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ON THEOREM GENERATORS IN PLANE GEOMETRY

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ABSTRACT. In this article we will construct some methods which can help to find new theorems in Euclidean plane geometry.

1. INTRODUCTION

In this article our main interest is to construct theorem generators (**TG**) which can be seen as methods for producing new theorems. We will see that in plane geometry to make **TG** it's natural to compare to any theorem it's parameter space and constructing morphisms between this spaces as objects, then uncover from parameter spaces it's corresponding theorems.
In this article we will use next well known facts from basic algebraic geometry (over complex number field).

Theorem 1.1. *Let $f : X \to Y$ — be dominating morphism of algebraic varieties. Then exists a nonempty open subset $U \subseteq Y$, such that for every $y \in f(X) \cap U$, $\dim f^{-1}(y) = \dim X - \dim Y$.*

Theorem 1.2. *Let given that Y is arbitrary algebraical variety and $\pi : Y \times \mathbb{P}^n \to Y$ — projection on Y. Then image of any closed set is also closed.*

To construct main example of **TG** we will need next important experimental fact (**EF**): most of rational morphisms between irreducible algebraical varieties with same dimensions are dominant. To understand why it is true see theorem 1. This experimental fact means that if we consider two irreducible algebraical varieties X, Y, where $\dim X = \dim Y$ and chose some rational morphism ϕ between them, then in quite random situation, we will get that ϕ is dominating.

2. CONSTRUCTION OF $\mathbf{TG}_{Dual}$ RELATED TO PROJECTIVE DUALITY

Well known that any theorem on plane with only points and lines can be translated to it's projectively dual theorem, where to any point we correspond line and to any line we correspond point, to any thee points which lies on same line we correspond three lines which have same intersection point. So we can look on it as simple example of **TG**.
Now we can give example of more general $\mathbf{TG}_{Dual}$. Definition is next:

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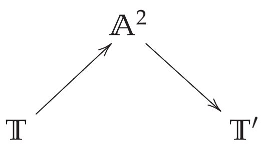
(1) First we consider some set of algebraical curves X of fixed degree on plane $\mathbb{C}^2$, for example X is points on plane, or X is conics on plane. Then we get that set X has parameter space equivalent to some affine space $\mathbb{A}^n$.
(2) Second we place some geometric theorem $\mathbb{T}$ inside space $\mathbb{A}^n$.
(3) We know that there exists correspondence $\mathbb{A}^n \leftrightarrow X$, so we can correspond to points in theorem $\mathbb{T}$ their images as finite set of curves of type X on plane.
(4) We can transfer information (of theorem $\mathbb{T}$) about points in $\mathbb{A}^n$ to theorem about curves in X.

Next we will show how this $\mathbf{TG}_{Dual}$ works.

Example 1. *If we consider X as lines on $\mathbb{C}^2$, then we get correspondent to it parameter space $\mathbb{A}^2$. So for any theorem $\mathbb{T}$ about points and lines we can place it in parameter space $\mathbb{A}^2$. Easy to see that if we use correspondence $X \leftrightarrow \mathbb{A}^2$, then we can transport theorem $\mathbb{T}$ to theorem $\mathbb{T}'$ about points in X. So we get next dictionary table :*

Set X	*Parameter space $\mathbb{A}^2$*
Three lines in X have same intersection point	*Three points in $\mathbb{A}^2$ lie on same line*

And diagram :



So easy to check that we get another interpretation of projective duality by such construction.

