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# INTEGRAL FORMULAS FOR A METRIC-AFFINE MANIFOLD WITH TWO COMPLEMENTARY ORTHOGONAL DISTRIBUTIONS

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ABSTRACT. We obtain integral formulas for a metric-affine space equipped with two complementary orthogonal distributions. The integrand depends on the Ricci and mixed scalar curvatures and invariants of the second fundamental forms and integrability tensors of the distributions. The formulas under some conditions yield splitting of manifolds (including submersions and twisted products) and provide geometrical obstructions for existence of distributions and foliations (or compact leaves of them).

**Keywords**: metric; connection; distribution; submersion; twisted product; mixed scalar curvature; mean curvature; umbilical; harmonic; statistical; divergence; splitting.

**MSC (2010)** Primary 53C12; Secondary 53C44.

## INTRODUCTION

Integral formulas provide obstructions for existence of distributions and foliations (or compact leaves of them) with given geometric properties and have applications in different areas of geometry and analysis, see survey in [1, 13]. Distributions on manifolds appear in various situations, e.g. as fields of tangent planes of foliations or kernels of differential forms [3]. The first known integral formula by G. Reeb [11] for a codimension-1 foliated closed Riemannian manifold $(M, g)$ tells us that the total mean curvature $H$ of the leaves is zero (thus, either $H \equiv 0$ or $H(x)H(y) < 0$ for some points $x, y \in M$). The proof is based on the divergence theorem and the identity $\operatorname{div} N = (\dim M)\, h_{sc}$, where $N$ is a unit normal to the leaves and $h_{sc}$ their scalar second fundamental form. The formula poses a generalization (which is a consequence of Green's theorem applied to $N$) to the case of second order mean curvature $\sigma_2$:

$$\int_M (2\,\sigma_2 - \mathrm{Ric}_{N,N})\,\mathrm{d\,vol}_g = 0. \tag{1}$$

Moreover, (1) admits a leaf-wise counterpart for a closed leaf $M'$ with induced metric $g'$. Both formulas have many applications: e.g., (1) implies nonexistence of umbilical foliations on a closed manifold of negative curvature. Later on [2] extended (1) to infinite series of formulas with higher order mean curvatures $\sigma_k$ $(k \geq 2)$. In further generalizations for complementary orthogonal distributions of any dimension [9,12,16], the integrand depends on second fundamental

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forms and integrability tensors of the distributions, their derivatives and curvature invariants, e.g. the Ricci curvature and the mixed scalar curvature.

No attempt has been made to develop integral formulas for general metric-affine spaces equipped with distributions. The Metric-Affine Geometry founded by E. Cartan uses an asymmetric connection $\overline{\nabla}$ with torsion (instead of the Levi-Civita connection $\nabla$); it generalizes Riemannian Geometry and appears in such context as homogeneous and almost Hermitian manifolds. The important distinguished cases are: Riemann-Cartan manifolds, where metric connections, i.e., $\overline{\nabla} g=0$, are used, see e.g. [7], and statistical manifolds, see [4, 6], where the torsion is zero and the tensor $\overline{\nabla} g$ is symmetric in all its entries. The theory of affine hypersurfaces in $\mathbb{R}^{n+1}$ is a natural source of statistical manifolds, see [8]. Riemann-Cartan spaces are central in gauge theory of gravity, where the torsion is represented by the spin tensor of matter.

In this paper, we obtain integral formulas for general metric-affine spaces equipped with distributions and for two distinguished classes. The formulas naturally generalize results for Riemannian case [9, 14–16] with the Ricci and mixed scalar curvatures, under some conditions they yield splitting of ambient manifolds (including submersions and twisted products) and provide geometrical obstructions for existence of distributions and foliations or compact leaves of them.

## 1. Preliminaries

Let $M^{n+p}$ be a connected smooth manifold with a pseudo-Riemannian metric $g$ of index $q$ and complementary orthogonal non-degenerate distributions $\mathcal{D}^\top$ and $\mathcal{D}^\perp$ (subbundles of the tangent bundle $TM$ of ranks $\dim_{\mathbb{R}}\mathcal{D}_x^\top=n$ and $\dim_{\mathbb{R}}\mathcal{D}_x^\perp=p$ for every $x\in M$). A distribution $\mathcal{D}^\top$ is non-degenerate, if $g_x$ $(x\in M)$ is a non-degenerate bilinear form on $\mathcal{D}_x^\top\subset T_xM$ for every $x\in M$; in this case, $\mathcal{D}^\perp$ is also non-degenerate. When $q=0$, $g$ is a Riemannian metric (resp. a Lorentz metric when $q=1$), see [3]. Let $^\top$ and $^\perp$ denote $g$-orthogonal projections onto $\mathcal{D}^\top$ and $\mathcal{D}^\perp$, respectively; for any $X\in\mathfrak{X}_M$ we write $X=X^\top+X^\perp$. We will define several tensors for one of distributions (say, $\mathcal{D}^\top$; similar tensors for the second distribution can be defined using $^\perp$ notation). The following convention is adopted for the range of indices:

$$a,b\ldots\in\{1\ldots n\},\quad i,j\ldots\in\{1\ldots p\}.$$

One may show that the local adapted orthonormal frame $\{E_a,\,\mathcal{E}_i\}$, where $\{E_a\}\subset\mathcal{D}^\top$, and $\epsilon_i=g(\mathcal{E}_i,\mathcal{E}_i)\in\{-1,1\}$, $\epsilon_a=g(E_a,E_a)\in\{-1,1\}$, always exists on $M$.

Let $T^\top,h^\top:\mathcal{D}^\top\times\mathcal{D}^\top\to\mathcal{D}^\perp$ be the integrability tensor and the second fundamental form of $\mathcal{D}^\top$,

$$T^\top(X,Y):=(1/2)\,[X,\,Y]^\perp,\quad h^\top(X,Y):=(1/2)\,(\nabla_X Y+\nabla_Y X)^\perp.$$

The mean curvature vector of $\mathcal{D}^\top$ is $H^\top=\sum_a\epsilon_a h^\top(E_a,E_a)$. We call $\mathcal{D}^\top$ *umbilical*, *harmonic*, or *totally geodesic*, if $h^\top=\frac{1}{n}H^\top\,g^\top$, $H^\top=0$, or $h^\top=0$, resp. The Weingarten operator $A^\top$ (of $\mathcal{D}^\top$) and the operator $T^{\top\sharp}$ are defined by

$$g(A_Z^\top X,Y)=g(h^\top(X^\top,Y^\top),Z^\perp),\quad g(T_Z^{\top\sharp}X,Y)=g(T^\top(X^\top,Y^\top),Z^\perp).$$

We use the following convention for various tensors: $T_i^{\top\sharp}:=T_{\mathcal{E}_i}^{\top\sharp}$, $A_i^\top:=A_{\mathcal{E}_i}^\top$, $\mathcal{T}_i=\mathcal{T}_{\mathcal{E}_i}$ etc.

