Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.5, (2016), Issue 2, pp.98-103

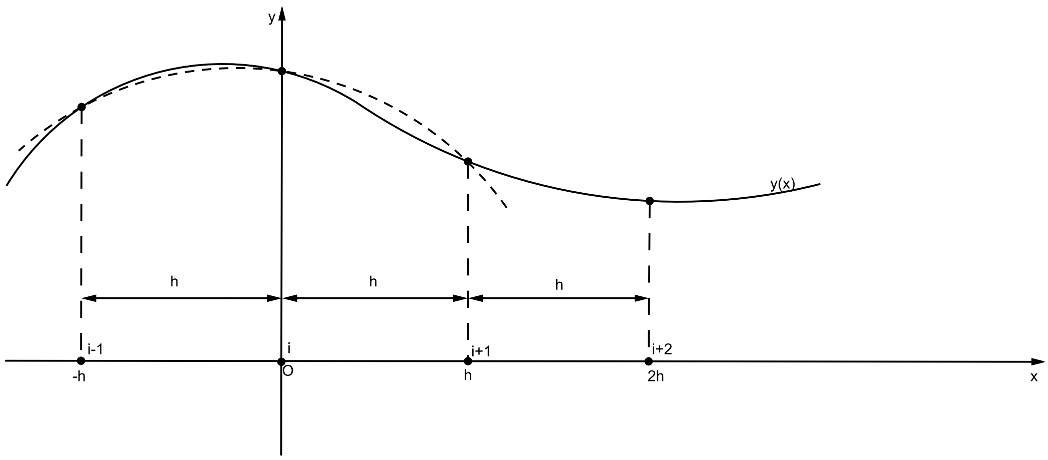
INTERPOLATION PARABOLAS AND QUADRATIC BÉZIER CURVES

LAURIAN-IOAN PIŞCORAN AND IOAN ŞCHIOPU

ABSTRACT. The most easy way to find approximative expressions for the derivatives of a function $y(x)$, which is known graphically or from charts in some points i, is to replace the function with a parabola passing to a certain number of points and to take the parabola derivatives as approximative values of the derivative of $y(x)$. In this paper we will investigate in which conditions the derivatives of a function $y(x)$ can be approximated using quadratic Bézier curves instead of using interpolation parabolas.

1. INTRODUCTION

In the following lines, we will recall some classical results regarding the interpolation parabolas presented in [1]. In the bellow picture is presented the interpolation parabola which contains some equidistant points.



For example, if we want to evaluate the second derivative of the function $y(x)$, when we know the values of the function in three consecutive points $i-1, i, i+1$ situated at equal distances on the Ox axis, we will denote by $y_{i-1}; y_i; y_{i+1}$ their values and then we will write the equation of the parabola which contains that knots, in the following way:

$$y_p = Ax^2 + Bx + C \tag{1.1}$$

2010 *Mathematics Subject Classification.* 65D17, 65D18.
Key words and phrases. Interpolation parabolas, Bézier curves.

Without the lose of generality, the authors of paper [1], have taken the abscissa of the point i in origin and they obtained:

$$\begin{aligned} y(-h) = y_{i-1} = Ah^2 - Bh + C \\ y(0) = y_i = C \\ y(h) = y_{i+1} = Ah^2 + Bh + C \end{aligned} \tag{1.2}$$

and then, one obtains:

$$y_{i-1} - 2y_i + y_{i+1} = 2Ah^2$$

Because the second derivative of the parabola (1.1) is 2A, the second derivative y_i'' of y in i, is approximated with

$$y_i'' = \frac{1}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \tag{1.3}$$

If the points are not displaced at equal distances, then the second derivative of the function for which we know the values in three distinct points displaced at the distance h and αh, can be obtained with the use of a parabola which contains those knots:

$$\begin{aligned} y(-h) = y_{i-1} = Ah^2 - Bh + C \\ y(0) = y_i = C \\ y(\alpha h) = y_{i+1} = \alpha^2 Ah^2 + \alpha Bh + C \end{aligned} \tag{1.4}$$

