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SOME FIXED POINT THEOREMS IN THE PROJECTIVE TENSOR PRODUCT OF 2-BANACH SPACES

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ABSTRACT. Let X and Y be two normed spaces which are also 2-Banach spaces. For the projective tensor product $X \otimes_\gamma Y$, we have defined a 2-norm for which $X \otimes_\gamma Y$ is a 2-Banach space. Considering the closed and bounded subspace D_X, D_Y and $D_{X\otimes_\gamma Y}$ of X, Y and $X \otimes_\gamma Y$ respectively, we take two pairs of mappings $T_1, S_1 : D_{X\otimes_\gamma Y} \to D_X$ and $T_2, S_2 : D_{X\otimes_\gamma Y} \to D_Y$ satisfying some specific characteristics. Using these two pairs we define a pair of self mappings T and S on $D_{X\otimes_\gamma Y}$. Some fixed point theorems for T and S on $D_{X\otimes_\gamma Y}$ are derived here with suitable example. Moreover, taking $X \otimes_\gamma Y$ as a 2-Banach algebra, we establish another fixed point theorem for a self mapping $\hat{T}$ on $D_{X\otimes_\gamma Y}$ with a proper example.

1. INTRODUCTION

In 1963, Gähler [7] introduced the notion of 2-metric spaces and and their topological structures. The concept of 2-normed linear spaces can be found in Gähler's paper [8] in 1965. Gähler investigated a lot of results in this space and also proved that if $(X, \|.\|)$ is a linear normed space then a 2-norm can be defined on X.

In 1969, White [19] introduced the concept of 2-Banach space and gave some important definitions and examples. Since then a lot of applications of such spaces can be found in various aspects.

Many authors viz., Iśeki [10], Khan and Khan [11], Rhoads [15], Hadžić [9] etc. developed the applications of 2-metric spaces and 2-Banach spaces in the field of fixed point theory.

In this paper, we consider the projective tensor product space as a 2-Banach space and derive some fixed point theorems here.

PRELIMINARIES

Definition 1.1 [8] Let X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(ii) $\|x, y\| = \|y, x\|$ for all $x, y \in X$,
(iii) $\|\alpha x, y\| = |\alpha|\|x, y\|$, α being real, $x, y \in X$,
(iv) $\|x, y + z\| \leqslant \|x, y\| + \|x, z\|$, for all, $x, y, z \in X$

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Then $\|.,.\|$ is called a 2-norm on X and $(X, \|.,.\|)$ is called a linear 2-normed space.

In 1988, N. Mohammad, and A.H. Siddiqui [13] introduced the concept of 2-Banach Algebra.

A 2-Banach algebra $(X, \|.,.\|)$ is a real algebra (of dimension greater than 2) which is a 2-Banach space (with respect to 2-norm topology) and in addition, the following condition holds: $\|a, bc\| \leqslant M\|a, b\|\|a, c\|,\ M > 0\ \forall a, b, c \in X$.

Definition 1.2 A sequence $\{x_n\}$ in a 2-normed space $(X, \|.,.\|)$ is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$ for all a in X.

Definition 1.3 A sequence $\{x_n\}$ in a 2-normed space X is called a convergent sequence if there is an x in X such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$ for all a in X.

Definition 1.4 A 2-normed space in which every Cauchy sequence is convergent is called a 2-Banach space.

Definition 1.5 Let X and Y be two linear 2-normed spaces. An operator $T : X \to Y$ is said to be continuous at $x \in X$ if for every sequence $\{x_n\}$ in X, $\{x_n\} \to x$ as $n \to \infty$ implies $\{T(x_n)\} \to T(x)$ in Y as $n \to \infty$.

Example [19] Let P_n denotes the set of all real polynomials of degree $\leqslant n$, on the interval $[0, 1]$. Clearly, P_n is a linear vector space over the reals with respect to usual addition and scalar multiplication,. Let $\{x_0, x_1, ..., x_{2n}\}$ be distinct fixed points in $[0, 1]$ and $g, h \in P_n$. Define the following 2-norm on P_n :

$$\|g, h\| = \begin{cases} \sum_{i=0}^{2n} |g(x_i)h(x_i)|, & \text{if } g \text{ and } h \text{ are linearly independent} \\ 0, & \text{if } g \text{ and } h \text{ are linearly dependent} \end{cases}$$

then $(P_n, \|.,.\|)$ is a 2-Banach space.

Algebric tensor product: [6] Let X, Y be normed spaces over F with dual spaces X^* and Y^* respectively. Given $x \in X, y \in Y$, Let $x \otimes y$ be the element of $BL(X^*, Y^*; F)$ (which is the set of all bounded bilinear forms from $X^* \times Y^*$ to F), defined by

$$x \otimes y(f, g) = f(x)g(y),\ (f \in X^*, g \in Y^*)$$

The algebraic tensor product of X and Y, $X \otimes Y$ is defined to be the linear span of $\{x \otimes y : x \in X, y \in Y\}$ in $BL(X^*, Y^*; F)$.

Projective tensor product: [6] Given normed spaces X and Y, the projective tensor norm γ on $X \otimes Y$ is defined by

$$\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\| : u = \sum_i x_i \otimes y_i\}$$

where the infimum is taken over all (finite) representations of u.

Lemma: [6] $X \otimes_\gamma Y$ can be represented as a linear subspace of $BL(X^*, Y^*; F)$ consisting of all elements of the form $u = \sum_i x_i \otimes y_i$ where $\sum_i \|x_i\|\|y_i\| < \infty$. Moreover, $\|u\|_\gamma = \inf\{\sum_i \|x_i\|\|y_i\|\}$ over all such representations of u.

Let X and Y be both normed as well as 2-normed spaces. For the projective tensor product $X \otimes_\gamma Y$ we define another 2-norm as follows:
for $u = \sum_i x_i \otimes y_i$, $v = \sum_j p_j \otimes q_j$ and $w = \sum_k g_k \otimes h_k$

$$\|u,v\| = \begin{cases} \frac{1}{2}\sum_{i,j}\left(\|x_i,p_j\|\|y_i\|\|q_j\| + \|y_i,q_j\|\|x_i\|\|p_j\|\right), \\ \quad if\ u\ and\ v \text{ are linearly independent} \\ 0, \quad if\ u\ and\ v \text{ are linearly dependent} \end{cases}$$

