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MINIMAL HALF-LIGHTLIKE SUBMANIFOLDS OF A SEMI-RIEMANNIAN PRODUCT MANIFOLD

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Abstract. In this paper, we study minimal half-lightlike submanifolds of a semi-Riemannian product manifold. We introduce a classes half-lightlike submanifolds of called screen semi-invariant half-lightlike submanifolds. We defined some special distribution of screen semi-invariant half-lightlike submanifold. We give some equivalent conditions for minimal half-lightlike submanifolds with respect to the Levi-Civita and quarter symmetric non-metric connection of semi-Riemannian manifolds and some results.

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1. Introduction

The theory of degenerate submanifolds of semi-Riemannian manifolds is one of a important topics of diferential geometry. The geometry of lightlike submanifolds a semi-Riemannian manifold was presented in [7] (see also [8]) by K.L. Duggal and A. Bejancu. Differential Geometry of Lightlike Submanifolds was presented in [17] by K. L. Duggal and B. Sahin. In [12],[13], [14], [15], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlie submanifolds. In [16], M. Atceken and E. Kilic introduced semi-invariant lightlike submanifolds of a semi-Riemannian product manifold. In [18], E. Kilic and B. Sahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product and gave some examples and results for lightlike submanifolds. In [19] E. Kilic and O. Bahadir studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection. In [20] O. Bahadir give some equivalent conditions for integrability of distributions with respect to Levi Civita connection of semi-Riemannian manifolds and some results.

In [1], Hayden introduced a metric connection with nonzero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and nonmetric connection have been studied by many authors ([2]-[6]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by Golab [3]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\widetilde{T}$ is of the form:

$$\widetilde{T}(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \tag{1.1}$$

for any vector fields X, Y on a manifold, where is a $1-$ form and φ is a tensor of type $(1,1)$
In this paper, we study minimal half-lightlike submanifolds of a semi-Riemannian product manifold. In Section 3, we introduce screen semi-invariant half-lightlike submanifolds of a semi-Riemannian product manifold. We defined some special distribution of screen semi-invariant half-lightlike submanifold. For these special distributions, we give some equivalent conditions on minimal half-lightlike submanifolds with respect to the Levi-Civita connection. In Section 4, we consider minimal half-lightlike submanifolds of a semi-Riemannian product manifold with quarter symmetric non-metric connection determined by the product structure. We compute some results with respect to the quarter-symmetric non-metric connection for special distributions.

2. Half-lightlike submanifolds

Let $(\widetilde{M},\widetilde{g})$ be an $(m+2)$-dimensional $(m>1)$ semi-Riemannian manifold of index $q\geq 1$ and M a submanifold of codimension 2 of $\widetilde{M}$. If $\widetilde{g}$ is degenerate on the tangent bundle TM on M, then M is called a lightlike submanifold of $\widetilde{M}$ [17]. Denote by g the induced degenerate metric tensor of $\widetilde{g}$ on M. Then there exists locally (or globally) a vector field $\xi\in\Gamma(TM)$, $\xi\neq 0$, such that $g(\xi,X)=0$ for any $X\in\Gamma(TM)$. For any tangent space T_xM, $(x\in M)$, we consider

$$T_xM^{\perp}=\{u\in T_x\widetilde{M}:\widetilde{g}(u,v)=0,\forall v\in T_xM\},\tag{2.1}$$

a degenerate 2-dimensional orthogonal (but not complementary) subspace of $T_x\widetilde{M}$. The radical subspace $Rad\ T_xM=T_xM\cap T_xM^{\perp}$ depends on the point $x\in M$. If the mapping

$$Rad\ TM:x\in M\longrightarrow Rad\ T_xM\tag{2.2}$$

defines a radical distribution on M of rank $r>0$, then the submanifold M is called $r-$lightlike submanifold. If $r=1$, then M is called half-lightlike submanifold of $\widetilde{M}$ [17]. Then there exist $\xi,u\in T_xM^{\perp}$ such that

$$\widetilde{g}(\xi,v)=0,\quad \widetilde{g}(u,u)\neq 0,\forall v\in T_xM^{\perp}.\tag{2.3}$$