One of the simplest curvature invariants of a pseudo-Riemannian manifold $(M,g)$ endowed with two complementary orthogonal distributions $(\mathcal{D}^\top,\mathcal{D}^\perp)$ is the *mixed scalar curvature*, i.e., an averaged sectional curvature of planes that non-trivially intersect the distribution $\mathcal{D}^\top$ and its orthogonal complement $\mathcal{D}^\perp$, see [16]:

$$\mathrm{S}_{\rm mix} = \sum\nolimits_{a,i} \epsilon_a\epsilon_i\, g(R_{a,i}E_a,\, \mathcal{E}_i). \tag{2}$$

Here $R_{X,Y} = [\nabla_Y, \nabla_X] + \nabla_{[X,Y]}$ is the curvature tensor of $\nabla$. The following formula [16] has found many applications:

$$\begin{aligned}\operatorname{div}(H^\perp + H^\top) &= \mathrm{S}_{\rm mix} + \langle h^\perp, h^\perp\rangle - g(H^\perp, H^\perp) - \langle T^\perp, T^\perp\rangle \\ &+ \langle h^\top, h^\top\rangle - g(H^\top, H^\top) - \langle T^\top, T^\top\rangle\,,\end{aligned} \tag{3}$$

see survey in [1, 13]. We use inner products of tensors, e.g.

$$\begin{aligned}\langle h^\top, h^\top\rangle &= \sum\nolimits_{i,j} \epsilon_i\epsilon_j\, g(h^\top(\mathcal{E}_i,\mathcal{E}_j), h^\top(\mathcal{E}_i,\mathcal{E}_j)),\\ \langle T^\top, T^\top\rangle &= \sum\nolimits_{i,j} \epsilon_i\epsilon_j\, g(T^\top(\mathcal{E}_i,\mathcal{E}_j), T^\top(\mathcal{E}_i,\mathcal{E}_j))\,.\end{aligned}$$

Let $\mathfrak{X}_M$ (resp., $\mathfrak{X}_{\mathcal{D}^\top}$) be the module over $C^\infty(M)$ of all vector fields on $M$ (resp. on $\mathcal{D}^\top$). A *metric-affine space* is a manifold $M$ endowed with a metric $g$ of certain signature and a linear connection $\overline\nabla : \mathfrak{X}_M \times \mathfrak{X}_M \to \mathfrak{X}_M$ on $TM$ that is

$$\overline\nabla_{fX_1+X_2} Y = f\overline\nabla_{X_1} Y + \overline\nabla_{X_2} Y, \quad \overline\nabla_X(fY + Z) = X(f)Y + f\overline\nabla_X Y + \overline\nabla_X Z\,.$$

The Levi-Civita connection $\nabla$ is a unique torsion free connection on $(M, g)$ preserving $g$. It can be taken as a center of affine space of all connections on $M$. The difference $\mathcal{T} := \overline\nabla - \nabla$ is called the *contorsion tensor*. Define the (1,2)-tensors $\mathcal{T}^*$ and $\widehat{\mathcal{T}}$ by

$$g(\mathcal{T}^*_X Y, Z) = g(\mathcal{T}_X Z, Y), \quad \widehat{\mathcal{T}}_X Y = \mathcal{T}_Y X, \quad X, Y, Z \in \mathfrak{X}_M\,.$$

Remark that generally $(\widehat{\mathcal{T}})^* \ne \widehat{\mathcal{T}^*}$. Indeed,

$$g((\widehat{\mathcal{T}})^*_X Y, Z) = g(\widehat{\mathcal{T}}_X Z, Y) = g(\mathcal{T}_Z X, Y), \quad g(\widehat{\mathcal{T}^*_X} Y, Z) = g(\mathcal{T}^*_Y X, Z) = g(\mathcal{T}_Y Z, X).$$

A connection $\overline\nabla = \nabla + \mathcal{T}$ is *metric compatible* if $\overline\nabla g = 0$; in this case,

$$\mathcal{T}^* = -\mathcal{T}.$$

If $\overline\nabla$ is torsionless and tensor $\overline\nabla g$ is symmetric in all its entries then $\overline\nabla$ is called a *statistical connection* in literature. In this case, the contorsion tensor has the following symmetries:

$$\widehat{\mathcal{T}} = \mathcal{T}, \quad \mathcal{T}^* = \mathcal{T}.$$

Comparing the curvature tensor $\bar R_{X,Y} = [\overline\nabla_Y, \overline\nabla_X] + \overline\nabla_{[X,Y]}$ of $\overline\nabla$ with $R$, we find

$$\bar R_{X,Y} - R_{X,Y} = (\nabla_Y\mathcal{T})_X - (\nabla_X\mathcal{T})_Y + [\mathcal{T}_Y, \mathcal{T}_X], \quad X, Y \in \mathfrak{X}_M\,. \tag{4}$$

Define two *mean curvature type vectors of* $\mathcal{T}$ by $H^\top_{\mathcal{T}} := \sum_a \epsilon_a \mathcal{T}_a E_a$ and $H^\perp_{\mathcal{T}} := \sum_i \epsilon_i \mathcal{T}_i \mathcal{E}_i$.

*Remark.* One can examine the extrinsic geometry also in terms of $\overline\nabla$. For example,

$$\bar h^\top(X, Y) = h^\top(X, Y) + \frac{1}{2}\,(\mathcal{T}_X Y + \mathcal{T}_Y X)^\perp, \quad X, Y \in \mathcal{D}^\top,$$

is the second fundamental form of $\mathcal{D}^\top$ w.r.t. $\overline\nabla$, and $\bar H^\top = H^\top + (H^\top_{\mathcal{T}})^\perp$ is its mean curvature vector.

**Definition.** The following function on a metric-affine manifold $(M,g,\overline{\nabla})$ endowed with two complementary orthogonal distributions $(\mathcal{D}^\top,\mathcal{D}^\perp)$:

$$\bar{\rm S}_{\,\rm mix}=\frac12\sum\nolimits_{a,i}\epsilon_a\,\epsilon_i\big(g(\bar R_{a,i}E_a,\mathcal{E}_i)+g(\bar R_{i,a}\,\mathcal{E}_i,E_a)\big) \tag{5}$$

is called the *mixed scalar curvature* w.r.t. $\overline{\nabla}$, see (2) for the Riemannian case.

Definition (5) does not depend on the order of distributions and the choice of a frame. Thus, by (4) and (2),

$$\begin{aligned}\bar{\rm S}_{\,\rm mix} &= \frac12\sum\nolimits_{a,i}\epsilon_a\epsilon_i\big(g((\nabla_i\mathcal{T})_aE_a,\mathcal{E}_i)-g((\nabla_a\mathcal{T})_iE_a,\mathcal{E}_i)+g((\nabla_a\mathcal{T})_i\,\mathcal{E}_i,E_a)\\ &- g((\nabla_i\mathcal{T})_a\,\mathcal{E}_i,E_a)+g([\mathcal{T}_i,\,\mathcal{T}_a]E_a,\mathcal{E}_i)+g([\mathcal{T}_a,\,\mathcal{T}_i]\,\mathcal{E}_i,E_a)\big)+{\rm S}_{\,\rm mix}.\end{aligned} \tag{6}$$

The Divergence Theorem for a vector field $\xi$ on $(M,g)$ states that

$$\int_M(\operatorname{div}\xi)\,{\rm d}\operatorname{vol}_g=0, \tag{7}$$

when either $\xi$ has compact support or $M$ is closed. The $\mathcal{D}^\perp$*-divergence* of $\xi$ is defined by $\operatorname{div}^\perp\xi=\sum_i\epsilon_i\,g(\nabla_i\,\xi,\mathcal{E}_i)$, and for $\xi\in\mathfrak{X}_{\mathcal{D}^\perp}$ we have $\operatorname{div}^\perp\xi=\operatorname{div}\xi+g(\xi,\,H^\top)$.

## 2. Integral formulas

The main idea of proving integral formulas (as in the case of formulas discussed in the introduction) is to calculate the divergence of a vector field and use the Stokes Theorem. In [9], this approach was applied to vector fields $Z_k=(A^\top_{H^\top})^kH^\perp+(A^\perp_{H^\perp})^kH^\top$, on a Riemannian space, namely, for $k=0,1$. We add to $Z_0=H^\perp+H^\top$ (in Section 2.1) and $Z_1=A^\top_{H^\top}H^\perp+A^\perp_{H^\perp}H^\top$ (in Section 2.2) certain vector fields on a metric-affine space, compute their divergence and find integral formulas. We also work with a closed manifold $M$ equipped with vector fields, distributions, etc. defined on the complement to the "set of singularities" $\Sigma$, which is a (possibly empty) union of closed submanifolds of variable codimension $\ge 2$. The singular case is important since many manifolds admit no smooth (e.g. codimension-one) distributions, while they admit such distributions and tensors outside some $\Sigma$. Example of such distributions provide "open book decompositions" on manifolds, see discussion in [9].

