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ON 4-TH ROOT FINSLER METRICS WITH SPECIAL NON-RIEMANNIAN CURVATURES PROPERTIES

T. KHOSHDANI AND N. ABAZARI

ABSTRACT. The theory of 4-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory. In this paper, we find the condition under which an 4-th root Finsler metric is Berwald metric, Landsberg metric and stretch metric. Then we characterize $S3$-like 4-th root metrics. Finally, we show that there is no $S4$-like 4-th root Finsler metric.

1. INTRODUCTION

The theory of m-th root metrics has been developed by H. Shimada [13], and applied to Biology as an ecological metric [1]. It is regarded as a direct generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric. The third and fourth root metrics are called the cubic metric and quartic metric, respectively.
Recently studies show that the theory of m-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory [2] [7] [10] [11] [6] [16] [18] [19] [20]. For quartic metrics, a study of the geodesics and of the related geometrical objects is made by S. Lebedev [5]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and N. Brinzei in two papers [3], [4]. Tensorial connections for such spaces have been recently studied by L. Tamassy [14]. In [16], Tayebi-Najafi characterize locally dually flat and Antonelli m-th root Finsler metrics. They show that every m-th root Finsler metric of isotropic mean Berwald curvature reduces to a weakly Berwald metric. In [17], they prove that every m-th root Finsler metric of isotropic Landsberg metric reduces to a Landsberg metric. Then, they show that every m-th root Finsler metric with almost vanishing H-curvature satisfies $\mathbf{H} = 0$. Recently, Tayebi-Nankali-Peyghan define some non-Riemannian curvature properties for Cartan spaces and consider Cartan space with the m-th root metric [18]. They prove that every m-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively.
Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let F be a scalar function on TM defined by $F = \sqrt[4]{A}$, where A is given by

$$A := a_{ijkl}(x) y^i y^j y^k y^l, \tag{1}$$

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with a_{ijkl} symmetric in all its indices [13]. Then F is called an 4-th root Finsler metric. Let F be an 4-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put

$$A_i := \frac{\partial A}{\partial y^i}, \quad \text{and} \quad A_{ij} := \frac{\partial^2 A}{\partial y^i \partial y^j}.$$

Suppose that the matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold

$$g_{ij} = \frac{A^{-\frac{3}{2}}}{16}[4AA_{ij} - 2A_iA_j], \tag{2}$$

$$g^{ij} = A^{\frac{-1}{2}}[4AA^{ij} + \frac{2}{3}y^iy^j], \tag{3}$$

$$y^iA_i = 4A, \;\; y^iA_{ij} = 3A_j, \;\; y_i = \frac{1}{4}A^{\frac{-1}{2}}A_i, \tag{4}$$

$$A^{ij}A_{jk} = \delta^i_k, \;\; A^{ij}A_i = \frac{1}{3}y^j, \;\; A_iA_jA^{ij} = \frac{4}{3}A, \tag{5}$$

$$A_0 := A_{x^m}y^m, \;\; A_{0l} := A_{x^my^l}y^m. \tag{6}$$

Let (M, F) be a Finsler manifold The second derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ are the components of an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F^2_x$ at $y \in T_xM_0$ are a symmetric trilinear form $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and Cartan torision, respectively. The rate of change of the Cartan torision along geodesics is the Landsberg curvature $\mathbf{L}_y$ on T_xM for any $y \in T_xM_0$. F is said to be Landsbergian if $\mathbf{L} = 0$. Here, We characterize conditions that 4-the root Finsler metrics be Berwald and Landsbergian and metrics.
Let us define the following

$$a_i := \frac{1}{F^3}a_{ijkl}(x)y^jy^ky^l, \quad a_{ij} := \frac{1}{F^2}a_{ijkl}(x)y^ky^l, \quad a_{ijk} := \frac{1}{F}a_{ijkl}(x)y^l. \tag{7}$$

Then, we have the following.