Based on the fact that $y_i'' = 2A$, and using the relation (1.4), one obtains:

$$y_i'' = \frac{1}{h^2}\frac{2}{\alpha(1+\alpha)}\left[\alpha y_{i-1} - (1+\alpha)y_i + y_{i+1}\right] \tag{1.5}$$

The Bézier quadratic curves attached to a set of three points p_0, p_1, p_2 (also called control points), can be written in the following way:

$$B(t) = \sum_{k=0}^{n} b_{k,n}(t) p_k$$

where $n = 2$ and $b_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ represent the Bernstein polynomials. The equation of a rational Bézier curve is:

$$B_1(t) = \frac{\sum\limits_{k=0}^{2} b_{k,2}(t) p_k w_k}{\sum\limits_{k=0}^{2} b_{k,2}(t) w_k} = \frac{w_0(1-t)^2 p_0 + 2w_1 p_1 t(1-t) + t^2 w_2 p_2}{(1-t)^2 w_0 + 2w_1 t(1-t) + t^2 w_2} \tag{1.6}$$

In the last years, there has been a growing interest in the study of the Bézier curves, especially Bézier rational curves. Some of the paper in this topic are [3]-[13]. The importance of these types of curves is revealed in GCAD (graphical computer asisting design), and in other important science fields. Our aim is to investigate in which conditions the derivatives of a function $y(x)$ can be approximated using quadratic Bézier curves instead of using interpolation parabolas.

2. Main Results

In the next lines, we will compute the equation of the Bézier curve attached in the same points as in (2), namely:

$$(-h, Ah^2 - Bh + C), (0, C), (h, Ah^2 + Bh + C)$$

We know from Bézier curves properties that the Bézier curve must contain the first, respectively the last control point of this set. The Bézier quadratic curve with the above control points can be computed:

$$B(h) = \sum_{k=0}^{2} b_{k,2}(t)p_k = (1-t)^2(-h, Ah^2 - Bh + C) + 2t(1-t)(0, C) + t^2(h, Ah^2 + Bh + C)$$

This equation led us to the parametric form of the Bézier quadratic curve:

$$\begin{cases} x(t) = h(2t-1) \\ y(t) = Ah^2 - Bh + C - 2t(Ah^2 - Bh) + 2t^2h^2A \end{cases} \tag{2.1}$$

and from these parametric equations, we get:

$$y''(t) = 4h^2A \tag{2.2}$$

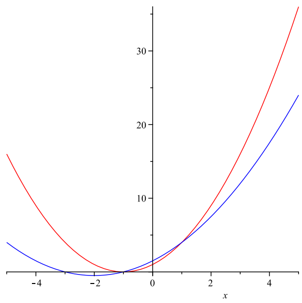
On the other hand, from (1.3), we know that

$$y_i'' = 2A. \tag{2.3}$$

The equality between the above equations holds if and only if $h = \pm\frac{\sqrt{2}}{2}$.

Example 2.1. If we take $h = 1; A = 1; B = 2; C = 1$, then we get the following control points: $p_0(-1, 0), p_1(0, 1), p_2(1, 4)$. The parabola which contains this knots is: $y(x) = (x+1)^2$. The Bézier curve which have the control points p_0, p_1, p_2, is given by: $y(x) = \frac{x^2+4x+3}{2}$.

The graphs are plotted bellow: with red for the parabola and with blue for the quadratic Bézier curve.