**Lemma 1** (see Lemma 2 in [9])**.** *Let $\Sigma_1$, $\operatorname{codim}\Sigma_1\ge 2$, be a closed submanifold of a Riemannian manifold $(M,g)$, and $\xi$ a vector field on $M\setminus\Sigma_1$ such that $\|\xi\|_g\in L^2(M,g)$. Then* (7) *holds.*

### 2.1. Integral formulas with $\bar{\rm S}_{\,\rm mix}$.

The divergence of $Z_0$, see (3), was calculated in [16] and the integral formula

$$\int_M\big\{\,{\rm S}_{\,\rm mix}-\|T^\top\|^2-\|T^\perp\|^2+\|h^\top\|^2+\|h^\perp\|^2-\|H^\top\|^2-\|H^\perp\|^2\big\}\,{\rm d}\operatorname{vol}_g=0 \tag{8}$$

was obtained for a closed Riemannian manifold $M$. We will denote by $\langle B,C\rangle_{|V}$ the inner product of tensors $B,C$ restricted on $V=(\mathcal{D}^\top\times\mathcal{D}^\perp)\cup(\mathcal{D}^\perp\times\mathcal{D}^\top)$.

**Lemma 2.** *We have*

$$
\begin{aligned}
&\operatorname{div}\big((H^\top_{\mathcal T} - H^\top_{\mathcal T^*})^\perp + (H^\perp_{\mathcal T} - H^\perp_{\mathcal T^*})^\top\big) = 2\,(\bar{\rm S}_{\rm mix} - {\rm S}_{\rm mix})\\
&-g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}) - g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}) - g(H^\top_{\mathcal T} - H^\perp_{\mathcal T} + H^\perp_{\mathcal T^*} - H^\top_{\mathcal T^*},\, H^\top - H^\perp)\\
&-\langle \mathcal T - \mathcal T^* + \widehat{\mathcal T} - \widehat{\mathcal T^*},\, A^\perp - T^{\perp\sharp} + A^\top - T^{\top\sharp}\rangle + \langle \mathcal T^*,\, \widehat{\mathcal T}\rangle_{\,|\,V}.
\end{aligned}\tag{9}
$$

*For statistical manifolds,* (9) *reads*

$$\bar{\rm S}_{\rm mix} - {\rm S}_{\rm mix} - g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T}) + (1/2)\,\langle \mathcal T,\, \mathcal T\rangle_{\,|\,V} = 0.$$

*For Riemann-Cartan manifolds,* (9) *reads*

$$
\begin{aligned}
&\operatorname{div}\big((H^\top_{\mathcal T})^\perp + (H^\perp_{\mathcal T})^\top\big) = \bar{\rm S}_{\rm mix} - {\rm S}_{\rm mix} - g(H^\top_{\mathcal T} - H^\perp_{\mathcal T},\, H^\top - H^\perp)\\
&+g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T}) - \langle \mathcal T + \widehat{\mathcal T},\, A^\perp - T^{\perp\sharp} + A^\top - T^{\top\sharp}\rangle - (1/2)\,\langle \mathcal T,\, \widehat{\mathcal T}\rangle_{\,|\,V}.
\end{aligned}
$$

*Proof.* Using (6) we get $\bar{\rm S}_{\rm mix} - {\rm S}_{\rm mix} = \frac12\,(Q_1 + Q_2)$, where

$$
\begin{aligned}
Q_1 &= \textstyle\sum_{a,i} \epsilon_a\, \epsilon_i\,[\, g((\nabla_i \mathcal T)_a E_a, \mathcal E_i) - g((\nabla_a \mathcal T)_i E_a, \mathcal E_i) + g([\mathcal T_i,\, \mathcal T_a] E_a, \mathcal E_i)\,],\\
Q_2 &= \textstyle\sum_{a,i} \epsilon_a\, \epsilon_i\,[\, g((\nabla_a \mathcal T)_i \mathcal E_i, E_a) - g((\nabla_i \mathcal T)_a \mathcal E_i, E_a) + g([\mathcal T_a,\, \mathcal T_i]\, \mathcal E_i, E_a)\,].
\end{aligned}
$$

Let $(\nabla_i E_a)^\top = 0$ and $(\nabla_a \mathcal E_i)^\perp = 0$ at a point $x \in M$, thus $-\nabla_i E_a = \sum_j \epsilon_j\, g((A^\perp_a + T^{\perp\sharp}_a)\mathcal E_i, \mathcal E_j)\mathcal E_j$ and $-\nabla_a\, \mathcal E_i = \sum_b \epsilon_b\, g((A^\top_i + T^{\top\sharp}_i) E_a, E_b) E_b$. We calculate at $x$:

$$
\begin{aligned}
g((\nabla_i \mathcal T)_a E_a, \mathcal E_i) &= g(\mathcal T_i E_a + \mathcal T_a \mathcal E_i,\, (A^\perp_a - T^{\perp\sharp}_a)\mathcal E_i) + \operatorname{div}^\perp(H^\top_{\mathcal T}),\\
g((\nabla_a \mathcal T)_i E_a, \mathcal E_i) &= g(\mathcal T^*_a \mathcal E_i + \mathcal T^*_i E_a,\, (A^\top_i - T^{\top\sharp}_i) E_a) + \operatorname{div}^\top(H^\perp_{\mathcal T^*}),\\
g((\nabla_a \mathcal T)_i \mathcal E_i, E_a) &= g(\mathcal T_a \mathcal E_i + \mathcal T_i E_a,\, (A^\top_i - T^{\top\sharp}_i) E_a) + \operatorname{div}^\top(H^\perp_{\mathcal T}),\\
g((\nabla_i \mathcal T)_a \mathcal E_i, E_a) &= g(\mathcal T^*_i E_a + \mathcal T^*_a \mathcal E_i,\, (A^\perp_a - T^{\perp\sharp}_a)\mathcal E_i) + \operatorname{div}^\perp(H^\top_{\mathcal T^*}),\\
g([\mathcal T_i,\, \mathcal T_a] E_a, \mathcal E_i) &= g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}) - g(\mathcal T_i E_a, \mathcal T^*_a \mathcal E_i),\\
g([\mathcal T_a,\, \mathcal T_i]\, \mathcal E_i, E_a) &= g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}) - g(\mathcal T^*_i E_a, \mathcal T_a \mathcal E_i),
\end{aligned}
$$

(omitting $\sum_{a,i} \epsilon_a\, \epsilon_i$) and find

$$
\begin{aligned}
Q_1 &= \operatorname{div}^\perp(H^\top_{\mathcal T}) - \operatorname{div}^\top(H^\perp_{\mathcal T^*}) + \textstyle\sum_{a,i} \epsilon_a\, \epsilon_i \big[g(\mathcal T_i E_a + \mathcal T_a \mathcal E_i,\, (A^\perp_a - T^{\perp\sharp}_a)\mathcal E_i)\\
&- g(\mathcal T^*_a \mathcal E_i + \mathcal T^*_i E_a,\, (A^\top_i - T^{\top\sharp}_i) E_a) - g(\mathcal T_i E_a, \mathcal T^*_a \mathcal E_i)\big] + g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}),\\
Q_2 &= \operatorname{div}^\top(H^\perp_{\mathcal T}) - \operatorname{div}^\perp(H^\top_{\mathcal T^*}) + \textstyle\sum_{a,i} \epsilon_a\, \epsilon_i \big[g(\mathcal T_a \mathcal E_i + \mathcal T_i E_a,\, (A^\top_i - T^{\top\sharp}_i) E_a)\\
&- g(\mathcal T^*_i E_a + \mathcal T^*_a \mathcal E_i,\, (A^\perp_a - T^{\perp\sharp}_a)\mathcal E_i) - g(\mathcal T_a \mathcal E_i, \mathcal T^*_i E_a)\big] + g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}).
\end{aligned}\tag{10}
$$