Theorem 1.1. *A Finsler space with the 4-th root metric $F = \sqrt[4]{A}$ is a Berwald space (resp. Landsberg space) if and only if $a_{ijk|h} = 0$ (resp. $a_{ijk|0} = 0$).*

Now, let (M, F) be a Finsler manifold. The v-curvature tensor of the Cartan connection is given by following

$$S_{lijk} = C^r_{lk}C_{rij} - C^r_{lj}C_{rik},$$

where C^i_{jk} denotes the Cartan torsion of F. A Finsler space (M, F) is called $S3$-like if there exists such a scalar $S = S(x, y)$ on TM such that the v-curvature tensor S_{hijk} is written in the following form

$$F^2S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}), \tag{8}$$

where $h_{ij} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric (see [9]). Every 3-dimensional Finsler space is $S3$-like (see [9]).

Theorem 1.2. *Let F be an 4-th root Finsler metric on an n-dimensional manifold M. Then, F is a S3-like metric if and only if*

$$S = \frac{16Q^{rp}}{[\tilde{h}_{lj}\tilde{h}_{ik} - \tilde{h}_{lk}\tilde{h}_{ij}]}[\mathbb{A}_{plk}\mathbb{A}_{rij} - \mathbb{A}_{plj}\mathbb{A}_{rik}]A^{-2}, \tag{9}$$

where

$$\widetilde{h}_{ij} := 4AA_{ij} - 3A_iA_j, \tag{10}$$

$$Q^{rp} := 4A^{rp} + \frac{2}{3}y^r y^p, \tag{11}$$

$$\mathbb{A}_{ijk} := A^2A_{ijk} + \frac{3}{4}A_iA_jA_k - \frac{1}{2}A(A_iA_{jk} + A_jA_{ik} + A_kA_{ij}). \tag{12}$$

Then, the scalar function S is a rational function in y.

A Finsler manifold (M, F) is called $S4$-like if there exists a scalar function $M = M(x, y)$ on TM such that v-curvature tensor is written in the following form

$$F^2S_{ijkl} = h_{ik}M_{jl} + h_{jl}M_{ik} + h_{jl}M_{ik} - h_{il}M_{jk} - h_{jk}M_{il}.$$

Every four-dimensional Finsler space is $S4$-like (see [9])

Theorem 1.3. *There is no $S4$-like 4-th root Finsler metric.*

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2. Preliminaries

Let M be a n-dimensional C^∞ manifold. A Finsler metric on M is a function $F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 = TM \setminus \{0\}$, (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C}=\mathbf{0}$ if and only if F is Riemannian.

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k$, where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$.

In local coordinates (x^i, y^i), the vector filed $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ is a global vector field on TM_0, where $G^i = G^i(x, y)$ are local functions on TM_0 given by

$$G^i := \frac{1}{4} g^{il} \Big\{ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \Big\}, \;\; y \in T_x M. \tag{13}$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F) [15].

For $y \in T_x M_0$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \to T_x M$ and $\mathbf{E}_y : T_x M \otimes T_x M \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y) u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y) u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}$ is called Berwald curvature. Then F is called a Berwald metric if $\mathbf{B} = 0$.

3. Proof of Theorem 1.1

A Finsler space is called a Berwald space (resp. Landsberg space) if $C_{ijk|h} = 0$ (resp. $C_{ijk|0} = 0$) holds [9].

Proof of Theorem 1.1: Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric and $S3$-like. The Cartan torsion of of F is given by

$$C_{ijk} = \frac{1}{4} A^{\frac{-5}{2}} \mathbb{A}_{ijk}, \tag{14}$$

where

$$\mathbb{A}_{ijk} = A^2 A_{ijk} + \frac{3}{4} A_i A_j A_k - \frac{1}{2} A (A_i A_{jk} + A_j A_{ik} + A_k A_{ij}). \tag{15}$$

So, we have

$$4A^{\frac{5}{2}} C_{ijk} = A^2 A_{ijk} + \frac{3}{4} A_i A_j A_k - \frac{1}{2} A (A_i A_{jk} + A_j A_{ik} + A_k A_{ij}). \tag{16}$$

By (7), we have

$$A_i := 4F^3 a_i, \;\; A_{ij} := 12F^2 a_{ij} \;\; , \;\; A_{ijk} := 24F a_{ijk}. \tag{17}$$