(i) If u and v are linearly dependent then by definition, $\|u,v\| = 0$.
If $\|u,v\| = 0$ then $\|x_i,p_j\|\|y_i\|\|q_j\| = 0$ and $\|y_i,q_j\|\|x_i\|\|p_j\| = 0\ \forall i,j$.
If $\|x_i,p_j\| \neq 0$ then $\|y_i\| = 0$ or $\|q_j\| = 0 \Rightarrow y_i = 0$ or $q_j = 0\ \forall i,j$.
(Clearly then $x_i \neq 0$ and $p_j \neq 0$ otherwise $\|x_i,p_j\| = 0$)
$\Rightarrow u = 0\ or\ v = 0$ which shows that u and v are linearly dependent.
Similarly, if $\|y_i,q_j\| \neq 0$ then $u = 0$ or $v = 0$ and thus u and v are linearly dependent.
Now, let $\|x_i,p_j\| = 0$ and $\|y_i,q_j\| = 0\ \forall\ i,j$.
$\Rightarrow x_i\ and\ p_j$ are linearly dependent and $y_i\ and\ q_j$ are linearly dependent $\forall\ i,j$(being 2-norms).
This also implies that x_i, x_j's are linearly dependent $\forall\ i,j$, and y_i, y_j's are linearly dependent $\forall\ i,j$.
Therefore, u can be expressed as: $u = \alpha(x_1 \otimes y_1)$, for some scalar α.
Similarly, p_i, p_j's are linearly dependent $\forall\ i,j$, and q_i, q_j's are linearly dependent $\forall\ i,j$.
So, we can take $v = \beta(p_1 \otimes q_1)$, for some scalar β. But x_1, p_1 and y_1, q_1 are linearly dependent. Hence, u and v are linearly dependent.

(ii) $\|u,v\| = \|v,u\|$

(iii) $\|u,\alpha v\| = |\alpha|\|u,v\|$, α is a scalar

(iv) $\|u,v+w\| \leqslant \|u,v\| + \|u,w\|$ (all these properties follow by definition.)

(Moreover, if X and Y are two 2-Banach spaces, then $X \otimes_\gamma Y$ is also a 2-Banach space for the above 2-norm).

Let D_X, D_Y and $D_{X\otimes_\gamma Y}$ denote closed and bounded subsets of X, Y and $X \otimes_\gamma Y$ respectively. Let $T_1, S_1 : D_{X\otimes_\gamma Y} \to D_X$ and $T_2, S_2 : D_{X\otimes_\gamma Y} \to D_Y$ be such that for any $u, v \in D_{X\otimes_\gamma Y}$ and $a \otimes b \in D_{X\otimes_\gamma Y}$ with $\|a\|\|b\| \geqslant 1$ and positive k, k',

(A) $\|T_1(u) - T_1(v)\| \leqslant \frac{1}{M_2N_1}(k\|u-v, a\otimes b\| - \psi(k\|u-v, a\otimes b\|))$

(B) $\|T_2(u) - T_2(v)\| \leqslant \frac{1}{M_1N_2}(k'\|u-v, a\otimes b\| - \psi(k'\|u-v, a\otimes b\|))$

(C) $\|T_1(u) - T_1(v), a\| \leqslant \frac{1}{N_2^2}(k\|u-v, a\otimes b\| - \psi(k\|u-v, a\otimes b\|))$

(D) $\|T_2(u) - T_2(v), b\| \leqslant \frac{1}{N_1^2}(k'\|u-v, a\otimes b\| - \psi(k'\|u-v, a\otimes b\|))$

where,

(a) $\psi : [0,\infty) \to [0,\infty)$ is continuous and non-decreasing, $\psi(0) = 0$,

(b) $\max[\|T_1u\|, \|S_1u\|] \leqslant N_1$ and $\max[\|T_2u\|, \|S_2u\|] \leqslant N_2$ (where $u \in D_{X\otimes_\gamma Y}$). (Here D_X, D_Y and $D_{X\otimes_\gamma Y}$ are bounded in norm by N_1, N_2 and N_1N_2 respectively.)

(c) $\max[\|T_1u, a\|, \|S_1u, a\|] \leqslant M_1$ and $\max[\|T_2u, b\|, \|S_2u, b\|] \leqslant M_2$ (where $u \in D_{X\otimes_\gamma Y}$). (Here D_X, D_Y and $D_{X\otimes_\gamma Y}$ are bounded in 2-norm by M_1, M_2 and M_1M_2 respectively.)

From the pairs (T_1, S_1) and (T_2, S_2), we define self mappings T and S on $D_{X\otimes_\gamma Y}$ as earlier such that $Tu = T_1u \otimes T_2u$ and $Su = S_1u \otimes S_2u\ \forall u \in D_{X\otimes_\gamma Y}$.

MAIN RESULTS

Theorem 2.1 For normed spaces X and Y which are also 2-Banach spaces, the mapping T derived by the pair of mappings (T_1, T_2) satisfying (A), (B), (C) and (D) has a unique fixed point in $D_{X\otimes_\gamma Y}$ if

(i) $p, q \in D_{X\otimes_\gamma Y}$ *with* $\|p\| > \|q\| \Rightarrow \|x, p\| > \|x, q\|\ \forall\ x \in D_{X\otimes_\gamma Y}$ and

(ii) $k + k^{/} \leqslant 1$

Proof. For $u, v \in D_{X\otimes_\gamma Y}$ and an arbitrary $a \otimes b \in D_{X\otimes_\gamma Y}$ with $\|a\|\|b\| \geqslant 1$, we have,

$$\begin{aligned}
\|Tu - Tv, a\otimes b\| &= \|T_1u \otimes T_2u - T_1v \otimes T_2v, a\otimes b\| \\
&\leqslant \|(T_1u - T_1v)\otimes T_2u, a\otimes b\| + \|T_1v \otimes (T_2u - T_2v), a\otimes b\| \\
&= \frac{1}{2}[\|T_1u - T_1v, a\|.\|T_2u\|\|b\| + \|T_2u, b\|\|T_1u - T_1v\|\|a\| \\
&\qquad +\|T_2u - T_2v, b\|.\|T_1v\|\|a\| + \|T_1v, a\|\|T_2u - T_2v\|\|b\|]
\end{aligned}$$

$$\begin{aligned}
\leqslant\ & \frac{1}{2}[\frac{1}{N_2^2}[k\|u-v, a\otimes b\| - \psi(k\|u-v, a\otimes b\|)].N_2N_2 \\
&+\frac{1}{M_2N_1}[k\|u-v, a\otimes b\| - \psi(k\|u-v, a\otimes b\|)]M_2N_1 \\
&+\frac{1}{N_1^2}[k^{/}\|u-v, a\otimes b\| - \psi(k^{/}\|u-v, a\otimes b\|)].N_1N_1 \\
&+\frac{1}{M_1N_2}[k^{/}\|u-v, a\otimes b\| - \psi(k^{/}\|u-v, a\otimes b\|)]M_1N_2]
\end{aligned}$$

$$\begin{aligned}
&= (k+k^{/})\|u-v, a\otimes b\| - \psi(k\|u-v, a\otimes b\|) - \psi(k^{/}\|u-v, a\otimes b\|) \\
&\leqslant \|u-v, a\otimes b\| - [\psi(k\|u-v, a\otimes b\|) + \psi(k^{/}\|u-v, a\otimes b\|)]\ (\text{for } k+k^{/} \leqslant 1)
\end{aligned}$$

Let $x_0 \in D_{X\otimes_\gamma Y}$ be fixed. We take $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$ Now,

$$\begin{aligned}
\|x_{n+1} - x_n, a\otimes b\| &= \|Tx_n - Tx_{n-1}, a\otimes b\| \\
&\leqslant \|x_n - x_{n-1}, a\otimes b\| - \psi(k\|x_n - x_{n-1}, a\otimes b\|) \\
&\qquad -\psi(k^{/}\|x_n - x_{n-1}, a\otimes b\|) \\
&\leqslant \|x_n - x_{n-1}, a\otimes b\|
\end{aligned}$$

Hence $\{\|x_{n+1} - x_n, a\otimes b\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say r.