From (10), using equalities

$$
\begin{aligned}
&\operatorname{div}^\perp(H^\top_{\mathcal T}) = \operatorname{div}((H^\top_{\mathcal T})^\perp) + g(H^\top_{\mathcal T}, H^\top - H^\perp),\\
&\operatorname{div}^\top(H^\perp_{\mathcal T^*}) = \operatorname{div}((H^\perp_{\mathcal T^*})^\top) - g(H^\perp_{\mathcal T^*}, H^\top - H^\perp),\\
&\operatorname{div}^\perp(H^\top_{\mathcal T^*}) = \operatorname{div}((H^\top_{\mathcal T^*})^\perp) + g(H^\top_{\mathcal T^*}, H^\top - H^\perp),\\
&\operatorname{div}^\top(H^\perp_{\mathcal T}) = \operatorname{div}((H^\perp_{\mathcal T})^\top) - g(H^\perp_{\mathcal T}, H^\top - H^\perp),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\operatorname{div}\big((H^\top_{\mathcal T} - H^\top_{\mathcal T^*})^\perp + (H^\perp_{\mathcal T} - H^\perp_{\mathcal T^*})^\top\big) = 2\,(\bar{\rm S}_{\rm mix} - {\rm S}_{\rm mix})\\
&- g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}) - g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}) - g(H^\top_{\mathcal T} - H^\perp_{\mathcal T} + H^\perp_{\mathcal T^*} - H^\top_{\mathcal T^*},\, H^\top - H^\perp)\\
&- \sum\nolimits_{a,i}\epsilon_a\,\epsilon_i\big[\,g((\mathcal T_i - \mathcal T_i^*)E_a + (\mathcal T_a - \mathcal T_a^*)\mathcal E_i,\, (A_a^\perp - T_a^{\perp\sharp})\,\mathcal E_i + (A_i^\top - T_i^{\top\sharp})E_a)\\
&- g(\mathcal T_a\,\mathcal E_i,\, \mathcal T_i^* E_a) - g(\mathcal T_a^*\,\mathcal E_i,\, \mathcal T_i\, E_a)\,\big],
\end{aligned}
$$

and hence, (9). □

In the next theorem we generalize (8).

**Theorem 3.** *Let $(M, g, \overline\nabla)$ be a closed metric-affine space and $\mathcal D^\top$ a distribution defined on the complement to the "set of singularities" $\Sigma$, see Section 1. If $\|\xi\|_g \in L^2(M, g)$, where $\xi = H^\perp + H^\top + \frac12\,(H^\top_{\mathcal T} - H^\top_{\mathcal T^*})^\perp + \frac12\,(H^\perp_{\mathcal T} - H^\perp_{\mathcal T^*})^\top$, then the following integral formula holds:*

$$
\begin{aligned}
&\int_M\big\{\bar{\rm S}_{\rm mix} - \langle T^\top, T^\top\rangle - \langle T^\perp, T^\perp\rangle + \langle h^\top, h^\top\rangle + \langle h^\perp, h^\perp\rangle - g(H^\top, H^\top) - g(H^\perp, H^\perp)\\
&\quad - \frac12\,\big[\,g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}) + g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}) + g(H^\top_{\mathcal T} - H^\perp_{\mathcal T} + H^\perp_{\mathcal T^*} - H^\top_{\mathcal T^*},\, H^\top - H^\perp)\,\big]\\
&\quad - \frac12\,\langle \mathcal T - \mathcal T^* + \widehat{\mathcal T} - \widehat{\mathcal T^*},\, A^\perp - T^{\perp\sharp} + A^\top - T^{\top\sharp}\rangle + \frac12\,\langle \mathcal T^*,\, \widehat{\mathcal T}\rangle_{\,|\,V}\big\}\,{\rm d\,vol}_g = 0.
\end{aligned}
$$

*Proof.* By (3) and Lemma 2, we have on $M\setminus\Sigma$:

$$
\begin{aligned}
\operatorname{div}\xi &= \bar{\rm S}_{\rm mix} - \langle T^\top, T^\top\rangle - \langle T^\perp, T^\perp\rangle + \langle h^\top, h^\top\rangle + \langle h^\perp, h^\perp\rangle - g(H^\top, H^\top) - g(H^\perp, H^\perp)\\
&- \frac12\,\big[\,g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T^*}) + g(H^\perp_{\mathcal T},\, H^\top_{\mathcal T^*}) + g(H^\top_{\mathcal T} - H^\perp_{\mathcal T} + H^\perp_{\mathcal T^*} - H^\top_{\mathcal T^*},\, H^\top - H^\perp)\,\big]\\
&- \frac12\,\langle \mathcal T - \mathcal T^* + \widehat{\mathcal T} - \widehat{\mathcal T^*},\, A^\perp - T^{\perp\sharp} + A^\top - T^{\top\sharp}\rangle + \frac12\,\langle \mathcal T^*,\, \widehat{\mathcal T}\rangle_{\,|\,V}.
\end{aligned}
\tag{11}
$$

Thus, the claim follows from (11) and Lemma 1. □

**Corollary 4.** *Let $(M, g, \overline\nabla)$ be a closed statistical manifold and $\mathcal D^\top$ a distribution defined on the complement to the "set of singularities" $\Sigma$. If $\|H^\perp + H^\top\|_g \in L^2(M, g)$ then*

$$
\begin{aligned}
&\int_M\big\{\,\bar{\rm S}_{\rm mix} - \langle T^\top, T^\top\rangle - \langle T^\perp, T^\perp\rangle + \langle h^\top, h^\top\rangle + \langle h^\perp, h^\perp\rangle\\
&- \ \ g(H^\top, H^\top) - g(H^\perp, H^\perp) - g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T}) + (1/2)\,\langle \mathcal T,\, \mathcal T\rangle_{\,|\,V}\big\}\,{\rm d\,vol}_g = 0.
\end{aligned}
$$

**Corollary 5.** *Let $(M, g, \overline\nabla)$ be a closed Riemann-Cartan manifold and $\mathcal D^\top$ a distribution defined on the complement to the "set of singularities" $\Sigma$. If $\|\bar H^\perp + \bar H^\top\|_g \in L^2(M, g)$ then*

$$
\begin{aligned}
&\int_M\big\{\,\bar{\rm S}_{\rm mix} - \langle T^\top, T^\top\rangle - \langle T^\perp, T^\perp\rangle + \langle h^\top, h^\top\rangle + \langle h^\perp, h^\perp\rangle\\
&- \ \ g(H^\top, H^\top) - g(H^\perp, H^\perp) + g(H^\top_{\mathcal T},\, H^\perp_{\mathcal T}) - g(H^\top_{\mathcal T} - H^\perp_{\mathcal T},\, H^\top - H^\perp)\\
&- \ \ \langle \mathcal T - \widehat{\mathcal T},\, A^\perp - T^{\perp\sharp} + A^\top - T^{\top\sharp}\rangle - (1/2)\,\langle \mathcal T,\, \widehat{\mathcal T}\rangle_{\,|\,V}\big\}\,{\rm d\,vol}_g = 0.
\end{aligned}
$$

**Definition.** We say that $(M', g')$ is a *leaf of a distribution* $\mathcal D$ on $(M, g)$ if $M'$ is a submanifold of $M$ with induced metric $g'$ and $T_xM' = \mathcal D_x$ for any $x \in M'$. A leaf $(M', g')$ of $\mathcal D$ is said to be *umbilical*, *harmonic*, or *totally geodesic*, if $h^\top = \frac1n\,H^\top g^\top$, $H^\top = 0$, or $h^\top = 0$, resp., on $M'$.

The next theorem generalizes result in [16].