Furthermore, $\xi\in Rad\ T_xM$, and

$$\widetilde{g}(\xi,X)=\widetilde{g}(\xi,v)=0,\forall X\in\Gamma(TM),\ v\in\Gamma(TM^{\perp}).\tag{2.4}$$

Thus, $Rad\ TM$ is locally (or globally) spanned by ξ. By denote the complementary vector bundle $S(TM)$ of $Rad\ TM$ in TM which is called screen bundle of M. Thus we have the following decomposition

$$TM=Rad\ TM\perp S(TM),\tag{2.5}$$

where $\perp$ denotes the orthogonal-direct sum. In this paper, we assume that M is half-lightlike. Then there exists complementary non-degenerate distribution $S(TM^{\perp})$ of $Rad\ TM$ in $TM^{\perp}$ such that

$$TM^{\perp}=Rad\ TM\perp S(TM^{\perp}).\tag{2.6}$$

Choose $u\in S(TM^{\perp}$ as a unit vector field with $\widetilde{g}(u,u)=\epsilon=\pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\widetilde{M}$. We note that ξ and u belong to $S(TM)^{\perp}$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \perp S(TM^{\perp})^{\perp},$$

where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad\ TM$ on a coordinate neighborhood $U \subset M$, there exists a uniquely determined null vector field $N \in \Gamma(ltr(TM))$ satisfying

$$\widetilde{g}(\xi, N) = 1,\ \widetilde{g}(N, N) = \widetilde{g}(N, X) = \widetilde{g}(N, u) = 0, \forall X \in \Gamma(TM), \tag{2.7}$$

where N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \perp ltr(TM)$ are called the lightlike transversal vector field, lightlike transversal vector bundle and transversal vector bundle of M with respect to $S(TM)$, respectively. Then we have the following decomposition:

$$T\widetilde{M} = TM \oplus tr(TM) = S(TM) \perp \{Rad\ TM \oplus ltr(TM)\} \perp S(TM^{\perp}). \tag{2.8}$$

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ and P the projection of TM on $S(TM)$ with respect to the decomposition (2.5). Thus, for any $X \in \Gamma(TM)$, we can write $X = PX + \eta(X)\xi$, where η is a local differential 1-form on M given by $\eta(X) = \widetilde{g}(X, N)$. Then the Gauss and Weingarten formulas are given by

$$\begin{aligned}
\widetilde{\nabla}_X Y &= \nabla_X Y + D_1(X, Y)N + D_2(X, Y)u, && (2.9)\\
\widetilde{\nabla}_X U &= -A_U X + \nabla^t_X U, && (2.10)\\
\widetilde{\nabla}_X N &= -A_N X + p_1(X)N + p_2(X)u, && (2.11)\\
\widetilde{\nabla}_X u &= -A_u X + \varepsilon_1(X)N + \varepsilon_2(X)u, && (2.12)\\
\nabla_X PY &= \nabla^*_X PY + E(X, PY)\xi, && (2.13)\\
\nabla_X \xi &= -A^*_{\xi} X - p_1(X)\xi, && (2.14)
\end{aligned}$$

for any $X, Y \in \Gamma(TM)$, $u \in s(TM^{\perp})$, $U \in \Gamma(tr(TM))$, where ∇, ∇^* and ∇^t are induced linear connections on M, $S(TM)$ and $tr(TM)$, respectively, D_1 and D_2 are called the lightlike second fundamental and screen second fundemental form of M respectively, E is called the local second fundamental form on $S(TM)$. A_U, A_N, A^*_{ξ} and A_u are linear operators on TM and τ, ρ and ϕ are $1-$forms on TM. We note that, the induced connection ∇ is torsion-free but it is not metric connection on M and satisfies

