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ANOTHER SYNTHETIC PROOF OF DAO'S GENERALIZATION OF THE SIMSON LINE THEOREM AND ITS CONVERSE

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ABSTRACT. In this article we give a synthetic proof of Dao's generalization of the Simson line theorem as well as its converse.

1. INTRODUCTION

In [1], Dao Thanh Oai published without proof a remarkable generalization of the Simson line theorem.

Theorem 1.1 (Dao). *Let ABC be a triangle, P a point on its circumcircle (O) and H its orthocenter. A line l passing through the circumcenter O cuts AP, BP, CP at A_P, B_P, C_P, respectively. Denote A_0, B_0, C_0 the projection of A_P, B_P, C_P on BC, CA, AB respectively. Then the three points A_0, B_0, C_0 are collinear on a line that passes through the midpoint of PH .*

We call this line the Dao's line respect to point P and line l. When the line l passes through P, then the line $\overline{A_0B_0C_0}$ is the Simson line of ΔABC and P.
Three proofs, by Telv Cohl and Luis Gonzalez, Tran Quang Huy can be found in [2]. Nguyen Le Phuoc and Nguyen Chuong Chi have given a synthetic proof in [4]. Nguyen van Linh give another synthetic proof in [3]. Leonard Mihai Giugiuc give a complex proof in [5]. Ngo Quang Duong give another proof with many properties of the Dao line in [6].
In this article we give another synthetic proof of the Dao line theorem and its converse.

2. A PROOF OF THEOREM 1

Lemma 2.1. *Let ABC be a triangle and P a point on its circumcircle (O). Suppose there is a line l that meet PA, PC at A_P, C_P, respectively that the circle with diameter A_PC and the circle with diameter C_PA meet at a point X with X lying on (O). Than the line l passes through O.*

Proof of lemma 2: Let XA_P meet (O) again at C', then CC' is a diameter of (O). Let XC_P meet (O) again at T, then AT is also a diameter of (O). By Pascal in $TAPCC'X$ we get Ω, A_P, C_P - where $\Omega \equiv TA \cap CC'$ - collinear but $\Omega \equiv O$, hence we have the required.

Lemma 2.2. *Let ABC a triangle and P a point on its circumcircle (O). Suppose there is a line l passing through the circumcenter O that meet PA, PC at A_P, C_P, respectively. Then the circle with diameter AC_P and the circle with diameter CA_P intersect at two points X, Y, one of them (say X) lies on (O), the second, (say Y), lies on the nine-point circle of ΔPAC.*