**Theorem 6.** *Let a distribution $\mathcal{D}^\top$ on a metric-affine space $(M, g, \overline{\nabla})$ has a compact leaf $(M', g')$ with condition*

$$2H^\top = (H^\top_{\mathcal{T}^*} - H^\top_{\mathcal{T}})^\perp \tag{12}$$

*on its neighborhood. Then the following integral formula along the leaf holds:*

$$\begin{aligned}&\int_{M'}\big\{\bar{\rm S}_{\rm mix} - \langle T^\perp, T^\perp\rangle + \langle h^\perp, h^\perp\rangle + \langle h^\top, h^\top\rangle + \frac12\,\big[g(H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*}, H^\perp)\\ &\quad + g(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*}, H^\top) - g(H^\top_{\mathcal{T}}, H^\perp_{\mathcal{T}^*}) - g(H^\perp_{\mathcal{T}}, H^\top_{\mathcal{T}^*}) + \langle \mathcal{T}^*, \widehat{\mathcal{T}}\rangle_{\,|V}\\ &\quad - \langle \mathcal{T} - \mathcal{T}^* + \widehat{\mathcal{T}} - \widehat{\mathcal{T}^*}, A^\perp - T^{\perp\sharp} + A^\top\rangle\big]\big\}\,{\rm d}\,{\rm vol}_{g'} = 0.\end{aligned} \tag{13}$$

*Proof.* Using $T^\top = 0$, (3), Lemma 2 and equalities

$$\begin{aligned}&{\rm div}^\perp(H^\perp) = -g(H^\perp, H^\perp), \quad {\rm div}^\top(H^\top) = -g(H^\top, H^\top),\\ &{\rm div}^\perp((H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top) = -g(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*}, H^\perp),\\ &{\rm div}^\top((H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*})^\perp) = -g(H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*}, H^\top),\end{aligned}$$

we have

$$\begin{aligned}&{\rm div}^\top\big(H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top\big) + {\rm div}^\perp\big(H^\top + \frac12\,(H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*})^\perp\big)\\ &= \bar{\rm S}_{\rm mix} + \langle h^\perp, h^\perp\rangle + \langle h^\top, h^\top\rangle - \langle T^\perp, T^\perp\rangle\\ &+ \frac12\,\big[g(H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*}, H^\perp) + g(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*}, H^\top) - g(H^\top_{\mathcal{T}}, H^\perp_{\mathcal{T}^*})\\ &- g(H^\perp_{\mathcal{T}}, H^\top_{\mathcal{T}^*}) + \langle \mathcal{T}^*, \widehat{\mathcal{T}}\rangle_{\,|V} - \langle \mathcal{T} - \mathcal{T}^* + \widehat{\mathcal{T}} - \widehat{\mathcal{T}^*}, A^\perp - T^{\perp\sharp} + A^\top\rangle\big].\end{aligned} \tag{14}$$

By conditions (12), the ${\rm div}^\perp$-term in (14) vanishes along $M'$. Thus, (13) follows from the Divergence theorem for $\xi = H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$ on $M'$. □

**Corollary 7.** *Let a distribution $\mathcal{D}^\top$ on a statistical manifold $(M, g, \overline{\nabla})$ admits a compact leaf $(M', g')$ with $H^\top = 0$ on its neighborhood. Then the following integral formula holds:*

$$\begin{aligned}&\int_{M'}\big\{\bar{\rm S}_{\rm mix} - \langle T^\perp, T^\perp\rangle + \langle h^\perp, h^\perp\rangle + \langle h^\top, h^\top\rangle\\ &- g(H^\top_{\mathcal{T}},\, H^\perp_{\mathcal{T}}) + (1/2)\,\langle \mathcal{T}, \mathcal{T}\rangle_{\,|V}\big\}\,{\rm d}\,{\rm vol}_{g'} = 0.\end{aligned}$$

**Corollary 8.** *Let a distribution $\mathcal{D}^\top$ on a Riemann-Cartan manifold $(M, g, \overline{\nabla})$ admits a compact leaf $(M', g')$ with $\bar H^\top = 0$ on its neighborhood. Then the following integral formula holds:*

$$\begin{aligned}&\int_{M'}\big\{\bar{\rm S}_{\rm mix} - \langle T^\perp, T^\perp\rangle + \langle h^\perp, h^\perp\rangle + \langle h^\top, h^\top\rangle + g(H^\top_{\mathcal{T}},\, H^\perp_{\mathcal{T}}) + g(H^\perp_{\mathcal{T}},\, H^\top)\\ &+ g(H^\top_{\mathcal{T}},\, H^\perp) - \langle \mathcal{T} + \widehat{\mathcal{T}}, A^\perp - T^{\perp\sharp} + A^\top\rangle - (1/2)\,\langle \mathcal{T},\, \widehat{\mathcal{T}}\rangle_{\,|V}\big\}\,{\rm d}\,{\rm vol}_{g'} = 0.\end{aligned}$$

*Remark.* For $\mathcal{D}^\perp$ spanned by a unit vector field $N$, set $\overline{\rm Ric}_{N,N} = \sum_a \epsilon_a\, g(\bar R_{N,E_a}N, E_a)$. Note that generally $g(\bar R_{N,E_a}N, E_a) \ne g(\bar R_{E_a,N}E_a, N)$. Let $\epsilon_N = 1$. Similarly to Lemma 2, we have

$$\begin{aligned}
&{\rm div}((\mathcal{T}_N N)^\top - (H^\top_{\mathcal{T}^*})^\perp) = \overline{\rm Ric}_{N,N} - {\rm Ric}_{N,N} + Q\,,\\
&Q = g(H^\top_{\mathcal{T}^*} + \mathcal{T}_N N,\, H^\perp - ({\rm Tr}\ A^\top)N) - \langle (\widehat{\mathcal{T}})^*_N + \mathcal{T}^*_N, A^\top_N + T^{\top\sharp}_N\rangle_{\,|\mathcal{D}^\top}\\
&\quad + g((\widehat{\mathcal{T}})^*_N N + \mathcal{T}_N N, H^\perp) - g(H^\top_{\mathcal{T}^*}, \mathcal{T}_N N) + \langle \widehat{\mathcal{T}}_N, \mathcal{T}^*_N\rangle_{\,|\mathcal{D}^\top}.
\end{aligned}$$

The above yield the integral formula, which for $\mathcal{T} = 0$ reduces to (1) with $\sigma_2 = {\rm Tr}\,(A^\top_N)^2$:

$$\int_M (2\,\sigma_2 - \overline{\rm Ric}_{N,N} - Q)\,{\rm d}\,{\rm vol}_g = 0.$$

2.2. **Integral formula with the Ricci curvature $\overline{\rm Ric}_{H^\top,H^\perp}$.** The divergence of $Z_1$ for a Riemannian manifold endowed with orthogonal complementary distributions $\mathcal{D}^\top$ and $\mathcal{D}^\perp$ has been calculated in [9]:

$$ {\rm div}(A^\top_{H^\top} H^\perp + A^\perp_{H^\perp} H^\top) = {\rm Ric}_{H^\top,H^\perp} + Q_1, \tag{15}$$

where

$$\begin{aligned}
Q_1 &= g(H^\top, \nabla_{H^\perp} H^\top) + g(H^\perp, \nabla_{H^\top} H^\perp)\\
&\quad + g({\rm Tr}^\perp_g(\nabla_\cdot\, T^\top)(\cdot\,, H^\perp), H^\top) + g({\rm Tr}^\top_g(\nabla_\cdot\, T^\perp)(\cdot\,, H^\top), H^\perp)\\
&\quad + \langle A^\top_{H^\top}, \nabla_\cdot\, H^\perp\rangle + \langle A^\perp_{H^\perp}, \nabla_\cdot\, H^\top\rangle - g(A^\top_{H^\perp} H^\top, H^\top) - g(A^\perp_{H^\top} H^\perp, H^\perp)\\
&\quad + 2\sum\nolimits_a \epsilon_a \big[\, g(A^\top_{(\nabla_a H^\top)^\perp} H^\perp, E_a) + g(\nabla_{T^\top(H^\perp,E_a)} E_a, H^\top)\,\big]\\
&\quad + 2\sum\nolimits_i \epsilon_i \big[\, g(A^\perp_{(\nabla_i H^\perp)^\top} H^\top, \mathcal{E}_i) + g(\nabla_{T^\perp(H^\top,\mathcal{E}_i)} \mathcal{E}_i, H^\perp)\,\big].
\end{aligned} \tag{16}$$