The normalized supporting element l_i, the angular metric tensor h_{ij} and the fundamental tensor g_{ij} are given in the following form

$$l_i = a_i \;\; , \;\; h_{ij} = 3(a_{ij} - a_i a_j) \;\; , \;\; g_{ij} = 3a_{ij} - 2a_i a_j. \tag{18}$$

Throughout this paper, we use the Cartan connection on Finsler manifolds. The horizontal and vertical derivatives of a Finsler tensor field are denoted by "|" and ", ", respectively. Then the following hold

$$F_{|i} = 0 \;\; , \;\; l_{i|j} = 0 \;\; , \;\; h_{ij|k} = 0. \tag{19}$$

It follows from (18) that the Cartan tensor $C_{ijk} (= \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k})$ is given in the form

$$C_{ijk} = \frac{3}{F} (a_{ijk} - a_{ij} a_k - a_{ik} a_j - a_{kj} a_i + 2a_i a_j a_k). \tag{20}$$

By taking a horizontal derivative of (20) and using (19) we get

$$a_{i|j} = 0 \ , \ a_{ij|k} = 0 \ , \ a_{ijk|h} = \frac{F}{3} C_{ijk|h}. \tag{21}$$

Then

$$C_{ijk|h} = \frac{1}{4F^2} A_{ijk|h} = \frac{3}{F} a_{ijk|h}. \tag{22}$$

Then, according to (22), the 4-th root metric $F = \sqrt[4]{A}$ is a Berwald metric if and only if $a_{ijk|h} = 0$.
On the other hand, the components of Landsberge curvature $L_{ijk} (= y^h C_{ijk|h})$ are given by

$$L_{ijk} = \frac{3}{F} a_{ijk|0}. \tag{23}$$

Here, the subscript 0 means the contraction for the supporting element y^i. By (23), the 4-th root metric $F = \sqrt[4]{A}$ is a Landsberg metric if and only if $a_{ijk|0} = 0$. This complete the proof. □

As a generalization of Landsberg curvature, Berwald introduced the notion of stretch curvature and denoted it by $\mathbf{\Sigma}_y$. He showed that $\mathbf{\Sigma} = 0$ if and only if the length of a vector remains unchanged under the parallel displacement along an infinitesimal parallelogram. Then, this curvature investigated by Matsumoto in [8]. Define the stretch curvature $\mathbf{\Sigma}_y : T_xM \otimes T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{\Sigma}_y(u, v, w, z) := \Sigma_{ijkl}(y) u^i v^j w^k z^l$, where

$$\Sigma_{ijkl} := 2(L_{ijk|l} - L_{ijl|k}).$$

A Finsler metric is said to be stretch metric if $\mathbf{\Sigma} = 0$. Every Landsberg metric is a stretch metric [21].

Corollary 3.1. *A Finsler space with the* 4*-th root metric* $F = \sqrt[4]{A}$ *is a stretch metric if and only if* $a_{ijk|0|s} = a_{ijs|0|k}$.

Proof. By (23), we have

$$L_{ijk|s} = \frac{3}{F} a_{ijk|0|s}. \tag{24}$$

Therefore, the components of stretch curvature are given by the following

$$\Sigma_{ijkl} = \frac{3}{F} (a_{ijk|0|s} - a_{ijs|0|k}). \tag{25}$$

Then, by (25) it follows that the 4-th root metric $F = \sqrt[4]{A}$ is a stretch metric if and only if $a_{ijk|0|s} = a_{ijs|0|k}$. □

4. Proof of Theorem 1.2

In this section, we are going to prove the Theorem 1.2.