$$(\nabla_X g)(Y, Z) = D_1(X, Y)\eta(Z) + D_1(X, Z)\eta(Y), \tag{2.15}$$

for any $X, Y, Z \in \Gamma(TM)$. However the connection ∇^* on $S(TM)$ is metric. From the above statements, we have

$$D_1(X, PY) = g(A^*_{\xi} X, PY), g(A^*_{\xi} X, N) = 0, D_1(X, \xi) = 0, \widetilde{g}(A_N X, N) = 0 \tag{2.16}$$

$$E(X, PY) = g(A_N X, PY), \epsilon D_2(X, Y) = g(A_u X, Y) - \varepsilon_1(X)\eta(Y), \tag{2.17}$$

$$\epsilon\rho(X) = \widetilde{g}(A_u X, N),\ p_1(X) = -\eta(\nabla_X \xi),\ \ p_2(X) = \epsilon\eta(A_u X),\ \varepsilon_1(X) = -\epsilon D_2(X, \xi) \tag{2.18}$$

for any $X, Y \in \Gamma(TM)$. From (2.16) and (2.17), A^*_{ξ} and A_N are $\Gamma(S(TM))-$valued shape operators related to D_1 and E, respectively and $A^*_{\xi}\xi = 0$.

Example 2.1. [17] Consider a surface M in R^4_2 given by the equation

$$x_3 = \frac{1}{\sqrt{2}}(x_1 + x_2),\ x_4 = \frac{1}{2}log(1 + (x_1 - x_2)^2)$$

Then $TM = Sp\{U_1, U_2\}$ and $TM^{\perp} = Sp\{\xi, u\}$ where

$$\begin{aligned} U_1 &= \sqrt{2}(1+(x_1-x_2)^2)\partial_1 + (1+(x_1-x_2)^2)\partial_3 + \sqrt{2}(x_1-x_2)\partial_4, \\ U_2 &= \sqrt{2}(1+(x_1-x_2)^2)\partial_2 + (1+(x_1-x_2)^2)\partial_3 - \sqrt{2}(x_1-x_2)\partial_4, \\ \xi &= \partial_1 + \partial_2 + \sqrt{2}\partial_3, \\ u &= 2(x_2-x_1)\partial_2 + \sqrt{2}(x_2-x_1)\partial_3 + (1+(x_1-x_2))\partial_4. \end{aligned}$$

By direct calculations we check that $RadTM$ is a distribution on M of rank 1 spanned by ξ. Hence M is a half-lightlike submanifold of R^4_2. We obtain the null canonical affine normal bundle

$$ltrTM = Span\{N = -\frac{1}{2}\partial_1 + \frac{1}{2}\partial_2 + \frac{1}{\sqrt{2}}\partial_3\}.$$

Definition 2.1. [17] Let (M, g) be a half-lightlike submanifold of a semi-Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then, we say that M is a minimal half-lightlike submanifold if

$$\sum_{i=1}^{n-1} D_1(e_i, e_i) = 0, \ \sum_{i=1}^{n-1} D_2(e_i, e_i) = 0 \quad and \quad \varepsilon_1(\xi) = 0,$$

where $\{e_i\}_{i=1}^{n-1}$ is an orthonormal basis of $s(TM)$.

For basic information on the geometry of lightlike submanifolds, we refer to [7], [17].

Let $(\widetilde{M}$ be an $n-$ dimensional diferentiable manifold with a tensor field F of type $(1,1)$ on $\widetilde{M}$ such that $F^2 = I$. Then M is called an almost product manifold with almost product structure F. If we put $\pi = \frac{1}{2}(I+F)$, $\sigma = \frac{1}{2}(I-F)$ then we have

$$\pi + \sigma = I, \ \pi^2 = \pi, \ \sigma^2 = \sigma, \ \pi\sigma = \sigma\pi = 0, \ F = \pi - \sigma.$$

Thus π and σ define two complementary distributions and the eigenvalue of F are ∓ 1. If an almost product manifold $\widetilde{M}$ admits a semi-Riemannian metric $\widetilde{g}$ such that

$$\widetilde{g}(FX, FY) = \widetilde{g}(X, Y), \ \widetilde{g}(FX, Y) = \widetilde{g}(X, FY), \forall X, Y \in \Gamma(\widetilde{M}),$$

then $(\widetilde{M}, \widetilde{g})$ is called semi-Riemannian almost product manifold. If, for any X, Y vector fields on $\widetilde{M}$, $(\widetilde{\nabla}_X F)Y = 0$, that is

$$\widetilde{\nabla}_X FY = F\widetilde{\nabla}_X Y,$$

then M is called an semi-Riemannian product manifold, where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}$.