Taking $n \to \infty$, we get

$$\begin{aligned}
r &\leqslant r - (\psi(kr) + \psi(k^{/}r)), (\text{by continuity of } \psi) \\
\Rightarrow \psi(kr) + \psi(k^{/}r) &\leqslant 0,
\end{aligned}$$

this is possible only when $r = 0$. So,

$$\lim_{n\to\infty} \|x_{n+1} - x_n, a\otimes b\| = 0.$$

Let $p \otimes q \in D_{X\otimes_\gamma Y}$ be such that $\|p \otimes q\| < 1$. If

$$\lim_{n\to\infty}(x_{n+1} - x_n) \neq k(p \otimes q) \text{ for any } k, \text{ then}$$

$\lim_{n\to\infty} \|x_{n+1} - x_n, p \otimes q\| \neq 0$. Again $\|a \otimes b\| \geqslant 1$ and $\|p \otimes q\| < 1$. Therefore,

$$\|a \otimes b\| > \|p \otimes q\| \Rightarrow \|u, a \otimes b\| > \|u, p \otimes q\| \ \forall\ u \in D_{X\otimes_\gamma Y} \text{ ((by condition (i))). So,}$$

$$\|x_{n+1} - x_n, a \otimes b\| > \|x_{n+1} - x_n, p \otimes q\| \ n = 0, 1, 2, ...$$

Taking limit as $n \to \infty$, we get,

$$\lim_{n\to\infty} \|x_{n+1} - x_n, a \otimes b\| \geqslant \lim_{n\to\infty} \|x_{n+1} - x_n, p \otimes q\| \neq 0, \text{ a contradiction.}$$

So, $\lim_{n\to\infty}(x_{n+1} - x_n) = k(p \otimes q)$ for some k.
Thus, $\lim_{n\to\infty} \|x_{n+1} - x_n, p \otimes q\| = 0$ for any $p \otimes q \in D_{X\otimes_\gamma Y}$ with $\|p \otimes q\| < 1$.
So, $\lim_{n\to\infty} \|x_{n+1} - x_n, u\| = 0\ \forall u \in D_{X\otimes_\gamma Y}$. Now, for any integer $p > 0$,

$$\begin{aligned}\|x_n - x_{n+p}, u\| &\leqslant \|x_n - x_{n+1}, u\| + \|x_{n+1} - x_{n+2}, u\| + ... + \|x_{n+p-1} - x_{n+p}, u\| \\ &\to 0 \ as\ n \to \infty\end{aligned}$$

showing that $\{x_n\}$ is a Cauchy sequence in $D_{X\otimes_\gamma Y}$. Let it converge to some $z \in D_{X\otimes_\gamma Y}$. Now,

$$\begin{aligned}\|z - Tz, u\| &\leqslant \|z - x_{n+1}, u\| + \|x_{n+1} - Tz, u\| \\ &= \|z - x_{n+1}, u\| + \|Tx_n - Tz, u\| \\ &\leqslant \|z - x_{n+1}, u\| + \|x_n - z, u\| - [\psi(k\|x_n - z, u\|) + \psi(k^/\|x_n - z, u\|)] \\ &\to\ 0\ as\ n \to \infty\end{aligned}$$

Hence, $\|z - Tz, u\| = 0 \Rightarrow z = Tz$.
To show the uniqueness:
Let z_1 and z_2 be two distinct fixed points for T in $D_{X\otimes_\gamma Y}$. Now,

$$\begin{aligned}\|z_1 - z_2, u\| &= \|Tz_1 - Tz_2, u\| \\ &\leqslant \|z_1 - z_2, u\| \\ &\quad - [\psi(k\|z_1 - z_2, u\|) + \psi(k^/\|z_1 - z_2, u\|)]\end{aligned}$$

$$\Rightarrow \psi(k\|z_1 - z_2, u\|) + \psi(k^/\|z_1 - z_2, u\|) \leqslant 0,$$

possible only for $z_1 = z_2$ (since u is arbitrary).
Thus, T has a unique fixed point in $D_{X\otimes_\gamma Y}$. □

Example 2.2 We define the following 2-norms(in the sense of White) on the normed spaces l^1 and $\mathbb{K}$:
$\|.,.\| : \mathbb{K} \times \mathbb{K} \to \mathbb{R}^+ \cup \{0\}$ such that

$$\|x, b\| = \begin{cases} |x||b|, & \text{if } x \text{ and } b \text{ are linearly independent} \\ 0, & \text{if } x \text{ and } b \text{ are linearly dependent}\end{cases}$$

and $\|.,.\| : l^1 \times l^1 \to \mathbb{R}^+ \cup \{0\}$ such that

$$\|\{x_{i_n}\}, \{b_{i_n}\}\| = \begin{cases} \|\{x_{i_n}\}\|\|\{b_{i_n}\}\|, & \text{if } \{x_{i_n}\}_n \text{ and } \{b_{i_n}\}_n \text{ are linearly independent} \\ 0, & \text{if } \{x_{i_n}\}_n \text{ and } \{b_{i_n}\}_n \text{ are linearly dependent}\end{cases}$$

Using these norms we can now define a 2-norm in the projective tensor product $l^1 \otimes_\gamma \mathbb{K}$, i.e., $l^1(\mathbb{K})$ as:
for $u = \sum_i x_i \otimes y_i$, $v = \sum_j p_j \otimes q_j \in l^1 \otimes_\gamma \mathbb{K}$

$$\begin{aligned}\|u, v\| &= \frac{1}{2} \sum_{i,j} \big(\|x_i, p_j\| \|y_i\| \|q_j\| + \|y_i, q_j\| \|x_i\| \|p_j\| \big) \\ &= \frac{1}{2} \sum_{i,j} \big(\|x_i\| \|p_j\| |y_i| |q_j| + \|x_i\| \|p_j\| |y_i| |q_j| \big) \\ &= \|x_i\| \|p_j\| |y_i| |q_j| \text{ (for linearly independent (}\{x_i\}\text{ and }\{p_j\}\text{) and (}\{y_i\}\text{ and }\{q_j\}\text{))}\end{aligned}$$

