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GOLDEN SECTIONS IN AN ISOSCELES TRIANGLE AND ITS CIRCUMCIRCLE

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ABSTRACT. Given an isosceles triangle and its circumcircle, we show that there are four lines parallel to the base, each intersecting the slant sides and the circumcircle symmetrically in four points exhibiting divisions in the golden ratio. We give a very simple construction of the four lines.

Given an isosceles triangle ABC with $AC = AB$, and its circumcircle (O), we solve the construction problem of a chord PQ of (O), extended if necessary, intersecting AC at Y and AB at Z, such that *some* segments with endpoints among P, Q, Y, Z are divided in the golden ratio by another point among them. We shall show that there are 4 such chords, and find all golden sections on each of them (Theorem 3 below). Recall that PQ is divided in the golden ratio by Y if

$$\frac{PY}{YQ} = \frac{PQ}{PY} = \varphi = \frac{\sqrt{5}+1}{2},$$

and the golden ratio φ satisfies $\varphi^2 = \varphi + 1$. We shall abbreviate the golden section of PQ by Y to $[PYQ]$.

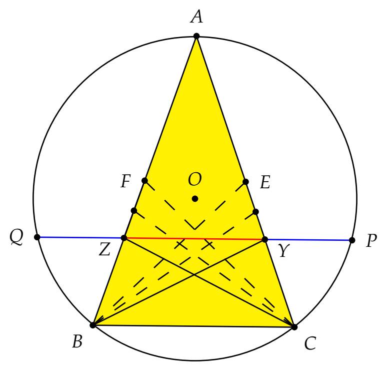

FIGURE 1

One of the four chords with golden section is very easy to describe: if BY and CZ are symmedians of ABC, and the line YZ intersects (O) at P and Q (so that P, Y, Z, Q are in linear order),

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then $[ZYP]$ and $[YZQ]$ (see Figure 1 in which the symmedians BY and CZ are constructed as the reflections of the medians BE and CF in the respective angle bisectors). We shall give in Proposition 2 below an easy construction of the remaining three chords.

The problem we solve in this note generalizes results of Odom [2], Tran [4], and Dao [1]; see also [3]. We use the method of homogeneous barycentric coordinates with reference to triangle ABC. For basic notations and results, see [5]. Suppose $BC = a$ and $AC = AB = b$. The circumcircle has barycentric equation

$$a^2yz + b^2x(y+z) = 0. \tag{1}$$

If $P = (u, v, w)$ is a point on (O), the line through P parallel to BC intersects (O) again at $Q = (u, w, v)$, AC at $Y = (u, 0, v + w)$, and AB at $Z = (u, v + w, 0)$. Note that P, Q, Y, Z have the same coordinate sum $u + v + w$.

Lemma 1. *For three points X_1, X_2, X_3 with equal coordinate sums, $[X_1X_2X_3]$ if and only if*

$$X_1 - (\varphi + 1)X_2 + \varphi X_3 = (0, 0, 0).$$

Proof. Let σ denote the common coordinate sum of X_1, X_2, X_3. $[X_1X_2X_3]$ if and only if $\frac{X_1X_2}{X_2X_3} = \varphi$. In absolute barycentric coordinates,

$$\frac{X_2}{\sigma} = \frac{\frac{X_1}{\sigma} + \varphi\frac{X_3}{\sigma}}{\varphi + 1} = \frac{X_1 + \varphi X_3}{(\varphi + 1)\sigma}.$$