Thus, on a closed manifold $(M, g)$ one has the integral formula, see [9, Theorem 1],

$$\int_M ({\rm Ric}_{H^\top,H^\perp} + Q_1)\,{\rm d}\,{\rm vol}_g = 0. \tag{17}$$

If the distributions are umbilical, integrable and have constant mean curvature then (16) reads

$$Q_1 = -(1/n + 1/p)\, g(H^\top, H^\top)\, g(H^\perp, H^\perp).$$

**Lemma 9.** *For the metric-affine case we have*

$$ {\rm div}(\mathcal{T}_{H^\perp} H^\top + \mathcal{T}_{H^\top} H^\perp) = -(\overline{\rm Ric}_{H^\top,H^\perp} - {\rm Ric}_{H^\top,H^\perp}) + Q_2, \tag{18}$$

*where* $\overline{\rm Ric}_{H^\top,H^\perp} = {\rm Sym}\big(\sum_a \epsilon_a\, g(\bar R_{H^\perp,E_a} H^\top, E_a) + \sum_i \epsilon_i\, g(\bar R_{H^\top,\mathcal{E}_i} H^\perp, \mathcal{E}_i)\big)$ *and*

$$\begin{aligned}
Q_2 = {\rm Sym}\Big( & g(\overline\nabla_{H^\perp} H^\top, H^\top_{\mathcal{T}^*}) + g(\overline\nabla_{H^\top} H^\perp, H^\perp_{\mathcal{T}^*}) - g(\mathcal{T}_{H^\perp} H^\top, H^\perp) - g(\mathcal{T}_{H^\top} H^\perp, H^\top)\\
& - \sum\nolimits_a \epsilon_a\, g\big(\nabla_{H^\perp}(\mathcal{T}_a H^\top) - \mathcal{T}_{(h^\top + T^\top)(H^\perp,E_a) + \nabla_a H^\perp} H^\top + \mathcal{T}_{H^\perp}(\overline\nabla_a H^\top),\, E_a\big)\\
& - \sum\nolimits_i \epsilon_i\, g\big(\nabla_{H^\top}(\mathcal{T}_i H^\perp) - \mathcal{T}_{(h^\perp + T^\perp)(H^\top,\mathcal{E}_i) + \nabla_i H^\top} H^\perp + \mathcal{T}_{H^\top}(\overline\nabla_i H^\perp),\, \mathcal{E}_i\big)\Big).
\end{aligned} \tag{19}$$

*Proof.* Using (4), we calculate

$$
\begin{aligned}
&\sum\nolimits_a \epsilon_a\, g(\bar R_{H^\perp,\,E_a}H^\top - R_{H^\perp,\,E_a}H^\top, E_a) =\\
&= \operatorname{Sym}\big(\operatorname{div}^\top(\mathcal{T}_{H^\perp}H^\top) + g(\nabla_{H^\perp}H^\top, H^\top_{\mathcal{T}^*}) + g(\mathcal{T}_{H^\perp}H^\top, H^\top_{\mathcal{T}^*})\\
&-\sum\nolimits_a \epsilon_a\big[g(\nabla_{H^\perp}(\mathcal{T}_a H^\top), E_a) - g(\mathcal{T}_{(h^\top+T^\top)(H^\perp,E_a)}H^\top, E_a)\\
&-g(\mathcal{T}_{\nabla_a H^\perp}H^\top, E_a) + g(\mathcal{T}_{H^\perp}(\nabla_a H^\top), E_a) + g(\mathcal{T}_{H^\perp}(\mathcal{T}_a H^\top), E_a)\big]\big).
\end{aligned}
\tag{20}
$$

Summing (20) with similar formula for $\sum_i \epsilon_i\, g(\bar R_{\mathcal{E}_i,H^\top}H^\perp - R_{\mathcal{E}_i,H^\top}H^\perp, \mathcal{E}_i)$ and using equalities

$$
\begin{aligned}
\operatorname{div}^\perp(\mathcal{T}_{H^\top}H^\perp) &= \operatorname{div}(\mathcal{T}_{H^\top}H^\perp) - g(H^\top, \mathcal{T}_{H^\top}H^\perp),\\
\operatorname{div}^\top(\mathcal{T}_{H^\perp}H^\top) &= \operatorname{div}(\mathcal{T}_{H^\perp}H^\top) - g(H^\perp, \mathcal{T}_{H^\perp}H^\top),
\end{aligned}
$$

yield (18)–(19). □

**Theorem 10.** *Let $(M, g, \overline\nabla)$ be a closed metric-affine space and $\mathcal{D}^\top$ a distribution defined on the complement to the "set of singularities" $\Sigma$. If $\|\xi\|_g \in L^2(M, g)$, where $\xi = A^\top_{H^\top}H^\perp + A^\perp_{H^\perp}H^\top + \mathcal{T}_{H^\top}H^\perp + \mathcal{T}_{H^\perp}H^\top$, then*

$$
\int_M \big\{\,\overline{\operatorname{Ric}}_{H^\top,H^\perp} + Q_1 + Q_2\big\}\,\mathrm{d}\,\mathrm{vol}_g = 0. \tag{21}
$$

*Proof.* From (15) and (18) we obtain

$$
\operatorname{div}(A^\top_{H^\top}H^\perp + A^\perp_{H^\perp}H^\top + \mathcal{T}_{H^\perp}H^\top + \mathcal{T}_{H^\top}H^\perp) = -\overline{\operatorname{Ric}}_{H^\top,H^\perp} + Q_1 + Q_2.
$$

Applying Lemma 1 to (16), (18) and (19), we obtain (21). □

For $\mathcal{T} = 0$, we have $Q_2 = 0$, and (21) reduces to (17). One may get a number of formulas from (21). The next one generalizes Proposition 3 in [9].

Recall that $\mathcal{D}^\top$ has constant mean curvature whenever its mean curvature vector $H^\top$ obeys $\nabla^\perp H^\top = 0$, where $\nabla^\perp$ is the connection in $\mathcal{D}^\perp$ induced by the Levi-Civita connection on $M$.

**Corollary 11.** *Let in conditions of Theorem 10, distributions $\mathcal{D}^\top$ and $\mathcal{D}^\perp$ be umbilical, integrable and have constant mean curvature. Then* (21) *reads*

$$
\begin{aligned}
&\int_M \big\{\overline{\operatorname{Ric}}_{H^\top,H^\perp} - (\frac{1}{n}+\frac{1}{p})\,g(H^\top,H^\top)\,g(H^\perp,H^\perp) + \operatorname{Sym}\big(g(\overline\nabla_{H^\top}H^\perp, H^\perp_{\mathcal{T}^*})\\
&+g(\overline\nabla_{H^\perp}H^\top, H^\top_{\mathcal{T}^*}) - g(H^\perp,\ \mathcal{T}_{H^\perp-H^\top}\,H^\top) - g(H^\top,\ \mathcal{T}_{H^\top-H^\perp}\,H^\perp)\\
&-\sum\nolimits_a \epsilon_a\, g(\overline\nabla_{H^\perp}(\mathcal{T}_a H^\top) - \mathcal{T}_{\nabla_a H^\perp}H^\top + \mathcal{T}_{H^\perp}(\nabla_a H^\top),\ E_a)\\
&-\sum\nolimits_i \epsilon_i\, g(\overline\nabla_{H^\top}(\mathcal{T}_i H^\perp) - \mathcal{T}_{\nabla_i H^\top}H^\perp + \mathcal{T}_{H^\top}(\nabla_i H^\perp),\ \mathcal{E}_i)\big)\big\}\,\mathrm{d}\,\mathrm{vol}_g = 0.
\end{aligned}
$$

## 3. Splitting results

In this section, we consider distributions $(\mathcal{D}^\top, \mathcal{D}^\perp)$ on a metric-affine manifold $(M, g, \overline\nabla)$, satisfying some geometrical conditions, and prove non-existence and splitting results, which follow from integral formulas of Section 2. In the sequel, we assume that $g > 0$. We omit similar results for (co)dimension one distributions and foliations and consequences for harmonic and Riemannian submersions, which follow from results below.