Let D_{l^1}, $D_{\mathbb{K}}$ and $D_{l^1 \otimes_\gamma \mathbb{K}}$ be the closed subsets of the normed spaces l^1, $\mathbb{K}$ and $l^1 \otimes_\gamma \mathbb{K}$, bounded by the constants K, K and K^2 respectively($K \geqslant 1$).
We define $T_1 : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1}$ by

$$T_1(\sum_i a_i \otimes x_i) = \frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}, \text{ where } a_i = \{a_{i_n}\}_n$$

and $T_2 : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{\mathbb{K}}$ by $\quad T_2(\sum_i a_i \otimes x_i) = \frac{1}{4} \sum_i \|a_i\|.|x_i|$.
For arbitrary $b_j = \{b_{j_n}\}_n \in D_{l^1}$, $b \in D_{\mathbb{K}}$ with $\|b_j\| |b| \geqslant 1$

$$\begin{aligned}\|T_1(\sum_i a_i \otimes x_i), b_j\| &= \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}, b_j\| = \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}\| \|b_j\| \text{ (for the 2-norm)} \\ &\leqslant \frac{1}{2K^5} \sum_i \|a_i\| |x_i| \|b_j\|\end{aligned}$$

So, taking the projective tensor norm in $l^1 \otimes_\gamma \mathbb{K}$ i.e., $l^1(\mathbb{K})$, we get,

$$\|T_1(\sum_i a_i \otimes x_i), b_j\| \leqslant \frac{1}{2K^5} \|\sum_i a_i \otimes x_i\| \|b_j\| \leqslant \frac{1}{2K^5} K^3 = \frac{1}{2K^2} (= M_1)$$

and

$$\begin{aligned}\|T_1(\sum_i a_i \otimes x_i)\| &= \|\frac{1}{2K^5} \sum_i \{a_{i_n} x_i\}\| \\ &\leqslant \frac{1}{2K^5} \sum_i \|a_i\| |x_i|\end{aligned}$$

So, taking the projective tensor norm, in $l^1 \otimes_\gamma \mathbb{K}$, we get,

$$\|T_1(\sum_i a_i \otimes x_i)\| \leqslant \frac{1}{2K^5} \|\sum_i a_i \otimes x_i\| \leqslant \frac{1}{2K^5} K^2 = \frac{1}{2K^3} (= N_1)$$

Similarly,

$$\begin{aligned}\|T_2(\sum_i a_i \otimes x_i), b\| &= \|\frac{1}{4}\sum_i \|a_i\|.|x_i|, b\| \\ &= \left|\frac{1}{4}\sum_i \|a_i\|.|x_i|\right| |b| \ \text{(for the 2-norm)} \\ &\leqslant \frac{1}{4}\sum_i \|a_i\||x_i||b|\end{aligned}$$

So, taking the projective tensor norm,

$$\|T_2(\sum_i a_i \otimes x_i), b\| \leqslant \frac{1}{4}\| \sum_i a_i \otimes x_i\||b| \leqslant \frac{K^3}{4}(= M_2)$$

and

$$\begin{aligned}\|T_2(\sum_i a_i \otimes x_i)\| &\leqslant \frac{1}{4}\| \sum_i a_i \otimes x_i\| \\ &\leqslant \frac{K^2}{4}(= N_2)\end{aligned}$$

For $u = \sum_i a_i \otimes x_i$ and $v = \sum_i d_i \otimes y_i$ in $D_{l^1 \otimes_\gamma \mathbb{K}}$, we have,

$$\begin{aligned}\|T_1 u - T_1 v, b_j\| &= \|\frac{1}{2K^5}\sum_i \{a_{i_n} x_i\} - \frac{1}{2K^5}\sum_i \{d_{i_n} y_i\}\|\|b_j\| \\ &\leqslant \|\frac{1}{2K^5}\sum_i a_i \otimes x_i - \frac{1}{2K^5}\sum_i d_i \otimes y_i\|\|b_j\|\|b_j\||b| \ (\because \|b_j\||b| \geqslant 1) \\ &= \frac{1}{2K^4}\|u - v, b_j \otimes b\| \\ &= \frac{\frac{1}{4}\|u - v, b_j \otimes b\|}{\frac{K^4}{2}} \leqslant 8\left[\frac{\frac{1}{2}\|u - v, b_j \otimes b\| - \frac{1}{2}\left[\frac{1}{2}\|u - v, b_j \otimes b\|\right]}{\frac{K^4}{2}}\right] \\ &= \frac{1}{N_2^2}\left[\frac{1}{2}\|u - v, b_j \otimes b\| - \psi\left(\frac{1}{2}\|u - v, b_j \otimes b\|\right)\right], \\ &\quad \text{where } \psi(t) = \frac{t}{2}, k = \frac{1}{2}\end{aligned}$$

and

$$
\begin{aligned}
\|T_1u - T_1v\| &= \|\frac{1}{2K^5}\sum_i\{a_{i_n}x_i\} - \frac{1}{2K^5}\sum_i\{d_{i_n}y_i\}\| \\
&= \frac{\frac{1}{4}\|u-v\|}{\frac{K^5}{2}} \leqslant \frac{\frac{1}{4}\|u-v\|\|b_j\||b|}{\frac{K^5}{2}} \\
&\leqslant 4K^5\left[\frac{\frac{1}{2}\|u-v,b_j\otimes b\| - \frac{1}{2}\left[\frac{1}{2}\|u-v,b_j\otimes b\|\right]}{\frac{K^5}{2}}\right] \text{ (for the 2-norm)} \\
&= \left[\frac{\frac{1}{2}\|u-v,b_j\otimes b\| - \frac{1}{2}\left[\frac{1}{2}\|u-v,b_j\otimes b\|\right]}{\frac{1}{2K^3}\frac{K^3}{4}}\right] \\
&= \frac{1}{M_2N_1}\left[\frac{1}{2}\|u-v,b_j\otimes b\| - \psi\left(\frac{1}{2}\|u-v,b_j\otimes b\|\right)\right], \\
&\qquad \text{where } \psi(t) = \frac{t}{2}, k = \frac{1}{2}
\end{aligned}
$$

Again,

$$
\begin{aligned}
\|T_2u - T_2v, b\| &= \|\frac{1}{4}\sum_i \|a_i\||x_i| - \frac{1}{4}\sum_i \|d_i\||y_i|, b\| \\
&\leqslant \frac{1}{4}\left|\sum_i \|a_i\||x_i| - \sum_i \|d_i\||y_i|\right| |b|\ \|b_j\||b|
\end{aligned}
$$