The result follows by cancelling σ. □

Consider, for example, the golden section $[PYZ]$. By Lemma 1,

$$(u, v, w) - (\varphi + 1)(u, 0, v + w) + \varphi(u, v + w, 0) = 0.$$

From the second components, we have $v + \varphi(v + w) = 0$, and $w = -\frac{(\varphi+1)v}{\varphi} = -\varphi v$. Solving

$$w = -\varphi v, \quad a^2vw + b^2u(v + w) = 0,$$

simultaneously, we have $u = -\frac{a^2v}{\varphi b^2}$. Therefore, in homogeneous barycentric coordinates,

$$\begin{aligned} P = &\ (a^2, -(2-\varphi)b^2, (\varphi - 1)b^2), \\ Q = &\ (a^2, (\varphi - 1)b^2, -(2-\varphi)b^2); \\ Y = &\ (a^2, 0, (2\varphi - 3)b^2), \\ Z = &\ (a^2, (2\varphi - 3)b^2, 0). \end{aligned}$$

More generally, consider a permutation of 3 symbols among P, Q, Y, Z, signifying a golden section of a segment.

(i) An interchange of $P \leftrightarrow Q$ with corresponding change $Y \to Z$ or $Z \to Y$ results in another golden section of another segment on the line. This is clear from symmetry. The same is true of an interchange of $Y \leftrightarrow Z$ with corresponding change $P \to Q$ or $Q \to P$.

(ii) An interchange of $P \leftrightarrow Q$ with the remaining Y or Z fixed is simply relabelling of the points P and Q. It corresponds to interchanging $v \leftrightarrow w$ in the coordinates. The same is true of a change $P \to Q$ or $Q \to P$ with X and Y fixed.

Making use of these, we divide the 24 permutations of 3 symbols among P, Q, Y, Z into 6 classes, each representing 4 segments in golden section on a line (see Table 1). The last column shows the relation between v and w, obtained by applying Lemma 1.

(1)	$[PYZ]$	$[QZY]$	$[PZY]$	$[QYZ]$	$(\varphi+1)v+\varphi w=0$
(1′)	$[QYP]$	$[PZQ]$	$[QZP]$	$[PYQ]$	$\varphi v+w=0$
(2)	$[ZYP]$	$[YZQ]$	$[YZP]$	$[ZYQ]$	$(\varphi+1)v+w=0$
(3)	$[PQY]$	$[QPZ]$	$[QPY]$	$[PQZ]$	$v-(\varphi+1)w=0$
(4)	$[YQP]$	$[ZPQ]$	$[YPQ]$	$[ZQP]$	$\varphi v-(\varphi+1)w=0$
(4′)	$[YPZ]$	$[ZQY]$	$[ZPY]$	$[YQZ]$	$v-\varphi w=0$

Table 1. Equivalent golden sections on a line.

Note that the equations in (1) and (1′) are the same since $\varphi^2=\varphi+1$. Therefore they define the same line. Similarly, (4) and (4′) also define the same line. There are altogether four lines. We determine the four points on each line by using the first representative given Table 1, and tabulate the results in Table 2.

(1)	$[PYZ]$	$w=-\varphi v$
		$P_1=(a^2,-(2-\varphi)b^2,(\varphi-1)b^2)$ $Q_1=(a^2,(\varphi-1)b^2,-(2-\varphi)b^2)$ $Y_1=(a^2,0,(2\varphi-3)b^2)$ $Z_1=(a^2,(2\varphi-3)b^2,0)$
(2)	$[ZYP]$	$w=-(\varphi+1)v$
		$P_2=(a^2,-(\varphi-1)b^2,\varphi b^2)$ $Q_2=(a^2,\varphi b^2,-(\varphi-1)b^2)$ $Y_2=(a^2,0,b^2)$ $Z_2=(a^2,b^2,0)$
(3)	$[PQY]$	$v=(\varphi+1)w$
		$P_3=(-a^2,(\varphi+2)b^2,(3-\varphi)b^2)$ $Q_3=(-a^2,(3-\varphi)b^2,(\varphi+2)b^2)$ $Y_3=(-a^2,0,5b^2)$ $Z_3=(-a^2,5b^2,0)$
(4)	$[YQP]$	$v=\varphi w$
		$P_4=(-a^2,(\varphi+1)b^2,\varphi b^2)$ $Q_4=(-a^2,\varphi b^2,(\varphi+1)b^2)$ $Y_4=(-a^2,0,(2\varphi+1)b^2)$ $Z_4=(-a^2,(2\varphi+1)b^2,0)$

Table 2. Four lines each with four points exhibiting golden section of segments.

