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η-RICCI SOLITONS ON η-EINSTEIN KENMOTSU MANIFOLDS

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ABSTRACT. The object of the present paper is to study η-Ricci solitons on η-Einstein Kenmotsu manifolds. It is shown that if the characteristic vector field ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein Kenmotsu manifold then ξ is (i) concurrent and (ii) Killing vector field.

1. INTRODUCTION

In 1982, Hamilton [15] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([27], [28]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphisms and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [16]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [29] Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et. al ([1], [2], [3], [23]), Bejan and Crasmareanu [4], Blaga [5], Chen and Deshmukh [10], Deshmukh et. al [14], He and Zhu [17], Hui et. al ([9], [19], [20], [21], [22]), Nagaraja and Premalatha [26], Tripathi [31] and many others.

In [12] Cho and Kimura studied Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined η-Ricci soliton, which satisfies the equation

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu \eta \otimes \eta = 0, \tag{2}$$

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where λ and μ are real constants. The η-Ricci solitons are studied on Hopf hypersurfaces in the paper [8]. Also it may be noted that a generalization of η-Einstein geometry is provided by $N(k)$-quasi-Einstein geometry and Ricci solitons for this framework are studied in [13].

Motivated by the above studies the object of the present paper is to study η-Ricci solitons on η-Einstein Kenmotsu manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of η-Ricci solitons on η-Einstein Kenmotsu manifolds. It is proved that if ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$ then ξ is (i) concurrent and (ii) Killing vector field.

2. PRELIMINARIES

A smooth manifold (M^n, g) ($n = 2m+1 > 3$) is said to be an almost contact metric manifold [7] if it admits a (1,1) tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{4}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{5}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following condition holds [24]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{7}$$

where ∇ denotes the Riemannian connection of g.
In a Kenmotsu manifold, the following relations hold [24]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{8}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{9}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{10}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{11}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{12}$$

$$S(\xi, \xi) = -(n-1), \text{ i.e., } Q\xi = -(n-1)\xi, \tag{13}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{14}$$

$$(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W \tag{15}$$

for all vector fields X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX, Y) = S(X, Y)$.

Definition. A Kenmotsu-manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{16}$$

where a and b are smooth functions on M.

Proposition 1. *In an η-Einstein Kenmotsu-manifold, the following relations hold:*

$$S(\phi X, Y) = ag(\phi X, Y) = -ag(X, \phi Y) = -S(X, \phi Y), \tag{17}$$

$$S(X, \xi) = (a + b)\eta(X), \quad S(\xi, \xi) = (a + b), \tag{18}$$

$$S(\phi X, \phi Y) = a[g(X, Y) - \eta(X)\eta(Y)]. \tag{19}$$

Proof. In view of (3) - (5), the proposition follows.

Definition. ([6], [25]) A vector field ξ is called torse forming if it satisfies

$$\nabla_X \xi = fX + \gamma(X)\xi, \tag{20}$$

for all vector fields X on M, where $f \in C^\infty(M)$ and γ is a 1-form.
A torse forming vector field ξ is called recurrent if $f = 0$.

Definition. [11] A vector field V is called concurrent vector field if it satisfies

$$\nabla_X V = 0 \tag{21}$$

for any vector field X on M.

3. η-RICCI SOLITONS ON η-EINSTEIN KENMOTSU MANIFOLDS

In this section, we study η-Ricci solitons on η-Einstein Kenmotsu manifolds $(M, g, \xi, \lambda, \mu, a, b)$ and in similar to Proposition 2.2 of the paper [6], we prove the following:

Theorem 2. *If $(M, g, \xi, \lambda, \mu, a, b)$ is an η-Ricci soliton on an η-Einstein Kenmotsu manifold, then*
(i) $a + b + \lambda + \mu = 0$,
(ii) *ξ is a geodesic vector field.*

Proof. Let $(M, g, \xi, \lambda, \mu, a, b)$ be an η-Ricci soliton on an η-Einstein Kenmotsu manifold. In view of (16) we have from (2) that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2[(a + \lambda)g(X, Y) + 2(b + \mu)\eta(X)\eta(Y)] = 0. \tag{22}$$

Putting $X = Y = \xi$ in (22) and using (3) and (4) we obtain $g(\nabla_\xi \xi, \xi) = -(a + b + \lambda + \mu)$, but $g(\nabla_X \xi, \xi) = 0$ for any vector field X on M, since ξ has a constant norm. Hence we get (i). Consequently (22) becomes

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2(a + \lambda)[g(X, Y) - \eta(X)\eta(Y)] = 0. \tag{23}$$