3. Screen Semi-Invariant half-lightlike submanifolds

Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$ For any $X \in \Gamma(TM)$ we can write

$$FX = fX + wX, \tag{3.1}$$

where f and w are the projections on of $\Gamma(T\widetilde{M})$ onto TM and $trTM$, respectively, that is, fX and wX are tangent and transversal components of FX. From (2.8) and (3.1), we can write

$$FX = fX + w_1(X)N + w_2(X)u, \tag{3.2}$$

where $w_1(X) = \widetilde{g}(FX, \xi)$, $w_2(X) = \epsilon\widetilde{g}(FX, u)$.

Definition 3.1. Let (M, g) be a half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If $FRad\ TM \subset S(TM)$, $Fltr(TM) \subset S(TM)$ and $F(S(TM^{\perp})) \subset S(TM)$ then we say that M is a screen semi-invaryant (SSI) half-lightlike submanifold.

If $FS(TM) = S(TM)$, then we say that M is a screen invaryant half-lightlike submanifold.

Now, let M be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set $L_1 = FRad\ TM$, $L_2 = Fltr(TM)$ and $L_3 = F(S(TM^{\perp}))$, then we can write

$$S(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3, \tag{3.3}$$

where L_0 is a $(m-4)-$dimensional distribution. Hence we have the following decompositions:

$$TM = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp Rad\ TM, \tag{3.4}$$

$$T\widetilde{M} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp S(TM^{\perp}) \perp \{Rad\ TM \oplus ltr(TM)\}. \tag{3.5}$$

Let (M, g) be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If we set

$$L = L_0 \perp L_1 \perp Rad\ TM \quad L^{\perp} = L_2 \perp L_3,$$

then we can write

$$TM = L \oplus L^{\perp}.$$

We note that the distribution L is a invariant distribution and the distribution $L^{\perp}$ is anti-invariant distribution with respect to F on M.

Example 3.1. [20] Let M_1 and M_2 be R_2^4 and R_2^3, respectively. Then $\widetilde{M} = M_1 \times M_2$ is a semi-Riemannian product manifold with metric tensor $\widetilde{g} = \pi^* g_1 + \sigma^* g_2$ and the product structure $F = \pi_* - \sigma_*$, where g_1 and g_2 are standard metric tensors of R_2^4 and R_2^3, π_* and σ_* are the projection maps of $\Gamma(T\widetilde{M})$ onto $\Gamma(TM_1)$ and $\Gamma(TM_2)$, respectively. We consider in $\widetilde{M}$ the submanifold M given by the following equations;

$$\begin{aligned} x_1 &= t_1 + t_2 - t_3, \\ x_2 &= t_1 + t_2 + t_3 + \sqrt{2}\arctan t_4, \\ x_3 &= \sqrt{2}(t_1 + t_2 + t_3) + \arctan t_4, \\ x_4 &= t_5, \\ x_5 &= t_1 - t_2 + t_3, \\ x_6 &= \arctan t_4, \\ x_7 &= t_1 - t_2 - t_3, \end{aligned}$$

where t_i are real parameters. Then we have

$$TM = Span\{U_1, U_2, U_3, U_4, U_5\},$$

where

$$\begin{aligned}
U_1 &= \frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\sqrt{2}\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_7},\\
U_2 &= \frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\sqrt{2}\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_5}-\frac{\partial}{\partial x_7},\\
U_3 &= -\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\sqrt{2}\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_5}-\frac{\partial}{\partial x_7},\\
U_4 &= \frac{\sqrt{2}}{(1+t_4^2)}\frac{\partial}{\partial x_2}+\frac{1}{(1+t_4^2)}\frac{\partial}{\partial x_3}+\frac{1}{(1+t_4^2)}\frac{\partial}{\partial x_6},\\
U_5 &= \frac{\partial}{\partial x_4}.
\end{aligned}$$