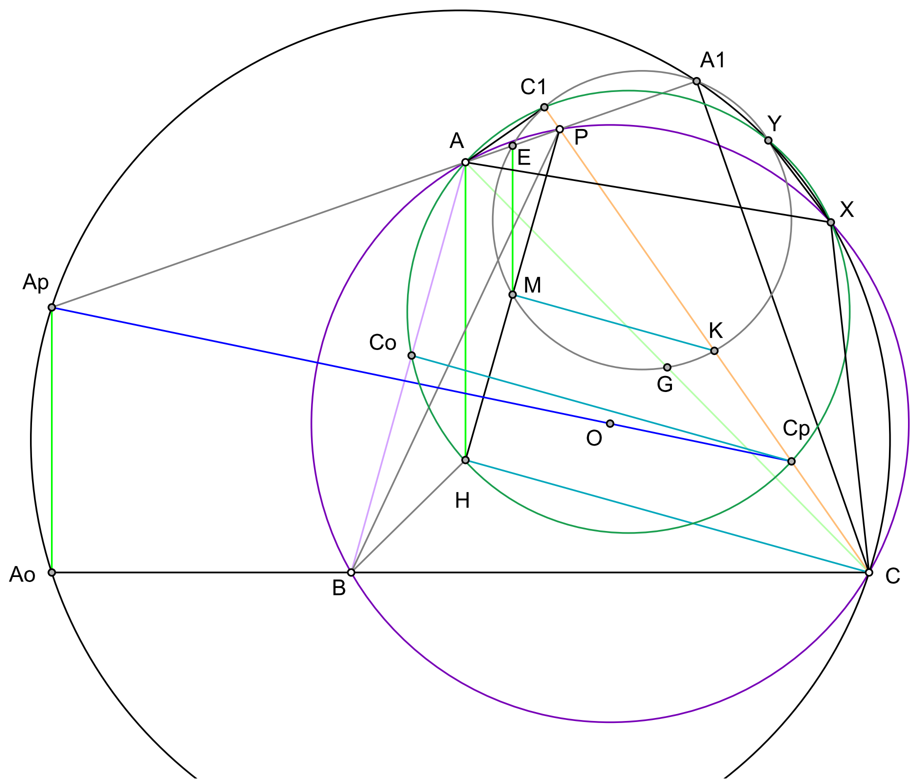

FIGURE 1

Proof of lemma 3, (See Figure 1). Let CO cut (O) again at C'; $C'A_P$ cut (O) at X; XC_P cut (O) again at L, then by Pascal in hexagon $APCC'XL$ we get A, O, L collinear. Hence X lies on the circle with diameter CA_P as well as the circle with diameter AC_P. Let the circle with diameter AC_P cut PC at A_1, the circle with diameter CA_P cut PA at C_1, then AA_1, CC_1 are two altitudes of ΔPAC. If they intersect at H_1 then H_1 obviously lies on the radical axis of the two spoken circles, thus H_1 lies on XY. Notice that $\angle A_1YC_1 = \angle A_1YX - \angle XYC_1 = 180^\circ - \angle A_1CX - (180^\circ - \angle XAH_1) = \angle XAH_1 - \angle A_1CX = 180^\circ - \angle AH_1C - \angle XAC - \angle ACH_1 - \angle A_1CX = 180^\circ - \angle AH_1C\hat{a}(180^\circ - \angle AXC) = 180^\circ - \angle AH_1C\hat{a}(180^\circ - \angle APC) = 180^\circ\hat{a}2.\angle AH_1C$. This means that if I is the midpoint of PH_1 then C_1IYA_1 cyclic, or Y lies on the nine-point circle of ΔPAC.

Remark 2.1. For convenience we supposed ΔABC is an acute-angled triangle. When ΔABC is obtuse-angled, the relationships between angles in the proof may be altered, but all the major conclusions hold.
The same applies for the main proofs of the Theorem and Converse theorem given below.

Remark 2.2. By Lemma 1, the converse of Lemma 2 holds.

Main proof of Theorem 1, (See Figure 2). Using Lemma 2, the circle with diameter AC_P and the circle with diameter CA_P meet at two point X, Y with X lying on the circumcirle of ΔABC and Y lying on the nine-point circle of ΔPAC. Let the nine-point circle of ΔPAC meets PA, PC again at A_1, C_1, respectively, then AA_1, CC_1 are of course two altitudes of ΔPAC. Let E, M, K be the midpoint of AP, PH, PC, respectively. As $\angle C_1YM =$

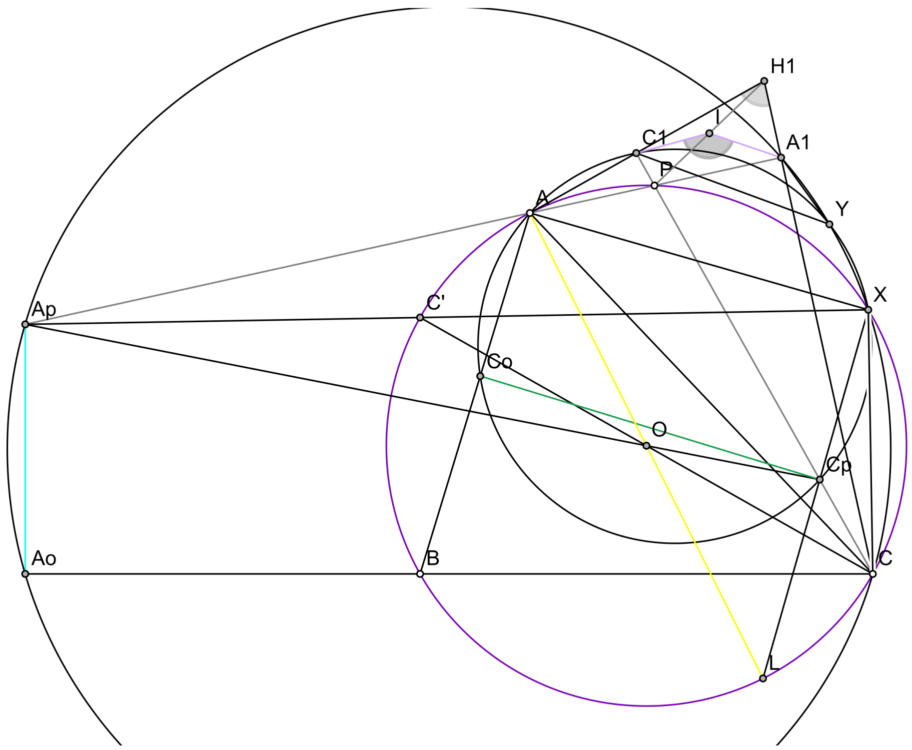

FIGURE 2

$\angle C_1KM = \angle C_1C_PC_0$, C_0, M, Y are collinear. We have: $\angle YXA = 180^\circ - \angle AC_1Y = 180^\circ - \angle AC_1C - \angle CC_1Y = 90^\circ - \angle CC_1Y = 90^\circ - \angle YMK = 90^\circ$â$(\angle EMK - \angle EMY) = 90^\circ + \angle EMY$â$(180^\circ - \angle ABC) = \angle ABC + \angle EMY$â$90^\circ$ (*) But also: $\angle YXA = \angle YXC - \angle AXC = (180^\circ$â$\angle CA_1Y) - (180^\circ$â$\angle ABC) = \angle ABC - \angle CA_1Y = \angle ABC - \angle YA_0C = \angle ABC$â$(90^\circ - \angle A_PA_0Y = \angle ABC + \angle A_PA_0Y$â$90^\circ$ (**) (*) and (**) give $\angle EMY = \angle A_PA_0Y$ or A_0, M, Y collinear. Hence A_0, C_0, M, Y are collinear. Similarly, we get A_0, B_0, M, Y are collinear. Then A_0, B_0, C_0, M, Y are all collinear.