We say that $(M,g)$ endowed with a distribution $\mathcal{D}^\top$ *splits* if $M$ is locally isometric to a product with canonical foliations tangent to $\mathcal{D}^\top$ and $\mathcal{D}^\perp$. Remark that if a simply connected manifold splits then it is the direct product.

The next conditions are introduced to simplify the presentation of results:

$$\mathcal{T}_X Y = 0 = \mathcal{T}_Y X,\quad \mathcal{T}^*_X Y = 0 = \mathcal{T}^*_Y X \quad (X\in\mathcal{D}^\top,\ Y\in\mathcal{D}^\perp), \tag{22}$$

$$g(H^\top_{\mathcal{T}} - H^\top_{\mathcal{T}^*},\ H^\perp) = 0. \tag{23}$$

For example, (22) provides vanishing of last two lines in (11).

Recent extensions of (7) to non-compact case are discussed in [14]. Applying S.T.Yau version of Stokes' theorem on a complete open Riemannian manifold $(M,g)$ yields the following.

**Lemma 12** (see Proposition 1 in [5])**.** *Let $(M,g)$ be a complete open Riemannian manifold endowed with a vector field $\xi$ such that* $\operatorname{div}\xi\ge 0$. *If the norm* $\|\xi\|_g\in L^1(M,g)$ *then* $\operatorname{div}\xi\equiv 0$.

3.1. **Harmonic distributions.** Note that condition (12) on $M$ when $\mathcal{T}=0$ reduces to $H^\top=0$, i.e., $\mathcal{D}^\top$ is harmonic. Next theorem generalizes Theorem 5 in [14].

**Theorem 13.** *Let $\mathcal{D}^\top$ and $\mathcal{D}^\perp$ be complementary orthogonal integrable distributions with conditions* (12), (22) *and* (23) *on a complete open metric-affine space $(M,g,\overline{\nabla})$. Suppose that the leaves $(M',g')$ of $\mathcal{D}^\top$ obey condition $\|\xi_{\,|M'}\|_{g'}\in L^1(M',g')$, where $\xi = H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$. If $\bar{\rm S}_{\rm mix}\ge 0$ then $M$ splits.*

*Proof.* By conditions, we have

$$\operatorname{div}^\top\xi = \bar{\rm S}_{\rm mix} + \langle h^\perp, h^\perp\rangle + \langle h^\top, h^\top\rangle. \tag{24}$$

Applying Lemma 12 to each leaf (a complete open manifold), and since $\bar{\rm S}_{\rm mix}\ge 0$, we get $\operatorname{div}\xi=0$. Thus, $h^\top = 0 = h^\perp$. By de Rham theorem, $(M,g)$ splits. □

Note that condition $\|\,\xi_{|M'}\|_{g'}\in L^1(M',g')$ is satisfied on any compact leaf $(M',g')$ of $\mathcal{D}^\top$.

Next two results generalize Theorem 2 and Corollary 4 in [16].

**Corollary 14.** *Let $(M,g,\overline{\nabla})$ be a metric-affine space endowed with a distribution $\mathcal{D}^\top$ with integrable normal bundle and conditions* (12), (22) *and* (23). *Then $\mathcal{D}^\top$ has no complete open leaves $(M',g')$ with the properties $\bar{\rm S}_{{\rm mix}\,|M'}>0$ and $\|\xi_{\,|M'}\|_{g'}\in L^1(M',g')$, where $\xi = H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$. In particular, there are no compact leaves with $\bar{\rm S}_{{\rm mix}\,|M'}>0$.*

*Proof.* Let $(M',g')$ be a complete open leaf obeying the conditions. By (14), we have (24) on $M'$. Applying Lemma 12 to the leaf, and since $\bar{\rm S}_{\rm mix}>0$, we get $\operatorname{div}\xi=0$. The above yields $h^\top=0=h^\perp$ and $\bar{\rm S}_{\rm mix}=0$ – a contradiction. □

**Corollary 15.** *A codimension one distribution $\mathcal{D}^\top$ of a metric-affine space $(M,g,\overline{\nabla})$ with the Ricci curvature $\overline{\rm Ric}>0$ and the properties* (12), (22) *and* (23) *has no compact leaves.*

*Proof.* For a codimension one $\mathcal{D}^\top$, we have $T^\top=0$ and $\epsilon_N\,\overline{\rm Ric}_{N,N}=\bar{\rm S}_{\rm mix}$, where $N$ is a unit normal to the leaves. Hence, the claim follows from Corollary 14. □

**Theorem 16.** *Let $(M,g,\overline{\nabla})$ be a complete open (or closed) metric-affine space endowed with complementary orthogonal harmonic foliations. Suppose that conditions* (22), $\|\xi\|_g\in L^1(M,g)$ *with $\xi=\frac12\,(H^\top_{\mathcal{T}}-H^\top_{\mathcal{T}^*})^\perp+\frac12\,(H^\perp_{\mathcal{T}}-H^\perp_{\mathcal{T}^*})^\top$, and*

$$g(H^\top_{\mathcal{T}},\ H^\perp_{\mathcal{T}^*}) + g(H^\perp_{\mathcal{T}},\ H^\top_{\mathcal{T}^*}) = 0$$

*are satisfied. If* $\bar{\rm S}_{\rm mix} \ge 0$ *then* $M$ *splits.*

*Proof.* Under conditions, from (11) we get
$$\operatorname{div}\xi = \bar{\rm S}_{\rm mix} + \langle h^\top, h^\top\rangle + \langle h^\perp, h^\perp\rangle.$$
By Lemma 12 and since $\bar{\rm S}_{\rm mix} \ge 0$, we obtain $\operatorname{div}\xi = 0$. Thus, $h^\top = 0 = h^\perp$. Hence, by de Rham decomposition theorem, $(M,g)$ splits. □

3.2. **Umbilical distributions.**

**Theorem 17.** *Let* $(M, g, \overline{\nabla})$ *be a metric-affine space endowed with two complementary orthogonal distributions* $(\mathcal{D}^\top, \mathcal{D}^\perp)$ *with conditions* (12), (22) *and* (23). *Then* $\mathcal{D}^\top$ *has no complete open umbilical leaves* $(M', g')$ *with the properties* $\bar{\rm S}_{{\rm mix}\,|M'} < 0$ *and* $\|\xi_{\,|M'}\|_{g'} \in L^1(M', g')$, *where* $\xi = H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$. *In particular, there are no compact umbilical leaves* $(M', g')$ *with* $\bar{\rm S}_{{\rm mix}\,|M'} < 0$.

*Proof.* Let $(M', g')$ be a complete open umbilical leaf obeying the conditions. From (14) we get
$$\operatorname{div}^\top\xi = \bar{\rm S}_{\rm mix} - \langle T^\perp, T^\perp\rangle - \frac{p-1}{p}\, g(H^\perp, H^\perp) - \frac{n-1}{n}\, g(H^\top, H^\top) \qquad (25)$$
on $M'$. Thus, applying Lemma 12 to the leaf, and since $\bar{\rm S}_{{\rm mix}\,|M'} < 0$, we get $\operatorname{div}^\top\xi = 0$. By the above, $H^\top = 0 = H^\perp$, $T^\perp = 0$ and $\bar{\rm S}_{\rm mix} = 0$ – a contradiction. □

**Corollary 18.** *A codimension one distribution* $\mathcal{D}^\top$ *on a metric-affine space* $(M, g, \overline{\nabla})$ *with the Ricci curvature* $\overline{\rm Ric} < 0$ *and conditions* (12), (22) *and* (23) *has no compact umbilical leaves.*

A submersion $f : (M, g) \to (\tilde M, \tilde g)$ is *conformal* if $f_*$ restricted to $(\ker f_*)^\perp$ is conformal map, see [14].