We easily check that the vector U_1 is a degenerate vector, M is a $1-$ lightlike submanifold of $\widetilde{M}$. We set $\xi = U_1$, then we have $Rad\ TM = Span\{\xi\}$ and $S(TM) = Span\{U_2, U_3, U_4, U_5\}$. We can easily obtain that

$$ltr(TM) = Span\{N = -\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2}+\sqrt{2}\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_5}+\frac{\partial}{\partial x_7}\},$$

and

$$S(TM^{\perp}) = Span\{u = \sqrt{2}\frac{\partial}{\partial x_2}+\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_6}\}.$$

Thus M is a half-lightlike submanifold of $\widetilde{M}$. Furthermore, we get

$$F\xi = U_2,\ \ FN = U_3,\ \ Fu = (1+t_4^2)U_4,\ \ FU_5 = U_5.$$

If we set $L_0 = Span\{U_5\}$, $L_1 = Span\{U_2\}$, $L_2 = Span\{U_3\}$, $L_3 = Span\{U_4\}$, then M is a screen semi-invariant half-lightlike submanifold of $\widetilde{M}$.

Now, Let M be minimal half-lightlike submanifold. If E_1 and E_2 is taken as follows:

$$E_1 = \frac{FN+F\xi}{2},\quad E_2 = \frac{FN-F\xi}{2}. \tag{3.6}$$

we have

$$g(E_1,E_1) = 1,\quad g(E_2,E_2) = -1,\quad g(E_1,E_2) = 0.$$

Then

$$\{e_1, e_2, ..., e_{n-4}, E_1, E_2, Fu\}$$

is an orthonormal basis of S(TM). Thus, If M is minimal, we obtain

$$D_1(E_1,E_1) = D_1(FN,FN)+D_1(F\xi,F\xi)+2D_1(FN,F\xi) = 0, \tag{3.7}$$
$$D_1(E_2,E_2) = D_1(FN,FN)+D_1(F\xi,F\xi)-2D_1(FN,F\xi) = 0. \tag{3.8}$$

from (3.7) and (3.8) we have

$$D_1(FN,F\xi) = 0,$$
$$D_1(FN,FN)+D_1(F\xi,F\xi) = 0. \tag{3.9}$$

Similarly, since $D_2(E_1,E_1) = 0 = D_2(E_2,E_2)$ we obtain

$$D_2(FN,F\xi) = 0,$$
$$D_2(FN,FN)+D_2(F\xi,F\xi) = 0. \tag{3.10}$$

from (3.9) and (3.10) the following corollary is given.

Corollary 3.1. *Let (M, g) be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If M is minimal, then the distribution $L_1 \perp L_2$ is mixed geodesic.*

Theorem 3.1. *Let (M, g) be a screen semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold $(\widetilde{M}, \widetilde{g})$. If M is minimal, then the following is provided.*
(i) *$D_1(FN, FN) + D_1(F\xi, F\xi) = 0$ and $D_2(FN, FN) + D_2(F\xi, F\xi) = 0$.*
(ii) *There is no component L_3 of $A_u Fu$, $\nabla_{Fu}\xi$ and $\nabla_\xi F\xi$.*
(iii) *There is no component L_2 of $\nabla_\xi Fu$, $\nabla_{FN}\xi$ and $A_u FN$.*