Proof of Theorem 1.2: Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric and $S3$-like. The angular metric of F are given by

$$h_{ij} = \frac{1}{16} \tilde{h}_{ij} A^{\frac{-3}{2}}, \tag{26}$$

where

$$\tilde{h}_{ij} = 4AA_{ij} - 3A_iA_j. \tag{27}$$

By (13), it is clear that C_{ijk} is an irrational function with respect to y. Also, we have

$$g^{mi} = A^{\frac{-1}{2}}Q^{mi}.$$

Then the components $C^m_{jk}(= g^{mi}C_{ijk})$ are given by

$$C^m_{jk} = \frac{1}{4}Q^{mi}\mathbb{A}_{ijk}A^{-3}. \tag{28}$$

Then C^m_{jk} is a rational function with respect to y. So, the v-curvature tensor $S_{lijk} = C^r_{lk}C_{rij} - C^r_{lj}C_{rik}$ of the Cartan Connection $C\Gamma$ for the 4-th root metric is given by the following

$$S_{lijk} = \frac{1}{16}Q^{rp}A^{\frac{-11}{2}}(\mathbb{A}_{plk}\mathbb{A}_{rij} - \mathbb{A}_{plj}\mathbb{A}_{rik}). \tag{29}$$

On the other hand, by assumption we have

$$S_{lijk} = \frac{S}{F^2}(h_{lj}h_{ik} - h_{lk}h_{ij}) \tag{30}$$

Substituting (26) into (30) implies that

$$S_{lijk} = \frac{1}{256}SA^{\frac{-7}{2}}(\tilde{h}_{lj}\tilde{h}_{ik} - \tilde{h}_{lk}\tilde{h}_{ij}). \tag{31}$$

By (29) and (31), we get

$$S = \frac{16Q^{rp}}{[\tilde{h}_{lj}\tilde{h}_{ik} - \tilde{h}_{lk}\tilde{h}_{ij}]}[\mathbb{A}_{plk}\mathbb{A}_{rij} - \mathbb{A}_{plj}\mathbb{A}_{rik}]A^{-2}, \tag{32}$$

It is trivial that S is a rational function in y. □

Corollary 4.1. *If $S = S(x)$ is isotropic, then F has vanishing vv-curvature.*

Proof. By assumption, the left-hand side of (32) is a function with respect to x while the other side is a function with respect to both x and y. Then, it implies that $S = 0$. Hence, $S_{ijkl} = 0$. □

5. Proof of Theorem 1.3

In this section, we are going to prove the Theorem 1.3. First, we remark that in [12], Schneider proved that every Finsler metric with vanishing v-curvature is Riemannian

Lemma 5.1. ([12]) Every Finsler metric with vanishing v-curvature is Riemannian.

Proof of Theorem 1.3: Let $F = \sqrt[4]{A}$ be an 4-th root Finsler metric that is $S4$-like, i.e.,

$$F^2S_{ijkl} = h_{ik}M_{jl} + h_{jl}M_{ik} - h_{il}M_{jk} - h_{jk}M_{il} \tag{33}$$

where $M_{ij} = M_{ij}(x, y)$ are scalar and symmetric functions on TM.
By putting (26) ,(29) into (33), we get

$$Q^{rp}[\mathbb{A}_{pil}\mathbb{A}_{rjk} - \mathbb{A}_{pik}\mathbb{A}_{rjl}] = \tilde{h}_{ik}M_{jl} + \tilde{h}_{jl}M_{ik} - \tilde{h}_{il}M_{jk} - \tilde{h}_{jk}M_{il} \tag{34}$$

Now, the right-hand side of (34) is an irrational function in y while its left-hand side is a rational function in y. Hence, the two sides should be equal to zero. So,

$$\mathbb{A}_{pil}\mathbb{A}_{rjk} - \mathbb{A}_{pik}\mathbb{A}_{rjl} = 0, \tag{35}$$

$$\tilde{h}_{ik}M_{jl} + \tilde{h}_{jl}M_{ik} - \tilde{h}_{il}M_{jk} - \tilde{h}_{jk}M_{il} = 0. \tag{36}$$

Then, by (29) it follows that $S_{ijkl} = 0$. By Schneider's lemma F is Riemannian. This is a contradict with our assumption that say F is a non-Riemmanian metric. Thus, there is no $S4$-like 4-th root metric. □

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Department of Mathematics
University of Mohaghegh Ardabili
P.O.Box 56199-11367
Ardabil, Iran.
E-mail address: `khoshdani@uma.ac.ir and khoshdani@yahoo.com`
E-mail address: `abazari@uma.ac.ir`