3. CONVERSE OF THEOREM 1 AND ITS PROOF

Theorem 3.1 ([7]). *Let ABC be a triangle, P a point on its circumcircle (O) and H its orthocenter. A line l passing through the midpoint of PH meets BC, CA, AB at A_0, B_0, C_0, respectively. The line passing through A_0 perpendicular to BC meets PA at A_P, the line passing through B_0 perpendicular to AC meets PB at B_P and the line passing through C_0 perpendicular to AB meets PC at C_P. Then A_P, B_P and C_P and the circumcenter O of ΔABC are collinear.*

Proof of Theorem 6, (See Figure 2). Let M, E, K be the midpoint of AH, AP, PC, respectively. Let the circle with diameter A_PC meet (O) - the circumcircle of ΔABC - again at X, meet A_PP again at A_1 and meet the line l at Y. Since $ME \parallel A_PA_0$, easy to see that quadrilateral MEA_1Y is cyclic. As $\angle PA_1K = \angle A_1PK = \angle ABC = 180^\circ - \angle AHC = 180^\circ - \angle EMK$, then quadrilateral EA_1KM is cyclic. If G is the midpoint of AC then easy to see that $\angle EGK = 180^\circ - \angle KPC_1 = 180^\circ - \angle PA_1K$, then EA_1KG is also cyclic. Hence E, A_1, Y, K, G, M, E are all concyclic, they lie on the nine-point circle of ΔPAC. Let this circle meet PC again at C_1. It is obvious that AA_1, CC_1 are two altitudes in ΔPAC

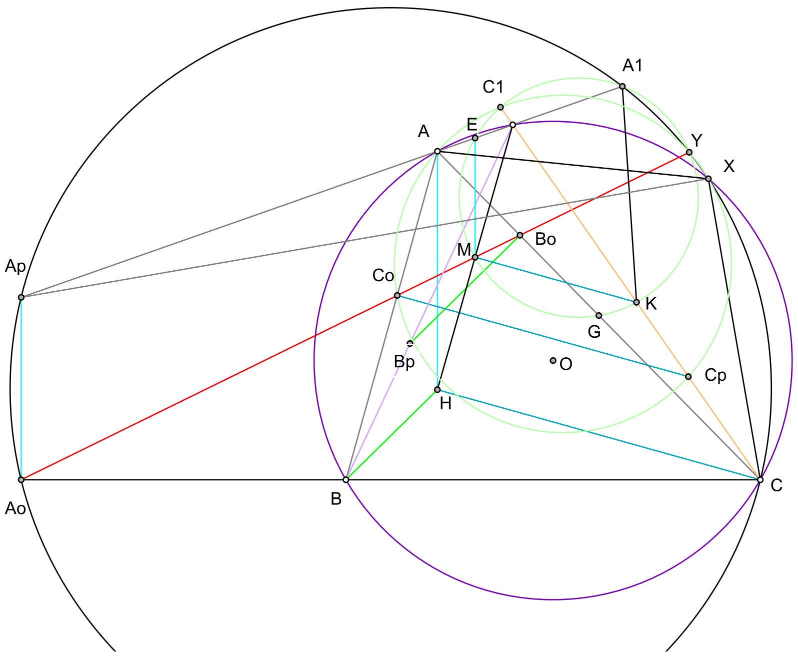

FIGURE 3

and quadrilateral $AC_1C_0C_P$ is cyclic. As $\angle C_1YC_0 = \angle C_1KM = \angle C_1C_PC_0$, quadrilateral $C_1YC_0C_P$ is cyclic. And $\angle C_1YX + \angle C_1AX = (\angle C_1YA_0 + \angle A_0YX) + \angle C_1AX = (\angle C_1CH + 180^\circ - \angle A_0CX) + \angle C_1AX = (180^\circ + \angle C_1CH - \angle A_0CX) + (90^\circ - \angle ACC_1 - \angle XAC) = 270^\circ + \angle ACH - \angle A_0CX - \angle XAC = 90^\circ + \angle ACH + \angle BAX - \angle XAC = 90^\circ + \angle ACH + \angle BAC = 180^\circ$, then quadrilateral AC_1YX is cyclic. Hence A, C_1, Y, X, C_P, C_0 are all concyclic, that is the circle with diameter A_PC and the circle with diameter C_PA meet at a point X with X lying on the circumcirle (O), than by Lemma 1, $\overline{A_PC_P}$ passes through O. Similarly, we have $\overline{A_PB_P}$ goes through O, that is P_A, P_B, P_C collinear on a line l passing through O.

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