**Theorem 19** (For $\mathcal{T} = 0$, see Corollary 5 in [14])**.** *Let* $(M, g, \overline{\nabla})$ *be a complete open metric-affine space,* $f : (M, g) \to (\tilde M, \tilde g)$ *a conformal submersion with umbilical fibres and conditions* (12), (22) *and* (23). *If* $\bar{\rm S}_{\rm mix} \le 0$ *and* $\|\xi_{|M'}\|_{g'} \in L^1(M', g')$, *where* $\xi = H^\perp + \frac12 (H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$, *on any fibre* $(M'g')$ *then* $(\ker f_*)^\perp$ *is integrable and* $M$ *splits.*

*Proof.* Set $\mathcal{D}^\top = \ker f$. Under conditions, we have (25). Applying Lemma 12 to every fibre (a complete open manifold), and since $\bar{\rm S}_{\rm mix} \le 0$, we get $\operatorname{div}\xi = 0$. The above yields vanishing of $H^\top$, $H^\perp$ and $T^\perp$. By de Rham decomposition theorem, $(M, g)$ splits. □

**Theorem 20** (For $\mathcal{T} = 0$, see Theorem 4 in [14])**.** *Let* $(M, g, \overline{\nabla})$ *be a complete open (or closed) metric-affine space endowed with complementary orthogonal umbilical distributions* $\mathcal{D}^\top$ *and* $\mathcal{D}^\perp$ *defined on the complement to the "set of singularities"* $\Sigma$. *If conditions* (22),
$$H^\top_{\mathcal{T}} = 0 = H^\perp_{\mathcal{T}}, \quad H^\top_{\mathcal{T}^*} = 0 = H^\perp_{\mathcal{T}^*} \qquad (26)$$
*and* $\|\xi\|_g \in L^1(M, g)$, *where* $\xi = H^\perp + H^\top$, *are satisfies and* $\bar{\rm S}_{\rm mix} \le 0$ *then* $M$ *splits.*

*Proof.* Under conditions, from (11) we get
$$\operatorname{div}\xi = \bar{\rm S}_{\rm mix} - \langle T^\top, T^\top\rangle - \langle T^\perp, T^\perp\rangle - \frac{p-1}{p}\, g(H^\perp, H^\perp) - \frac{n-1}{n}\, g(H^\top, H^\top). \qquad (27)$$
From (27) and Lemma 12 and since $\bar{\rm S}_{\rm mix} \le 0$, we get $\operatorname{div}\xi = 0$. The above yields vanishing of $T^\top, T^\perp, H^\top, H^\perp$. Hence, by de Rham decomposition theorem, $(M, g)$ splits. □

Umbilical integrable distributions appear on double-twisted products, see [10].

**Definition.** A *doubly-twisted product* $B\times_{(v,u)} F$ of two metric-affine spaces $(B, g_B, \mathcal{T}_B)$ and $(F, g_F, \mathcal{T}_F)$ is a manifold $M = B\times F$ with metric $g = g^\top + g^\perp$ and contorsion tensor $\mathcal{T} = \mathcal{T}^\top + \mathcal{T}^\perp$, where

$$\begin{aligned} g^\top(X,Y) &= v^2 g_B(X^\top, Y^\top), \quad g^\perp(X,Y) = u^2 g_F(X^\perp, Y^\perp),\\ (\mathcal{T}^\top)_X Y &= u^2(\mathcal{T}_B)_{X^\top} Y^\top, \quad (\mathcal{T}^\perp)_X Y = v^2(\mathcal{T}_F)_{X^\perp} Y^\perp, \end{aligned}$$

and the warping functions $u, v\in C^\infty(M)$ are positive. Indeed, $\mathcal{T}^* = \mathcal{T}^{*\top} + \mathcal{T}^{*\perp}$, where

$$(\mathcal{T}^{*\top})_X Y = u^2(\mathcal{T}^*_B)_{X^\top} Y^\top, \quad (\mathcal{T}^{*\perp})_X Y = v^2(\mathcal{T}^*_F)_{X^\perp} Y^\perp.$$

Let $\mathcal{D}^\top$ be tangent to the *fibers* $\{x\}\times F$ and $\mathcal{D}^\perp$ tangent to the *leaves* $B\times\{y\}$. The second fundamental forms and the mean curvature vectors of $B\times_{(v,u)} F$ are given by, see [10],

$$\begin{aligned} h^\perp &= -\nabla^\top(\log u)\, g^\perp, \quad h^\top = -\nabla^\perp(\log v)\, g^\top,\\ H^\perp &= -n\,\nabla^\top(\log u), \quad H^\top = -p\,\nabla^\perp(\log v). \end{aligned}$$

Thus, the leaves and the fibers of $B\times_{(v,u)} F$ are umbilical w.r.t. $\overline{\nabla}$ and $\nabla$. Conditions (22) are obviously satisfied for $B\times_{(v,u)} F$. Next corollaries extend results in [15].

**Corollary 21** (of Theorem 17)**.** *Let* (12), (23) *and* $\|\xi_{\,|M'}\|_{g'}\in L^1(M', g')$, *where* $\xi = H^\perp + \frac12\,(H^\perp_{\mathcal{T}} - H^\perp_{\mathcal{T}^*})^\top$, *are satisfied along the fibres of* $B\times_{(v,u)} F$, *where* $(F, g_F)$ *is complete open (or closed). If* $\bar{\rm S}_{\rm mix}\le 0$ *then* $M$ *is the direct product.*

*Proof.* Under conditions, from (25) we get

$$\operatorname{div}^\top \xi = \bar{\rm S}_{\rm mix} - \frac{p-1}{p}\, g(H^\perp, H^\perp) - \frac{n-1}{n}\, g(H^\top, H^\top).$$

Applying Lemma 12 to each fibre (a complete manifold), and since $\bar{\rm S}_{\rm mix}\le 0$, we get $\operatorname{div}\xi = 0$. Hence, $\bar{\rm S}_{\rm mix} = 0$ and $H^\top = 0 = H^\perp$, i.e., $\nabla^\top u = 0 = \nabla^\perp v$. By the above, ${\rm S}_{\rm mix} = 0$; thus, $u$ and $v$ are constant. By de Rham decomposition theorem, $(M, g)$ splits with the factors $(B, c_1\cdot g_B)$ and $(F, c_2\cdot g_F)$ for some $c_1, c_2 > 0$. □

**Corollary 22** (of Theorem 20)**.** *Let* $M = B\times_{(v,u)} F$ *be complete open (or closed) and conditions* (23), (26) *and* $\|\xi\|_g\in L^1(M, g)$, *where* $\xi = H^\perp + H^\top$ *are satisfied. If* $\bar{\rm S}_{\rm mix}\le 0$ *then* $M$ *is the direct product.*

*Proof.* Set $\mathcal{D}^\top = \pi_*(TF)$. Under conditions, we get, see (27),

$$\operatorname{div}\xi = \bar{\rm S}_{\rm mix} - \frac{p-1}{p}\, g(H^\perp, H^\perp) - \frac{n-1}{n}\, g(H^\top, H^\top).$$

Applying Lemma 12 to $M$, and since $\bar{\rm S}_{\rm mix}\le 0$, we get $\operatorname{div}\xi = 0$. Hence, $\bar{\rm S}_{\rm mix} = 0$ and $H^\top = 0 = H^\perp$, i.e., $\nabla^\top u = 0 = \nabla^\perp v$. By the above, ${\rm S}_{\rm mix} = 0$; thus, $u$ and $v$ are constant. By de Rham decomposition theorem, $(M, g)$ splits with the factors $(B,\, c_1\cdot g_B)$ and $(F,\, c_2\cdot g_F)$ for some positive $c_1, c_2\in\mathbb{R}$. □

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