and $z$ are linearly dependent;
(ii) $(x, x | z) = (z, z | x)$;
(iii) $(x, y | z) = (y, x | z)$;
(iv) $(\alpha x, y | z) = \alpha (x, y | z)$ for any real number $\alpha$;
(v) $(x + x', y | z) = (x, y | z) + (x', y | z)$, $x, x', y, z \in X$.

$(., | .)$ is called a 2-inner product and $(X, (., | .))$ a 2-inner product space.

Different results on 2-inner product spaces and linear 2-normed spaces can be found in [1], [20]. Here we can raise the following problem:

Can we establish some analogous fixed point theorems for self mappings on 2-inner product spaces and also their tensor products?

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DEPARTMENT OF MATHEMATICS, GAUHATI UNIVERSITY, GUWAHATI-781014, ASSAM, INDIA
E-mail address: dipankardasguw@yahoo.com

DEPARTMENT OF MATHEMATICS, GAUHATI UNIVERSITY, GUWAHATI-781014, ASSAM, INDIA,
E-mail address: nila_g2003@yahoo.co.in

DEPARTMENT OF MATHEMATICS, APPLIED MATHEMATICS AND HUMANITIES DEPARTMENT, SARDAR VALLABHIBHAI NATIONAL INSTITUTE OF TECHNOLOGY, ICHCHHANATH MAHADEV DUMAS ROAD, SURAT 395 007, GUJARAT, INDIA, [2CM] L. 1627 AWADH PURI COLONY BENIGANJ, PHASE -III, OPPOSITE - INDUSTRIAL TRAINING INSTITUTE (I.T.I.), AYODHYA MAIN ROAD FAZABAD 224 001, UTTAR PRADESH, INDIA,
E-mail address: vishnumaranmishra@gmail.com, vishnu narayanmishra@yahoo.co.in