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A NOTE ON REGULAR PENTAGONS ARISING FROM THE GOLDEN ARBELOS

HIROSHI OKUMURA

Abstract. Several regular pentagons are constructed from the golden arbelos.

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1. Introduction

Let us consider three semicircles α, β and γ with diameters AO, BO and AB, respectively constructed on the same side of AB for a point O on the segment AB in the plane. The configuration of α, β and γ is called an arbelos. Let a and b be the radii of α and β, respectively. The number $\phi = \left(1+\sqrt{5}\right)/2$ is called the golden number or the golden mean, which is closely related to the regular pentagon [6]. The arbelos is called a golden arbelos if $a/b = \phi^{\pm 1}$, and has been considered in several papers [1, 2, 3, 4, 5]. But its seems that no regular pentagon has been constructed for the golden arbelos. In this note we show that several regular pentagons are constructed from the golden arbelos.

2. Regular pentagons

We use the following properties of regular pentagons.
(1) The height of a regular pentagon inscribed in a circle of radius r equals $(1+\phi/2)r$.
(2) If $PQRST$ is a regular pentagon, the point of intersection of the lines QR and ST coincides with the reflection of the point P in the line RS.

The center of the semicircle α is denoted by O_α, and the circle formed by α and its reflection in the line AB is denoted by $\overline{\alpha}$. The notations O_β, O_γ, $\overline{\beta}$ and $\overline{\gamma}$ are defined similarly. The segments $O_\alpha O_\gamma$ and AO_β share the midpoint. We now assume $b/a = \phi$. Then the perpendiculars to AB at points O_α and O_γ touch the incircle of the arbelos [1, 2] (see Figure 1).

Theorem 2.1. *Let the perpendicular bisector of the segment AO_β intersect the semicircle γ in a point C, and let D be the reflection of C in the line AB. Then the following statements hold.*
(i) *The points D, A, C are consecutive vertices of a regular pentagon inscribed in the circle $\overline{\gamma}$.*
(ii) *The lines AC, DO_α and the semicircle α meet in a point E, also the lines EO_β, CD and α meet in a point F, and the points A, E, F are consecutive vertices of a regular pentagon inscribed in the circle $\overline{\alpha}$.*

(iii) *The lines* BC, DO_γ *and the semicircle* β *meet in a point* G, *and the points* O, G *are consecutive vertices of a regular pentagon inscribed in the circle* $\overline{\beta}$.
(iv) *If the line* DO_α *meets* γ *in a point* H, *then* $CGO_\gamma O_\alpha H$ *is a regular pentagon of side length* b.

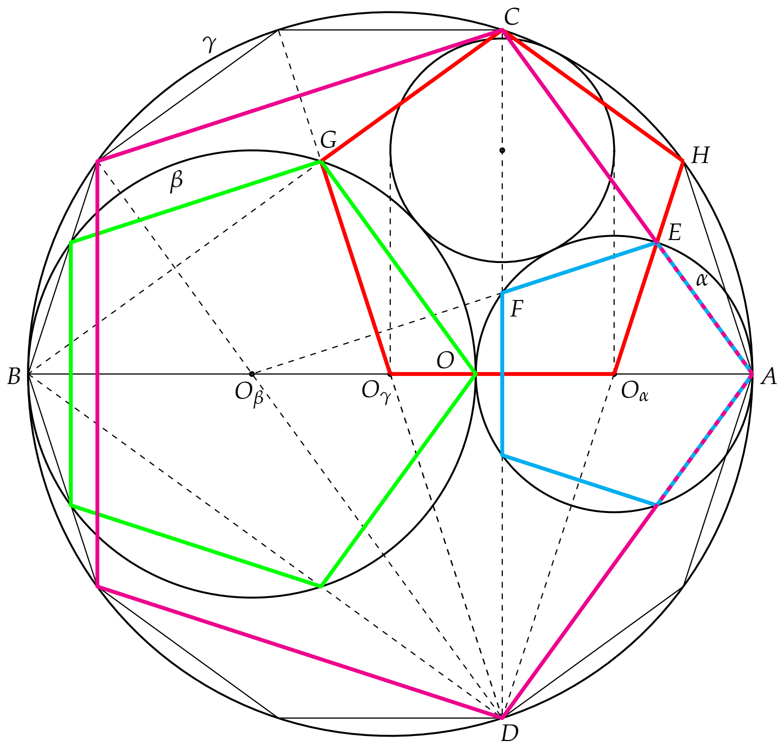

Figure 1.

Proof. We use a rectangular coordinate system with origin O such that A has coordinates $(2a, 0)$, and assume that the arbelos is constructed in the region $y > 0$. Since the point O_β has x-coordinate $-b$, the point C has x-coordinate $c_x = a - b/2 = (\sqrt{5}/2 - 1)b$ and y-coordinate $c_y = \sqrt{(2a - c_x)(c_x + 2b)} = (b/2)\sqrt{5 + 2\sqrt{5}} = (b/2)\tan 72°$. This implies $\angle AO_\gamma C = 72°$, since $|O_\alpha O_\gamma| = b$ and CD bisects $O_\alpha O_\gamma$. This proves (i). Let the segment AC intersect the semicircle α in a point E. Then A and E are consecutive vertices of a regular pentagon inscribed in $\overline{\alpha}$, since the triangles $CO_\gamma A$ and $EO_\alpha A$ are homothetic with center A. Let the line CD intersect α in a point F. Then F is a vertex of the same regular pentagon by (1), for the distance between A and CD is $2a - c_x = (1 + \phi/2)a$. The rest of (ii) follows from (2). Let G be the point of intersection of the segment BC and the semicircle β. Then G and O are the images of C and A, respectively by the homothety with center B carrying γ into β. Therefore O and G are consecutive vertices of a regular pentagon inscribed in β, i.e., $\angle OO_\beta G = 72°$. While B divides CG externally in the ratio $(a + b) : b = b : a$, i.e., G has y-coordinate $ac_y/b = (a/2)\tan 72°$. Hence G lies on the perpendicular bisector of $O_\beta O_\gamma$, since $|O_\beta O_\gamma| = a$. Hence the triangle $GO_\beta O_\gamma$ is isosceles with base angle $72°$ and equal side length b. But $\angle BCD = \angle BAD = 54°$ and $|O_\alpha O_\gamma| = b$. Therefore if H is the reflection of G in CD, then $CGO_\gamma O_\alpha H$ is a regular pentagon, where notice that the points O_α, E and H are collinear. Hence the lines DO_α and DO_γ pass through H and G, respectively by (2). Then H lies on γ, because $\angle CHD = \angle CAD$. Therefore (iii) and (iv) are proved. □

The points A, H, C and the point of intersection of the line DO_γ and the semicircle γ are consecutive vertices of a regular decagon of side length b inscribed in the circle $\overline{\gamma}$. For $|O_\alpha O_\gamma| = |O_\alpha H|$ and $\angle AO_\alpha H = 72^\circ$ imply $\angle AO_\gamma H = 36^\circ$, and obviously $\angle AO_\gamma G = 108^\circ$. While $\angle O_\gamma DO_\beta = 18^\circ$, for the triangles DAO_α and $DO_\beta O_\gamma$ are congruent. Therefore the point of intersection of the line DO_β and γ is also a vertex of the regular decagon.

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Department of Mathematics
Yamato University
2-5-1 Katayama Suita Osaka 564-0082, Japan
E-mail address: okumura.hiroshi@yamato-u.ac.jp