A NOTE ON REGULAR PENTAGONS ARISING FROM THE GOLDEN ARBELOS

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Abstract. Several regular pentagons are constructed from the golden arbelos.

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1. INTRODUCTION

Let us consider three semicircles $\alpha$, $\beta$, and $\gamma$ with diameters $AO$, $BO$ and $AB$, respectively, constructed on the same side of $AB$ for a point $O$ on the segment $AB$ in the plane. The configuration of $\alpha$, $\beta$, and $\gamma$ is called an arbelos. Let $a$ and $b$ be the radii of $\alpha$ and $\beta$, respectively. The number $\phi = \left(1 + \sqrt{5}\right)/2$ is called the golden number or the golden mean, which is closely related to the regular pentagon [6]. The arbelos is called a golden arbelos if $a/b = \phi \pm 1$, and has been considered in several papers [1, 2, 3, 4, 5]. But it seems that no regular pentagon has been constructed for the golden arbelos. In this note we show that several regular pentagons are constructed from the golden arbelos.

2. REGULAR PENTAGONS

We use the following properties of regular pentagons.

(1) The height of a regular pentagon inscribed in a circle of radius $r$ equals $(1 + \phi/2)r$.

(2) If $PQRST$ is a regular pentagon, the point of intersection of the lines $QR$ and $ST$ coincides with the reflection of the point $P$ in the line $RS$.

The center of the semicircle $\alpha$ is denoted by $O_\alpha$, and the circle formed by $\alpha$ and its reflection in the line $AB$ is denoted by $\overline{\alpha}$. The notations $O_\beta$, $O_\gamma$, $\overline{\beta}$, and $\overline{\gamma}$ are defined similarly. The segments $O_\alpha O_\gamma$ and $AO_\beta$ share the midpoint. We now assume $b/a = \phi$. Then the perpendiculars to $AB$ at points $O_\alpha$ and $O_\gamma$ touch the incircle of the arbelos [1, 2] (see Figure 1).

Theorem 2.1. Let the perpendicular bisector of the segment $AO_\beta$ intersect the semicircle $\gamma$ in a point $C$, and let $D$ be the reflection of $C$ in the line $AB$. Then the following statements hold.

(i) The points $D$, $A$, $C$ are consecutive vertices of a regular pentagon inscribed in the circle $\overline{\alpha}$.

(ii) The lines $AC$, $DO_\alpha$, and the semicircle $\alpha$ meet in a point $E$, also the lines $EO_\beta$, $CD$ and $\alpha$ meet in a point $F$, and the points $A$, $E$, $F$ are consecutive vertices of a regular pentagon inscribed in the circle $\overline{\alpha}$.
(iii) The lines $BC$, $DO_\gamma$ and the semicircle $\beta$ meet in a point $G$, and the points $O$, $G$ are consecutive vertices of a regular pentagon inscribed in the circle $\beta$.

(iv) If the line $DO_\alpha$ meets $\gamma$ in a point $H$, then $CGO_\gamma O_\alpha H$ is a regular pentagon of side length $b$.

**Figure 1.**

**Proof.** We use a rectangular coordinate system with origin $O$ such that $A$ has coordinates $(2a,0)$, and assume that the arbelos is constructed in the region $y > 0$. Since the point $O_\beta$ has $x$-coordinate $-b$, the point $C$ has $x$-coordinate $c_x = a - b/2 = (\sqrt{5}/2 - 1)b$ and $y$-coordinate $c_y = \sqrt{(2a - c_x)(c_x + 2b)} = (b/2)\sqrt{5} + 2\sqrt{5} = (b/2)\tan 72^\circ$. This implies $\angle AO_\gamma C = 72^\circ$, since $|O_\alpha O_\gamma| = b$ and $CD$ bisects $O_\beta O_\gamma$. This proves (i). Let the segment $AC$ intersect the semicircle $\alpha$ in a point $E$. Then $A$ and $E$ are consecutive vertices of a regular pentagon inscribed in $\alpha$, since the triangles $CO_\gamma A$ and $EO_\alpha A$ are homothetic with center $A$. Let the line $CD$ intersect $\alpha$ in a point $F$. Then $F$ is a vertex of the same regular pentagon by (1), for the distance between $A$ and $CD$ is $2a - c_x = (1 + \phi/2)a$. The rest of (ii) follows from (2). Let $G$ be the point of intersection of the segment $BC$ and the semicircle $\beta$. Then $G$ and $O$ are the images of $C$ and $A$, respectively by the homothety with center $B$ carrying $\gamma$ into $\beta$. Therefore $O$ and $G$ are consecutive vertices of a regular pentagon inscribed in $\beta$, i.e., $\angle O_\alpha O_\gamma G = 72^\circ$. While $B$ divides $CG$ externally in the ratio $(a+b):b = b:a$, i.e., $G$ has $y$-coordinate $ac_y/b = (a/2)\tan 72^\circ$. Hence $G$ lies on the perpendicular bisector of $O_\beta O_\gamma$, since $|O_\beta O_\gamma| = a$. Hence the triangle $GO_\beta O_\gamma$ is isosceles with base angle $72^\circ$ and equal side length $b$. But $\angle BCD = \angle BAD = 54^\circ$ and $|O_\alpha O_\gamma| = b$. Therefore if $H$ is the reflection of $G$ in $CD$, then $CGO_\gamma O_\alpha H$ is a regular pentagon, where notice that the points $O_\alpha$, $E$ and $H$ are collinear. Hence the lines $DO_\alpha$ and $DO_\gamma$ pass
A note on regular pentagons arising from the golden arbelos through $H$ and $G$, respectively by (2). Then $H$ lies on $\gamma$, because $\angle CHD = \angle CAD$. Therefore (iii) and (iv) are proved. □

The points $A$, $H$, $C$ and the point of intersection of the line $DO_\gamma$ and the semicircle $\gamma$ are consecutive vertices of a regular decagon of side length $b$ inscribed in the circle $\gamma$. For $|O_\alpha O_\gamma| = |O_\alpha H|$ and $\angle AO_\alpha H = 72^\circ$ imply $\angle AO_\gamma H = 36^\circ$, and obviously $\angle AO_\gamma G = 108^\circ$. While $\angle O_\gamma DO_\beta = 18^\circ$, for the triangles $DAO_\alpha$ and $DO_\beta O_\gamma$ are congruent. Therefore the point of intersection of the line $DO_\beta$ and $\gamma$ is also a vertex of the regular decagon.

REFERENCES


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