Remark 2.1. If we want to apply the Newton interpolation method for the above parabola, respectively Bézier quadratic curve, one obtain:

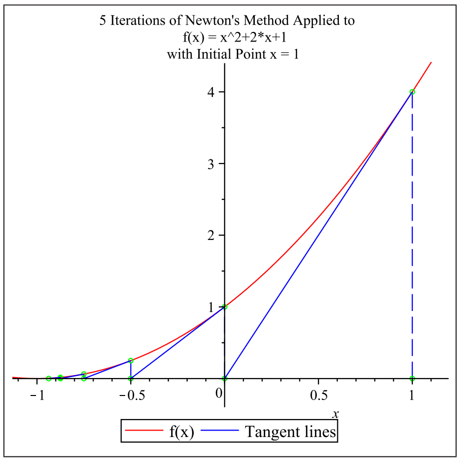


FIGURE 1. Interpolation for parabola

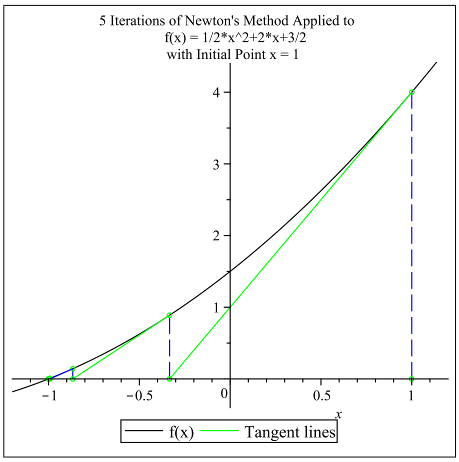


FIGURE 2. Interpolation for Bézier curve

Next for non-equidistant points,

$$(-h, Ah^2 - Bh + C), (0, C), (\alpha h, \alpha^2 Ah^2 + \alpha Bh + C) \tag{2.4}$$

we can compute the quadratic Bézier curve attached. After computations, we get the parametric equations for the quadratic Bézier curve:

$$\begin{cases} x(t) = -h(1-t)^2 + t^2\alpha h \\ y(t) = Ah^2 - Bh + C - 2t(Ah^2 - Bh) + t^2((1+\alpha)^2 Ah^2 + (\alpha - 1)Bh) \end{cases} \tag{2.5}$$

From (2.4), we can obtain now:

$$y(t) = y_{i-1} + 2t(y_i - y_{i-1}) + t^2(y_{i+1} - 2y_i + y_{i-1})$$

and from this, we get:

$$y''(t) = 2(y_{i+1} - 2y_i + y_{i-1})$$

Theorem 2.1. *The curvature of the Bézier curve (2.1) for $t = 0$ is given by*

$$K(0) = \frac{A}{\left(\sqrt{(1 + B - Ah)^2}\right)^3}$$

Proof. When we apply the classical formula

$$K(t) = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{\left(\sqrt{(x'(t))^2 + (y'(t))^2}\right)^3} \tag{2.6}$$

one obtains for the Bézier curve (2.1):

$$K(t) = \frac{A}{\left(\sqrt{(1 + Ah(2t-1) + B)^2}\right)^3} \tag{2.7}$$

After we replace $t = 0$ in the above equation, we get the desired result. □

Theorem 2.2. *The curvature of the Bézier curve (2.5) for $t = 0$, is:*

$$K(0) = \frac{\alpha A(2\alpha + 3)}{\left(2\sqrt{(1 + (Ah - B))^2}\right)^3}$$

Proof. Applying (2.6), one obtains:

$$K(t) = \frac{\left|(2h(1-t)+2t\alpha h)\left[2(1+\alpha)^2 Ah^2+2(\alpha-1)Bh\right]-(2\alpha h-2h)\left[-2ah^2+2Bh+2t\left((1+\alpha)^2 Ah^2+(\alpha-1)Bh\right)\right]\right|}{\left[\sqrt{(2h(1-t)+2t\alpha h)^2+(-2Ah^2+2Bh+2t[(1+\alpha)^2 Ah^2+(\alpha-1)Bh])^2}\right]^{3/2}}$$

Replacing $t = 0$ in the above equation, we get the desired result. □

The rational quadratic Bézier curve is:

$$B_1(t) = \frac{p_0 w_0 (1-t)^2 + 2t(1-t)p_1 w_1 + p_2 t^2}{(1-t)^2 w_0 + 2t(1-t)w_1 + t^2 w_2} \tag{2.8}$$