Setting $Y = \xi$ in (23) and using (4) and (6) we get $g(\nabla_\xi \xi, X) = 0$ for any vector field X on M and hence we have $\nabla_\xi \xi = 0$, i.e., ξ is a geodesic vector field. Thus we get (ii).
Also in similar to Proposition 2.3 of the paper [6], we prove the following:

Theorem 3. *If ξ is a torse forming η-Ricci soliton on an η-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$ then $f = -(a + \lambda)$, η is closed, $b = -a + (n - 1)(a + \lambda)^2$ and $\mu = -\lambda - (n - 1)(a + \lambda)^2$.*

Proof. Let ξ be a torse forming η-Ricci soliton on an η-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$. Then we have from (20) that $g(\nabla_X \xi, \xi) = f\eta(X) + \gamma(X)$ and hence we get $\gamma = -f\eta$. Consequently (20) becomes

$$\nabla_X \xi = f[X - \eta(X)\xi]. \tag{24}$$

Using (24) in (23), we get

$$(f + a + \lambda)[g(X, Y) - \eta(X)\eta(Y)] = 0, \tag{25}$$

for all vector fields X and Y and hence it follows that $f = -(a+\lambda)$. Thus we get from (24) that

$$\nabla_X \xi = -(a+\lambda)[X - \eta(X)\xi], \tag{26}$$

which means that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence we get $d\eta = 0$, i.e., η is closed. It is known that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{27}$$

In view of (26), (27) yields

$$R(X,Y)\xi = (a+\lambda)^2[\eta(X)Y - \eta(Y)X]. \tag{28}$$

From (28), we get

$$S(X,\xi) = (n-1)(a+\lambda)^2\eta(X). \tag{29}$$

From (18) and (29), we get $b = -a + (n-1)(a+\lambda)^2$ and $\mu = -\lambda - (n-1)(a+\lambda)^2$. Thus we get the theorem.

Corollary 4. *If ξ is a torse forming Ricci soliton on an η-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$ then the Ricci soliton is shrinking, steady and expanding according as $a + b > 0$, $a + b = 0$ and $a + b < 0$ respectively.*

Proof. In particular, if $\mu = 0$ then from Theorem 3.1 and Theorem 3.2, we get $\lambda + (n-1)(a+\lambda)^2 = 0$ and hence we obtain $\lambda = -(a+b)$. Hence the proof is complete.

Corollary 5. *If ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein Kenmotsu manifold $(M, g, \xi, \lambda, \mu, a, b)$ then ξ is (i) concurrent and (ii) Killing vector field.*

Proof. Since ξ is recurrent, therefore $f = 0$ and hence $a + \lambda = 0$. So, by virtue of (26) we get $\nabla_X \xi = 0$, for all X on M, which means that ξ is concurrent vector field. Also in that case

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$$

for all $X, Y \in \chi(M)$ that means ξ is Killing vector field.

As a generalization of ϕ-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [30], recently Hui [18] introduced the notion of ϕ-pseudo Ricci symmetric Kenmotsu manifolds.

Definition. A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n = 2m+1 > 3)$ is said to be ϕ-pseudo Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2((\nabla_X Q)(Y)) = 2A(X)QY + A(Y)QX + S(Y,X)\rho \tag{30}$$

for any vector fields X,and Y on M, where A is a non-zero 1-form.

If, in particular, $A = 0$, then (30) turns into the notion of ϕ-Ricci symmetric Kenmotsu manifold introduced by Shukla and Shukla [30].

In [18], it is proved that every ϕ-pseudo Ricci symmetric Kenmotsu manifold is an η-Einstein manifold and its Ricci tensor S is of the form (16), where $a = \frac{(n-1)}{A(\xi)-1}$ and $b = \frac{(n-1)A(\xi)}{1-A(\xi)}$, provided $1 - A(\xi) \neq 0$. So $a + b = -(n-1)$ and hence by Theorem 3.1 we get $\lambda + \mu = (n-1)$. Thus by virtue of Theorem 3.2, we can state the following:

Theorem 6. *If ξ is a torse forming η-Ricci soliton on a ϕ-pseudo Ricci symmetric Kenmotsu manifold M then $f = -(a+\lambda)$, η is closed and $1 + (a+\lambda)^2 = 0$.*

Also in view of corollary 3.1, we get $\lambda = -(a+b) = (n-1) > 0$. This leads to the following:

Corollary 7. *If ξ is a torse forming Ricci soliton on a ϕ-pseudo Ricci symmetric Kenmotsu manifold M then the Ricci soliton (g, ξ, λ) is always expanding.*

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