Taking the projective tensor norm,

$$
\begin{aligned}
\|T_2u - T_2v, b\| &\leqslant \frac{K}{4}\ |\|u\| - \|v\||\ |b|\ \|b_j\| \\
&\leqslant \frac{K}{4}\|u-v\|\|b_j\|\ |b| \\
&= \frac{K}{4}\|u-v,b_j\otimes b\| \text{ (for the 2-norm)} \\
\Rightarrow \|T_2u - T_2v, b\| &\leqslant 4K^6\left(\frac{1}{4}\|u-v,b_j\otimes b\| - \frac{1}{2}\left[\frac{1}{4}\|u-v,b_j\otimes b\|\right]\right) \ (\because K \geqslant 1) \\
&= \frac{\frac{1}{4}\|u-v,b_j\otimes b\| - \psi\left(\frac{1}{4}\|u-v,b_j\otimes b\|\right)}{N_1^2}, \\
&\qquad \text{where } \psi(t) = \frac{t}{2}, k' = \frac{1}{4}
\end{aligned}
$$

Also,

$$\|T_2u - T_2v\| = \|\frac{1}{4}\sum_i \|a_i\||x_i| - \frac{1}{4}\sum_i \|d_i\||y_i|\|$$
$$\leqslant \frac{1}{4}\left|\sum_i \|a_i\||x_i| - \sum_i \|d_i\||y_i|\right| \|b_j\|\,|b| \;\; (\because \|b_j\||b| \geqslant 1)$$

Taking the projective tensor norm ,

$$\|T_2u - T_2v\| \leqslant \frac{1}{4}\,|\|u\| - \|v\||\,\|b_j\|\,|b|$$
$$\leqslant \frac{1}{4}\|u-v\|\|b_j\|\,|b|$$
$$= \frac{1}{4}\|u-v, b_j \otimes b\| \leqslant \|u-v, b_j \otimes b\|$$
$$= \frac{\frac{1}{4}\|u-v, b_j \otimes b\| - \frac{1}{2}\left[\frac{1}{4}\|u-v, b_j \otimes b\|\right]}{\frac{K^2}{4}\frac{1}{2K^2}}$$
$$= \frac{\frac{1}{4}\|u-v, b_j \otimes b\| - \psi\left(\frac{1}{4}\|u-v, b_j \otimes b\|\right)}{M_1N_2}, \text{ where } \psi(t) = \frac{t}{2}, k^{/} = \frac{1}{4}$$

Therefore, (T_1, T_2) satisfies the conditions (A), (B), (C), (D). Also,

$$k + k^{/} = \frac{1}{2} + \frac{1}{4} < 1$$

Moreover, for $\sum_i p_i \otimes x_i, \sum_i q_i \otimes y_i$ *and* $\sum_j b_j \otimes c_j \in D_{l^1 \otimes_\gamma \mathbb{K}}$,

$$\|\sum_i p_i \otimes x_i\| > \|\sum_i q_i \otimes y_i\| \Rightarrow \sum_i \|p_i\||x_i| > \sum_i \|q_i\||y_i|$$
$$\Rightarrow \sum_j \|b_j\||c_j| \sum_i \|p_i\||x_i| > \sum_j \|b_j\||c_j| \sum_i \|q_i\||y_i|$$
$$\Rightarrow \sum_{i,j} \|b_j\|\|p_i\||c_j||x_i| > \sum_{i,j} \|b_j\|\|q_i\||c_j||y_i|$$
$$\Rightarrow \|\sum_j b_j \otimes c_j, \sum_i p_i \otimes x_i\| > \|\sum_j b_j \otimes c_j, \sum_i q_i \otimes y_i\|$$

So, by Theorem 2.1, the mapping $T : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ defined by

$$T(u) = T_1(u) \otimes T_2(u)$$

has a unique fixed point in $D_{l^1 \otimes_\gamma \mathbb{K}}$.

Corollary 2.3 If in Theorem 2.1 T_1 or T_2 satisfies one of the following conditions:

$$\|T_1(u) - T_1(v), a\| \leqslant \frac{k}{M_2}\|u-v, a \otimes b\|, \; \|b\| \geqslant 1$$
$$\|T_2(u) - T_2(v), b\| \leqslant \frac{k^{/}}{M_1}\|u-v, a \otimes b\|, \; \|a\| \geqslant 1$$

then T has a unique fixed point if $k + k^{/} \leqslant 1$.

Corollary 2.4 If T_1 and T_2 satisfy both the above conditions, then T has a unique fixed point if $k + k^{/} < 1$.

Now, we discuss common fixed points for the pair of mappings (T, S) derived by the pairs (T_1, S_1) and (T_2, S_2). For this, we replace the conditions (A), (B), (C) and (D) of the Theorem 2.1 by the following:

(E) $\|T_1(u) - S_1(v)\| \leqslant \dfrac{1}{M_2N_1}(k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|))$

(F) $\|T_2(u) - S_2(v)\| \leqslant \dfrac{1}{M_1N_2}(k^{/}\|u - v, a \otimes b\| - \psi(k^{/}\|u - v, a \otimes b\|))$

(G) $\|T_1(u) - S_1(v), a\| \leqslant \dfrac{1}{N_2^2}(k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|))$

(H) $\|T_2(u) - S_2(v), b\| \leqslant \dfrac{1}{N_1^2}(k^{/}\|u - v, a \otimes b\| - \psi(k^{/}\|u - v, a \otimes b\|))$

Theorem 2.5 The mappings T and S derived by (T_1, S_1) and (T_2, S_2) satisfying (E), (F), (G) and (H) have a common unique fixed point in $D_{X\otimes_\gamma Y}$, if $p, q \in D_{X\otimes_\gamma Y}$

(i) $\|p\| > \|q\| \Rightarrow \|x, p\| > \|x, q\|$ $(\forall x \in D_{X\otimes_\gamma Y})$ and
(ii) $k + k^{/} \leqslant 1$.