Proof. From (3.9) and (3.10) we have (i). For $N \in \Gamma(ltrTM)$ and $\xi \in RadTM$ we obtain

$$\begin{aligned} D_1(FN, F\xi) &= g(\widetilde{\nabla}_{FN} F\xi, \xi) = g(F\xi, \widetilde{\nabla}_{FN}\xi) = g(F\xi, \nabla_{FN}\xi) = 0, \\ D_2(FN, F\xi) &= g(\widetilde{\nabla}_{FN} F\xi, u) = g(F\xi, \widetilde{\nabla}_{FN} u) = -g(F\xi, A_u FN) = 0. \end{aligned}$$

Thus there is no component L_2 of $\nabla_{FN}\xi$ *and* $A_u FN$. Moreover we get

$$\varepsilon_1(\xi) = g(\widetilde{\nabla}_\xi u, \xi) = g(\widetilde{\nabla}_\xi Fu, F\xi) = g(\nabla_\xi Fu, F\xi) = 0$$

thus there is no component L_2 of $\nabla_\xi Fu$. we obtained (iii).
Also we obtain

$$\varepsilon_1(\xi) = g(\widetilde{\nabla}_\xi Fu, F\xi) = -g(Fu, \widetilde{\nabla}_\xi F\xi) = 0.$$

Thus there is no component L_3 of $\nabla_\xi F\xi$ and we have

$$D_1(Fu, Fu) = g(\widetilde{\nabla}_{Fu} Fu, \xi) = -g(Fu, \nabla_{FU}\xi) = 0,$$
$$D_2(Fu, Fu) = g(\widetilde{\nabla}_{Fu} Fu, u) = -g(Fu, \widetilde{\nabla}_{Fu} u) = g(Fu, A_u Fu) = 0.$$

thus there is no component L_3 of $\nabla_{Fu}\xi$ and $A_u Fu$. Proof is completed. □

4. Quarter-symmetric Non-metric Connections

Let (M, g, F) be a semi-Riemannian product manifold and $\widetilde{\nabla}$ be the Levi-Civita connection on M. If we set

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \pi(Y) FX \tag{4.1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{D}$ is a linear connection on $\widetilde{M}$, where u is a 1-form on $\widetilde{M}$ with U as associated vector field, that is

$$\pi(X) = \widetilde{g}(X, U).$$

The torsion tensor of $\widetilde{D}$ on $\widetilde{M}$ denoted by $\widetilde{T}$. Then we obtain

$$\widetilde{T}(X, Y) = \pi(Y) FX - \pi(X) FY, \tag{4.2}$$

and

$$(\widetilde{D}_X \widetilde{g})(Y, Z) = -\pi(Y)\widetilde{g}(FX, Z) - \pi(Z)\widetilde{g}(FX, Y), \tag{4.3}$$

for any $X, Y \in \Gamma(T\widetilde{M})$. Thus $\widetilde{D}$ is a quarter-symmetric non-metric connection on $\widetilde{M}$.
Let M be a half-lightlike submanifold of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$ with quarter-symmetric non-metric connection $\widetilde{D}$. Then the Gauss and Weingarten formulas with respect to $\widetilde{D}$ are given by, respectively,

$$\widetilde{D}_X Y = D_X Y + \widetilde{D}_1(X, Y) N + \widetilde{D}_2(X, Y) u. \tag{4.4}$$

$$\widetilde{D}_X N = -\widetilde{A}_N X + \widetilde{p}_1(X)N + \widetilde{p}_2(X)u, \tag{4.5}$$

and

$$\widetilde{D}_X u = -\widetilde{A}_u X + \widetilde{\varepsilon}_1(X)N + \widetilde{\varepsilon}_2(X)u, \tag{4.6}$$