If we take in equation (1.6) the standard form, i.e. $w_0 = w_2 = 1$ and $w_1 = w$, and the control points as in (1.2), one obtains:

$$B_1(t) = \frac{(-h, Ah^2 - Bh + C)(1-t)^2 + 2t(1-t)(0, C)w + (h, Ah^2 + Bh + C)t^2}{(1-t)^2 + 2t(1-t)w + t^2} \tag{2.9}$$

After computations, we get the parametric equations for the rational quadratic Bézier curve:

$$\begin{cases} x(t) = \frac{h(2t-1)}{2t(1-t)(w-1)+1} \\ y(t) = \frac{2(Ah^2+C)t^2+(1-2t)(Ah^2-Bh+C)+2t(1-t)Cw}{2t(1-t)(w-1)+1} \end{cases} \tag{2.10}$$

As we know from literature, see [13], the rational Bézier curves in standard form, can represent a parabola when $w = 1$. In our case for the rational quadratic Bézier curve, when $w = 1$, equations (2.10), became:

$$\begin{cases} x(t) = h(2t-1) \\ y(t) = 2(Ah^2 + C)t^2 + (1-2t)(Ah^2 - Bh + C) + 2t(1-t)C \end{cases} \tag{2.11}$$

and this equation can be reduced to (2.1) and can represent a parabola for some values of h, as we see before in the begining of this section.

References

[1] Salvadori M.G., Baron M.L., *Numerical methods in technics,* Ed. Tehnică, (in romanian) (1972).

[2] Southwell R.V., *Relaxation model in theoretical physics,* Oxford Univ. Press, (1946).

[3] Faux, I. D., Pratt, M. J., *Computational Geometry for Design and Manufacture*, Chichester, UK: Ellis Horwood Limited, (1979).

[4] Forrest, A. R., *The twisted cubic curves: A computer aided geometric design approach*, Computer-Aided Design, 12(4): 165-172, (1980).

[5] Farin, G., *Algorithms for rational Bézier curves*, Computer-Aided Design, 15(2): 73-77, (1983).

[6] Pigel, L., *Representation of quadric primitives by rational polynomials*, Computer Aided Geometric Design, 2(13): 151-155, (1985).

[7] Lee, E. T. Y., The rational Bézier representation for conic, in Geometric Modeling: Algorithms and New Trends (ed. Farin, G. E.), SIAM, 319, (1987).

[8] Pigel, L., Tiller, W., *A menagerie of rational B-spline circles*, IEEE Computer Graphics and Application, (9): 48-56, (1989).
[9] Wang, G. J., *Rational cubic circular arcs and their application in CAD*, Computers in Industry, 16(3): 283-288, (1991).
[10] Wang, G. J., Wang, G. Z., *The rational cubic Bézier representation of conics*, Computer Aided Geometric Design, 9(6): 447-455, (1992).
[11] Farin, G., *Curves and Surfaces for Computer Aided Geometric Design, A Practical Guide*, 3rd ed., Boston: Academic Press, (1993).
[12] Hu Q., Wang G., *Geometric meanings of the parameters on rational conic segments*, Science in China Ser. A Mathematics, Vol.48 No.9 1209-1222, (2005).
[13] Marsh D., *Applied Geometry for Computer Graphics and CAD*, Springer-Verlag London Limited, (2005).

NORTH UNIVERSITY CENTER OF BAIA MARE
TECHNICAL UNIVERSITY OF CLUJ NAPOCA
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
VICTORIEI 76, 430122 BAIA MARE, ROMANIA
E-mail address: plaurian@yahoo.com

ECONOMICAL COLLEGE OF NĂSĂUD
BULEVARDUL GRANICERILOR NR. 2A, 425200,
NĂSĂUD, ROMANIA
E-mail address: SchiopuIonutz@ymail.com