Proof. For $u, v \in D_{X\otimes_\gamma Y}$, and $a \otimes b \in D_{X\otimes_\gamma Y}$ with $\|a \otimes b\| \geqslant 1$

$$\begin{aligned}
\|Tu - Sv, a \otimes b\| &= \|T_1u \otimes T_2u - S_1v \otimes S_2v, a \otimes b\| \\
&\leqslant \frac{1}{2}[\|T_1u - S_1v, a\|\|T_2u\|\|b\| + \|T_2u, b\|\|T_1u - S_1v\|\|a\| \\
&\quad + \|T_2u - S_2v, b\|\|S_1v\|\|a\| + \|S_1v, a\|\|T_2u - S_2v\|\|b\|]
\end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{2}[\frac{1}{N_2^2}[k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)].N_2N_2 \\
&\quad + \frac{1}{M_2N_1}[k\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|)]M_2N_1 \\
&\quad + \frac{1}{N_1^2}[k^{/}\|u - v, a \otimes b\| - \psi(k^{/}\|u - v, a \otimes b\|)].N_1N_1 \\
&\quad + \frac{1}{M_1N_2}[k^{/}\|u - v, a \otimes b\| - \psi(k^{/}\|u - v, a \otimes b\|)]M_1N_2]
\end{aligned}$$

$$\begin{aligned}
&\leqslant (k + k^{/})\|u - v, a \otimes b\| - \psi(k\|u - v, a \otimes b\|) - \psi(k^{/}\|u - v, a \otimes b\|) \\
&\leqslant \|u - v, a \otimes b\| - (\psi(k\|u - v, a \otimes b\|) + \psi(k^{/}\|u - v, a \otimes b\|))
\end{aligned}$$

Let $x_0 \in D_{X\otimes_\gamma Y}$ be fixed. We take $x_1 = Sx_0$, $x_2 = Tx_1$, $x_3 = Sx_2$, $x_4 = Tx_3$, ... , $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$. Now,

$$\begin{aligned}
\|x_{2n+2} - x_{2n+1}, a\otimes b\| &= \|Tx_{2n+1} - Sx_{2n}, a\otimes b\| \\
&\leqslant \|x_{2n+1} - x_{2n}, a\otimes b\| - \psi(k\|x_{2n+1} - x_{2n}, a\otimes b\|) \\
&\qquad -\psi(k^{/}\|x_{2n+1} - x_{2n}, a\otimes b\|) \\
&\leqslant \|x_{2n+1} - x_{2n}, a\otimes b\| \qquad (1) \\
\text{Similarly,}\|x_{2n+1} - x_{2n}, a\otimes b\| &\leqslant \|x_{2n} - x_{2n-1}, a\otimes b\|,\ n = 0,1,2,...
\end{aligned}$$

Hence $\{\|x_{n+1} - x_n, a\otimes b\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say r.
Taking $n \to \infty$ in (1), we get $r = 0$. Now, as in Theorem 2.1, we can show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n, u\| = 0\ \forall u \in D_{X\otimes_\gamma Y}$$

Now, it can be easily shown that $\{x_n\}$ is a Cauchy sequence in $D_{X\otimes_\gamma Y}$. Let it converge to some $z \in D_{X\otimes_\gamma Y}$. Now, proceeding as in Theorem 2.1, we can show that $z = Sz$ and also $z = Tz$. Uniqueness can also be shown in a similar manner.
So, z is the common unique fixed point for T and S in $D_{X\otimes_\gamma Y}$. □

Next we consider algebraic tensor product space as a 2-Banach algebra.

Let X and Y be two unital Banach algebras which are also 2-Banach algebras. We show that $X\otimes Y$ with projective tensor norm is a 2-normed algebra (with the 2-norm already defined). For $u = \sum_{i=1}^n x_i \otimes y_i$, $v = \sum_{j=1}^m p_j \otimes q_j$ and $w = \sum_{r=1}^k e_r \otimes f_r$ in $X\otimes_\gamma Y$ we have,

$$\begin{aligned}
\|uw, v\| &= \|\sum_{i,r=1}^{\min(n,k)} x_i e_r \otimes y_i f_r, \sum_{j=1}^{m} p_j \otimes q_j\| \\
&= \frac{1}{2}\sum_{i,r,j=1}^{\min(m,n,k)} (\|x_i e_r, p_j\|\|y_i f_r\|\|q_j\| + \|y_i f_r, q_j\|\|x_i e_r\|\|p_j\|) \\
&\leqslant \frac{1}{2}\sum_{i,r,j=1}^{\min(m,n,k)} (K\|x_i, p_j\|\|e_r, p_j\|\|y_i f_r\|\|q_j\| + K^{/}\|y_i, q_j\|\|f_r, q_j\|\|x_i e_r\|\|p_j\|) \\
&\quad (\text{for some constant } K \text{ and } K^{/})
\end{aligned}$$

Let $\epsilon = \min_{j=1,2,\dots,m}(\|p_j\|, \|q_j\|)$, $p_j \neq 0, q_j \neq 0$. We choose $N(\geqslant 1)$ be such that $\max(K, K') \leqslant N\epsilon$. Now,

$$\begin{aligned}
\|uw, v\| &\leqslant \frac{N\epsilon}{2} \sum_{i,r,j=1}^{\min(m,n,k)} (\|x_i, p_j\| \|e_r, p_j\| \|y_i f_r\| \|q_j\| + \|y_i, q_j\| \|f_r, q_j\| \|x_i e_r\| \|p_j\|) \\
&= \frac{N}{2} \sum_{i,r,j=1}^{\min(m,n,k)} [\min_{j=1,2,\dots,m}(\|p_j\|, \|q_j\|)(\|x_i, p_j\| \|e_r, p_j\| \|y_i f_r\| \|q_j\|) \\
&\quad + \min_{j=1,2,\dots,m}(\|p_j\|, \|q_j\|)(\|y_i, q_j\| \|f_r, q_j\| \|x_i e_r\| \|p_j\|)] \\
&\leqslant \frac{N}{2} \sum_{i,r,j=1}^{\min(m,n,k)} (\|x_i, p_j\| \|y_i\| \|q_j\| . \|e_r, p_j\| \|f_r\| \|q_j\| + \|y_i, q_j\| \|x_i\| \|p_j\| . \|f_r, q_j\| \|e_r\| \|p_j\| \\
&\quad + \|x_i, p_j\| \|y_i\| \|q_j\| . \|f_r, q_j\| \|e_r\| \|p_j\| + \|y_i, q_j\| \|x_i\| \|p_j\| . \|e_r, p_j\| \|f_r\| \|q_j\|)
\end{aligned}$$

$$\begin{aligned}
&= M \sum_{i,r,j=1}^{\min(m,n,k)} \frac{1}{2}(\|x_i, p_j\| \|y_i\| \|q_j\| + \|y_i, q_j\| \|x_i\| \|p_j\|) \frac{1}{2}(\|e_r, p_j\| \|f_r\| \|q_j\| + \|f_r, q_j\| \|e_r\| \|p_j\|) \\
&= M \|u, v\| \|w, v\|, \text{ where } M = 2N
\end{aligned}$$

showing that $X \otimes Y$ is a 2-normed algebra. Taking the completion with respect to the 2-norm, we can make it a 2-Banach algebra.
Let (T_1, T_2) be a pair of mappings where $T_1 : D_{X\otimes_\gamma Y} \to D_{X\otimes_\gamma Y}$ and $T_2 : D_{X\otimes_\gamma Y} \to D_{X\otimes_\gamma Y}$ are such that for any $u, v, a, b \in D_{X\otimes_\gamma Y} \cup \{e\}$ with $\|a\| \geqslant 1$ and $\|b\| \geqslant 1$,

(I) $\|T_1(u) - T_1(v), a\| \leqslant \dfrac{1}{\sqrt{M}M_2} \sqrt{k\|u - v, ab\| - \psi(k\|u - v, ab\|)}$

(J) $\|T_2(u) - T_2(v), b\| \leqslant \dfrac{1}{\sqrt{M}M_1} \sqrt{k'\|u - v, ab\| - \psi(k'\|u - v, ab\|)}$,
(for constants M_1, $M_2 > 0$ and M is defined as above)
where,

(a) $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\psi(0) = 0$

(b) $\|T_1 u, a\| \leqslant \dfrac{M_1}{M}$ and $\|T_2 u, b\| \leqslant \dfrac{M_2}{M}$. [Here $D_{X\otimes_\gamma Y}$ is a closed and bounded subspace of $X \otimes_\gamma Y$.]