Note that BY_2 and CZ_2 are symmedians of the isosceles triangle ABC.

To construct the four lines, it is enough to construct the points P_i, $i=1,2,3,4$ (see Figure 2). This is very easy because of the following collinearity relations.

Proposition 2. *Let D be the midpoint of the base BC of the isosceles triangle ABC with circumcircle (O).*

(a) *P_2 is the intersection of (O) with the half-line Z_2Y_2.*
(b) *P_1 and P_4 are the intersections of (O) with the line DY_2.*
(c) *P_3 is the* (second) *intersection of (O) with the line DP_2.*

Proof. While (a) is clear, the following expressions give the divisions of the segments DY_2 by P_1, P_4, and P_2P_3 by D.

$$\begin{aligned}(a^2, -(2-\varphi)b^2, (\varphi-1)b^2) &= -(2-\varphi)b^2 \cdot (0,1,1) + 1 \cdot (a^2, 0, b^2),\\ (-a^2, (\varphi+1)b^2, \varphi b^2) &= (\varphi+1)b^2 \cdot (0,1,1) - 1 \cdot (a^2, 0, b^2),\\ 3b^2(0,1,1) &= (a^2, -(\varphi-1)b^2, \varphi b^2) + (-a^2, (\varphi+2)b^2, (3-\varphi)b^2).\end{aligned}$$

These prove (b) and (c). □

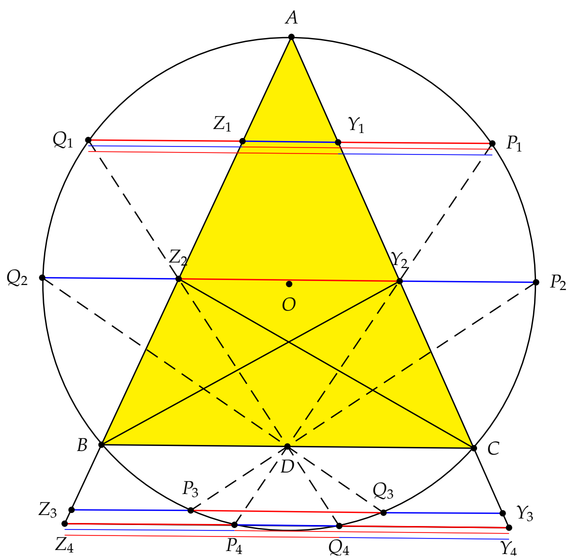

FIGURE 2

We summarize the results of this note in the following theorem.

Theorem 3. *Given an isosceles triangle ABC with $AB = AC$ and its circumcircle (O), let PQ be a chord of (O) parallel to BC, intersecting, extended if necessary, AC at Y and AB at Z. If any three of the points P, Q, Y, Z exhibit a golden section of a segment, then the chord is one of P_iQ_i, $i = 1, 2, 3, 4$, determined by P_i in* Proposition 2. *On these four lines, the segments in golden sections are indicated in* Table 3 *below.*

(1)	$[P_1Y_1Z_1]$	$[Q_1Z_1Y_1]$	$[Q_1Y_1P_1]$	$[P_1Z_1Q_1]$
(2)	$[Z_2Y_2P_2]$	$[Y_2Z_2Q_2]$		
(3)	$[P_3Q_3Y_3]$	$[Q_3P_3Z_3]$		
(4)	$[Y_4Q_4P_4]$	$[Z_4P_4Q_4]$	$[Y_4P_4Z_4]$	$[Z_4Q_4Y_4]$

Table 3. Golden section of segments on four lines
in an isosceles triangle with its circumcircle

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