for any $X, Y \in \Gamma(TM)$, where $D_X Y$, $\widetilde{A}_N X$, $\widetilde{A}_u X \in \Gamma(TM)$, $\widetilde{D}_1(X,Y) = \widetilde{g}(\widetilde{D}_X Y, \xi)$, $\widetilde{D}_2(X,Y) = \epsilon\widetilde{g}(\widetilde{D}_X Y, u)$, $\widetilde{p}_1(X) = \widetilde{g}(\widetilde{D}_X N, \xi)$, $\widetilde{p}_2(X) = \epsilon\widetilde{g}(\widetilde{D}_X N, u)$, $\widetilde{\varepsilon}_1(X) = \widetilde{g}(\widetilde{D}_X u, \xi)$, $\widetilde{\varepsilon}_2(X) = \epsilon\widetilde{g}(\widetilde{D}_X u, u)$. Here, $\widetilde{D}_1$ and $\widetilde{D}_2$ the lightlike second fundamental form and the screen second fundamental form of M with respect to $\widetilde{D}$ respectively. Both $\widetilde{A}_N$ and $\widetilde{A}_u$ are linear operators on $\Gamma(TM)$. From (4.1), (4.4), (4.5), (4.6) and the Gauss-Weingarten formulas we obtain

$$\begin{aligned} D_X Y &= \nabla_X Y + \pi(Y) f X, && (4.7)\\ \widetilde{D}_1(X,Y) &= D_1(X,Y) + \pi(Y) w_1(X), && (4.8)\\ \widetilde{D}_2(X,Y) &= D_2(X,Y) + \pi(Y) w_2(X), && (4.9) \end{aligned}$$

$$\begin{aligned} \widetilde{A}_N X &= A_N X - \pi(N) f X, && (4.10)\\ \widetilde{p}_1(X) &= p_1(X) + \pi(N) w_1(X), && (4.11)\\ \widetilde{p}_2(X) &= p_2(X) + \pi(N) w_2(X), && (4.12) \end{aligned}$$

and

$$\begin{aligned} \widetilde{A}_u X &= A_u X - \pi(u) f X, && (4.13)\\ \widetilde{\varepsilon}_1(X) &= \varepsilon_1(X) + \pi(u) w_1(X), && (4.14)\\ \widetilde{\varepsilon}_2(X) &= \varepsilon_2(X) + \pi(u) w_2(X). && (4.15) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$.

Definition 4.1. Let M be a half-lightlike submanifold with quarter symmetric non-metric connection of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. If

$$\sum_{i=1}^{n-1} \widetilde{D}_1(e_i, e_i) = 0, \quad \sum_{i=1}^{n-1} \widetilde{D}_2(e_i, e_i) = 0, \ \widetilde{\varepsilon_1}(\xi) = 0,$$

then we say that M is minimal with respect to quarter symmetric non-metric connection, in where $\{e_i\}_{i=1}^{n-1}$ is orthonormal basis of $s(TM)$.

Proposition 4.1. *Let M be a half-lightlike submanifold with quarter symmetric non-metric connection of a semi-Riemannian product manifold $\left(\widetilde{M}, \widetilde{g}\right)$. If M is minimal with respect to quarter symmetric non-metric connection. Then the following statements are provided:*
(i) $\widetilde{D}_1(FN, FN) + \widetilde{D}_1(F\xi, F\xi) = 0$ *and* $\widetilde{D}_1(FN, F\xi) + \widetilde{D}_1(F\xi, FN) = 0.$
(ii) $D_1(FN, F\xi) + D_1(F\xi, FN) = -\pi(F\xi)$ *and* $D_1(FN, FN) + D_1(F\xi, F\xi) = -\pi(FN).$
(iii) $\widetilde{D}_2(FN, F\xi) = \widetilde{D}_2(F\xi, FN) = D_2(FN, F\xi) = 0.$
(iv) $\widetilde{D}_2(FN, FN) + \widetilde{D}_2(F\xi, F\xi) = D_2(FN, FN) + D_2(F\xi, F\xi) = 0.$
(v) $D_1(Fu, Fu) = 0$ *and* $D_2(Fu, Fu) = -\pi(Fu).$
(vi) $\varepsilon_1(\xi) = 0.$