From the pair (T_1, T_2) we define a self mapping $\hat{T} : D_{X\otimes_\gamma Y} \to D_{X\otimes_\gamma Y}$ such that $\hat{T}u = T_1 u T_2 u$ for $u \in D_{X\otimes_\gamma Y}$.

Theorem 2.6 From the pair of mappings (T_1, T_2) satisfying (I) and (J) the mapping $\hat{T}$ defined above has a unique fixed point in $D_{X\otimes_\gamma Y}$ if

(i) For $p, q \in D_{X\otimes_\gamma Y}$, $\|p\| > \|q\| \Rightarrow \|u, p\| > \|u, q\| \ \forall u \in D_{X\otimes_\gamma Y}$,

(ii) $k + k' \leqslant 1$

Proof. For $u, v, a, b \in D_{X\otimes_\gamma Y}$ with $\|a\| \geqslant 1, \|b\| \geqslant 1$

$$\begin{aligned}
\|\hat{T}u - \hat{T}v, ab\| &= \|T_1uT_2u - T_1vT_2v, ab\| \\
&\leqslant \|(T_1u - T_1v)T_2u, ab\| + \|T_1v(T_2u - T_2v), ab\| \\
&\leqslant M\|T_1u - T_1v, ab\|\|T_2u, ab\| + M\|T_1v, ab\|\|T_2u - T_2v, ab\| \\
&\leqslant M^3\|T_1u - T_1v, a\|\|T_1u - T_1v, b\|\|T_2u, a\|\|T_2u, b\| \\
&\quad + M^3\|T_1v, a\|\|T_1v, b\|\|T_2u - T_2v, a\|\|T_2u - T_2v, b\| \\
&\leqslant M^3\left[\frac{1}{\sqrt{M}M_2}\sqrt{k\|u-v, ab\| - \psi(k\|u-v, ab\|)}\left(\frac{M_2}{M}\right)\right]^2 \\
&\quad + M^3\left[\frac{1}{\sqrt{M}M_1}\sqrt{k^/\|u-v, ab\| - \psi(k^/\|u-v, ab\|)}\left(\frac{M_1}{M}\right)\right]^2 \\
&= (k + k^/)\|u-v, ab\| - \psi(k\|u-v, ab\|) - \psi(k^/\|u-v, ab\|) \\
&\leqslant \|u-v, ab\| - (\psi(k\|u-v, ab\|) + \psi(k^/\|u-v, ab\|))
\end{aligned}$$

Let $x_0 \in D_{X\otimes_\gamma Y}$ be fixed. We take $x_{n+1} = \hat{T}x_n$, $n = 0, 1, 2, ...$ Now, for arbitrary $a \in D_{X\otimes_\gamma Y}$ with $\|a\| \geqslant 1$,

$$\begin{aligned}
\|x_{n+1} - x_n, a\| = \|x_{n+1} - x_n, ae\| &= \|\hat{T}x_n - \hat{T}x_{n-1}, ae\| \\
&\leqslant \|x_n - x_{n-1}, ae\| - \psi(k\|x_n - x_{n-1}, ae\|) \\
&\quad - \psi(k^/\|x_n - x_{n-1}, ae\|) \\
&\leqslant \|x_n - x_{n-1}, a\|
\end{aligned}$$

Hence $\{\|x_{n+1} - x_n, a\|\}$ is a monotonically decreasing sequence of non-negative real numbers and so, is convergent to some real, say r.
Taking $n \to \infty$, we get

$$\begin{aligned}
r &\leqslant r - (\psi(kr) + \psi(k^/r)), \text{(by continuity of } \psi) \\
\Rightarrow \psi(kr) + \psi(k^/r) &\leqslant 0,
\end{aligned}$$

this is possible only when $r = 0$. So, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Now, let $p \in D_{X\otimes_\gamma Y}$ be such that $\|p\| < 1$. So, $\|p\| < \|a\|$.
By condition (i), $\|x_{n+1} - x_n, p\| < \|x_{n+1} - x_n, a\|$, $n = 0, 1, 2, ...$ Taking limit as $n \to \infty$ on both sides, we get,

$$\lim_{n\to\infty} \|x_{n+1} - x_n, p\| = 0.$$

Hence, $\lim_{n\to\infty} \|x_{n+1} - x_n, u\| = 0\ \forall u \in D_{X\otimes_\gamma Y}$ So, $\{x_n\}$ is a Cauchy sequence in $D_{X\otimes_\gamma Y}$. $D_{X\otimes_\gamma Y}$ being closed, this sequence converges to some $z \in D_{X\otimes_\gamma Y}$. Now, as in Theorem 2.1, it can be shown that

$$\|z - \hat{T}z, u\| \to 0 \text{ as } n \to \infty,$$

which implies that $z = \hat{T}z$. Uniqueness can be shown in a similar manner. Thus, $\hat{T}$ has a unique fixed point in $D_{X\otimes_\gamma Y}$. □

Example 2.7 Let $D_{l^1\otimes_\gamma \mathbb{K}}$ be the closed and bounded (in 2-norm by some constant K) subspace of the 2-Banach algebra $l^1 \otimes_\gamma \mathbb{K}$ with the 2-norm defined earlier. We define the self mappings T_1 and T_2 on $D_{l^1\otimes_\gamma \mathbb{K}}$ by $T_1(\sum_i a_i \otimes x_i) = \dfrac{1}{2d} \sum_i a_i \otimes x_i$ and
$T_2(\sum_i a_i \otimes x_i) = \dfrac{1}{4d} \sum_i a_i \otimes x_i$, where $a_i = \{a_{i_n}\}_n \in l^1$, and d is a positive constant such that $d \geqslant \sqrt{2K}$, and $d \geqslant 2$.
For $p = \sum_k b_k \otimes y_k,\ q = \sum_j c_j \otimes y_j^{/} \in l^1 \otimes_\gamma \mathbb{K}$ with $\|p\| \geqslant 1,\ \|q\| \geqslant 1$,

$$\begin{aligned}
\|T_1(\sum_i a_i \otimes x_i), \sum_k b_k \otimes y_k\| &= \|\frac{1}{2d} \sum_i a_i \otimes x_i, \sum_k b_k \otimes y_k\| \\
&= \frac{1}{2d}\frac{1}{2} \sum_{i,k} (\|a_i, b_k\| \|x_i\| \|y_k\| + \|x_i, y_k\| \|a_i\| \|b_k\|) \\
&= \frac{1}{2d} \sum_{i,k} \|a_i\| \|b_k\| \|x_i\| \|y_k\| \ (\text{for the 2-norms in } l^1 \text{ and } \mathbb{K}\ ) \\
&\leqslant \frac{1}{2d} \sum_{i,k} \|a_i\| \|b_k\| |x_i| |y_k| (\sum_j \|c_j\| |y_j^{/}|) \ (\because \|q\| \geqslant 1) \\
&= \frac{1}{2d} \sum_{i,k,j} \|a_i\| \|b_k\| \|c_j\| |x_i| |y_k| |y_j^{/}| \\
&= \frac{1}{2d} \sum_{i,k,j} \|a_i\| \|b_k c_j\| |x_i| |y_k y_j^{/}|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2d} \| \sum_i a_i \otimes x_i, \sum_{k,j} b_k c_j \otimes y_k y_j^{/} \| \\
&= \frac{1}{2d} \| \sum_i a_i \otimes x_i, pq\| \\
&\leqslant \frac{K}{2d} \leqslant \frac{1}{2d}\frac{d^2}{2} \ (K \leqslant \frac{d^2}{2}) \\
&= \frac{\frac{1}{2}}{M} (= \frac{M_1}{M}),\ M = \frac{2}{d}, M_1 = \frac{1}{2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|T_2(\sum_i a_i \otimes x_i), \sum_k b_k \otimes y_k\| &\leqslant \frac{1}{4d} \| \sum_i a_i \otimes x_i, \sum_{k,j} b_k c_j \otimes y_k y_j^{/} \| \\
&= \frac{1}{4d} \| \sum_i a_i \otimes x_i, pq\| \leqslant \frac{K}{4d} \\
&= \frac{\frac{1}{4}}{M} (= \frac{M_2}{M})
\end{aligned}$$

For arbitrary $u, v \in D_{l^1 \otimes_\gamma \mathbb{K}}$, we have,

$$\begin{aligned}
\|T_1u - T_1v, p\| &= \|\frac{1}{2d}u - \frac{1}{2d}v, \sum_k b_k \otimes y_k\| \leqslant \frac{\frac{1}{4d}\|u-v, pq\|}{\frac{1}{2}} \\
&= \frac{\sqrt{\|\frac{1}{4d}(u-v), pq\|}\sqrt{\|\frac{1}{4d}(u-v), pq\|}}{\frac{1}{2}} \\
&\leqslant \frac{\sqrt{\|\frac{1}{4d}(u-v), pq\|}\sqrt{\frac{1}{4d}\|u, pq\| + \frac{1}{4d}\|v, pq\|}}{\frac{1}{2}} \\
&\leqslant \frac{\sqrt{\|\frac{1}{4d}(u-v), pq\|}\sqrt{\frac{K}{2d}}}{\frac{1}{2}} \leqslant \frac{\sqrt{\|\frac{1}{4d}(u-v), pq\|}\sqrt{\frac{d}{4}}}{\frac{1}{2}} \\
&\leqslant \frac{\sqrt{\frac{1}{4d}\|(u-v), pq\|}}{\frac{1}{\sqrt{\frac{d}{2}}}\sqrt{\frac{1}{2}}} \\
&\leqslant \sqrt{\frac{2}{d}}\frac{\sqrt{\frac{1}{4}\|(u-v), pq\|}}{\sqrt{\frac{2}{d}}},\ M = \frac{1}{\frac{d}{2}} = \frac{2}{d} \\
&\leqslant 4\frac{\sqrt{\frac{1}{4}\|(u-v), pq\|}}{\sqrt{M}},\ M = \frac{1}{\frac{d}{2}} = \frac{2}{d} \\
&\leqslant \frac{\sqrt{\frac{1}{4}\|(u-v), pq\|}}{\sqrt{M}M_2}
\end{aligned}$$

$$\|T_1u - T_1v, p\| \leqslant \frac{1}{\sqrt{M}M_2}\sqrt{\left[\frac{1}{2}\|u-v, pq\| - \psi\left(\frac{1}{2}\|u-v, pq\|\right)\right]},\ \text{where } \psi(t) = \frac{t}{2}, k = \frac{1}{2}$$

Similarly,

$$\|T_2u - T_2v, p\| \leqslant \frac{1}{\sqrt{M}M_1}\sqrt{\left[\frac{1}{2}\|u-v, pq\| - \psi\left(\frac{1}{2}\|u-v, pq\|\right)\right]},\ \text{where } \psi(t) = \frac{t}{2}, k^{/} = \frac{1}{2}$$

Therefore, T_1 and T_2 satisfy the conditions (I) and (J) respectively. Also

$$k + k^{/} = \frac{1}{2} + \frac{1}{2} = 1$$

Now, proceeding as in Example 2.2, we can show that for $p, q \in l^1 \otimes_\gamma \mathbb{K}$

$$\|p\| > \|q\| \Rightarrow \|u, p\| > \|u, q\| \; \forall u \in D_{l^1 \otimes_\gamma \mathbb{K}}$$

So, the mapping $\hat{T} : D_{l^1 \otimes_\gamma \mathbb{K}} \to D_{l^1 \otimes_\gamma \mathbb{K}}$ defined by $\hat{T}(u) = T_1 u T_2 u$ has a unique fixed point in $D_{l^1 \otimes_\gamma \mathbb{K}}$.

2. Concluding Remarks:

In 1973 [2], Gähler, et al. introduced the concept of 2-inner product spaces.
For a linear space X of dimension greater than 1 let $(.,.|.)$ be a real-valued function on $X \times X \times X$ which satisfies the following conditions:

(i) $(x, x|z) \geqslant 0; (x, x|z) = 0$ if and only if x and z are linearly dependent;
(ii) $(x, x|z) = (z, z|x)$;
(iii) $(x, y|z) = (y, x|z)$;
(iv) $(\alpha x, y|z) = \alpha(x, y|z)$ for any real number α;
(v) $(x + x^{/}, y|z) = (x, y|z) + (x^{/}, y|z), x, x^{/}, y, z \in X$.

$(.,.|.)$ is called a 2-inner product and $(X, (.,.|.))$ a 2-inner product space.

Different results on 2-inner product spaces and linear 2-normed spaces can be found in [1], [20].

Here we can raise the following problem:

Can we establish some analogous fixed point theorems for self mappings on 2-inner product spaces and also their tensor products?

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