Proof. Let $\{e_1, e_2, ..., e_{n-4}, E_1, E_2, Fu\}$ be quasi orthonormal basis of $s(TM)$, in where

$$E_1 = \frac{FN + F\xi}{2}, \quad E_2 = \frac{FN - F\xi}{2},$$

and $\{e_1, e_2, ..., e_{n-4}\}$ is orthonormal basis of L_0. Then we obtain

$$\widetilde{D}_1(E_1, E_1) = \widetilde{D}_1(FN, FN) + \widetilde{D}_1(F\xi, F\xi) + \widetilde{D}_1(FN, F\xi) + \widetilde{D}_1(F\xi, FN) = 0$$

and

$$\widetilde{D}_1(E_2, E_2) = \widetilde{D}_1(FN, FN) + \widetilde{D}_1(F\xi, F\xi) - \widetilde{D}_1(FN, F\xi) - \widetilde{D}_1(F\xi, FN) = 0.$$

Thus we get (i). from (4.8) and (i) we have

$$D_1(FN, F\xi) + D_1(F\xi, FN) + \pi(F\xi) = 0,$$

$$D_1(FN, FN) + D_1(F\xi, F\xi) + \pi(FN) = 0.$$

Hence we get (ii).
Similarly we get (iv). We know that $\widetilde{D}_2(FN, F\xi) + \widetilde{D}_2(F\xi, FN) = 0$. However we have

$$\widetilde{D}_2(F\xi, FN) = D_2(F\xi, FN) + \pi(FN)w_2(F\xi) = D_2(F\xi, FN),$$

$$\widetilde{D}_2(FN, F\xi) = D_2(FN, F\xi) + \pi(F\xi)w_2(FN) = D_2(FN, F\xi).$$

Thus we get (iii). We obtain

$$\widetilde{D}_1(Fu, Fu) = D_1(Fu, Fu) + \pi(Fu)w_1(Fu) = D_1(Fu, Fu) = 0,$$

and

$$\widetilde{D}_2(Fu, Fu) = D_2(Fu, Fu) + \pi(Fu)w_2(Fu) = 0.$$

Thus we get (v). From (4.14) we obtain

$$\widetilde{\varepsilon}_1(\xi) = \varepsilon_1(\xi) + \pi(u)w_1(\xi) = 0.$$

Thus proof is completed. □

Corollary 4.1. *Let M be a minimal semi-invariant half-lightlike submanifold of a semi-Riemannian product manifold* $\left(\widetilde{M}, \widetilde{g}\right)$*. Then the distribution L is minimal with respect to quarter symmetric non-metric connection.*

Proof. From (4.14) we have

$$\widetilde{\varepsilon}_1(\xi) = \varepsilon_1(\xi) + \pi(u)w_1(\xi) = 0.$$

From (4.8) we obtain

$$\widetilde{D}_1(e_i, e_i) = D_1(e_i, e_i) + \pi(e_i)w_1(e_i) = 0. \tag{4.16}$$

Similarly from (4.9) we get

$$\widetilde{D}_2(e_i, e_i) = D_2(e_i, e_i) + \pi(e_i)w_2(e_i) = 0. \tag{4.17}$$

From (4.16) and (4.17) proof is completed. □

Definition 4.2. A lightlike submanifold M is said to be irrotational if

$$\widetilde{\nabla}_X \xi \in \Gamma(TM),$$

for any $X \in \Gamma(TM)$, where $\xi \in \Gamma(TM)$

For a half-lightlike M, since $D_1(X,\xi) = 0$, the above definition is equivalent to

$$D_2(X,\xi) = 0 = \varepsilon_1(X), \ \forall X \in \Gamma(TM) \tag{4.18}$$

Then we obtain

$$\widetilde{D}_X\xi = \widetilde{\nabla}_X\xi + \pi(\xi)FX\,. \tag{4.19}$$

Thus we have the following proposition

Proposition 4.2. *Let M be irrotational semi-invariant half-lightlike submanifold. Then M is L− irrotational with respect to* $\widetilde{D}$.

Proof. For any $X \in \Gamma(L_4)$ from (4.19) we get $\widetilde{D}_X\xi \in \Gamma(TM)$. Thus proof is completed. □

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