Definition 1. *For any curve Γ in $\mathbb{C}^3$ we say that Γ is spherical, iff it lies on some sphere in $\mathbb{C}^3$.*

Example 2. *Let given 8 quadratic surfaces in space $\mathbb{C}^3$, name them as $\mathcal{C}_1, \ldots, \mathcal{C}_8$. Denote curves $\Gamma_{i,j} := \mathcal{C}_i \cap \mathcal{C}_j$. Let given that for any numbers $(i, j) \neq (4, 8)$, curves $\Gamma_{i,j}$ are all spherical. We will prove that then curve $\Gamma_{4,8}$ is spherical.*

To see how this example follows from $\mathbf{TG}_{Dual}$, consider parameter space $\mathbb{A}^k$ for set X of all quadratic surfaces in $\mathbb{C}^3$. So we know that quadratic surfaces have their representations in $\mathbb{A}^k$ as points. Set of all spheres on $\mathbb{C}^3$ can be represented in $\mathbb{A}^k$ as some $4D$ plane $P \subseteq \mathbb{A}^k$. So theorem in example 2 can be translated to next theorem $\mathbb{T}$ in space $\mathbb{A}^k$: Let given points $p_1, p_2, \ldots, p_8 \in \mathbb{A}^k$ and $4D$ plane $P \in \mathbb{A}^k$, such that for any pair of numbers $(i, j) \neq (4, 8)$, line through points p_i, p_j intersects plane P, then we can prove that line through points p_4, p_8 also intersects with plane P. To see why this theorem $\mathbb{T}$ is true, note that from conditions of theorem we can easily check that all points $\{p_i\}_{1 \leq i \leq 8}$ lie on same $5D$ plane Q, such that $P \subset Q$. So from well known theorems from algebraic geometry we get that line $p_4p_8 \subset Q$ intersect 4 dimensional plane $P \subseteq Q$, because $\dim Q = 5$ (intersection point can be infinite).

3. Categorification of plane geometry

To construct main example of **TG** we need to construct categorification of plane geometry. First we give some definitions from category theory, see [8].

Definition 2. *A source is a pair $(A, f_i)_I$ consisting of an object A and a family of morphisms $f_i : A \to A_i$ with domain A, indexed by some class I. A is called the domain of the source and the family $(A_i)_I$ is called the co-domain of the source.*

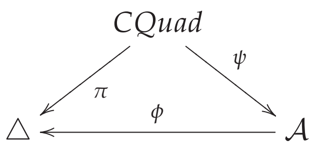
Everywhere in text we mean plane geometry over complex plane $\mathbb{C}^2$. To construct categorification of plane geometry first we need to construct objects. We give this definition in next examples :

A. We can say that set of all triangles on plane $\mathbb{C}^2$ form object $\triangle$ in category.
B. Sets of four points form object $Quad$. Also sets of four lines form object $CQuad$.
C. Sets of five conics on $\mathbb{C}^2$ can be seen as object in category.

So from these examples we can define objects in category as different finite sets of points and of curves of fixed type.
Second we need to define morphisms between this objects. Also we give this definition by examples :

A. For any triangle $ABC \in \triangle$ consider it's in-center and all 3 ex-centers, so we can say that this correspondence can be seen as morphism $f : \triangle \to Quad$.
B. For any complete quadrilateral we can correspond three midpoints of its diagonals, so we get morphism $f : CQuad \to \triangle$.
C. For any triangle ABC we can correspond it's circumcircle and get morphism $f_\omega : \triangle \to \circ$, where $\circ$ is circles on plane.

So from these examples we can see that morphism can be defined as "correspondence" to object from category of another object, which can be constructed from it.
So we constructed category **Plane**. Note that any theorem in plane geometry can be seen as commutative diagram in category **Plane**. Let's give example if we consider Gauss line theorem : three midpoints of diagonals of any complete quadrilateral $ABCDEF$ lie on same line. We can consider complete quadrilaterals on plane as object $CQuad \in$ **Plane**. To any complete quadrilateral we can correspond three midpoints of it's diagonals and get morphism to triangles $\pi : CQuad \to \triangle$. Also we can to any complete quadrilateral correspond two midpoints of diagonals AC, BD and line EF and so we get morphism $\psi : CQuad \to \mathcal{A}$, where $\mathcal{A}$ is sets of two points and one line on $\mathbb{C}^2$. Consider morphism $\phi : \mathcal{A} \to \triangle$, where for any two points X, Y and line l, $\phi(\{X \cup Y \cup l\}) := \{X, Y, XY \cap l\} \in \triangle$. So Gauss theorem is equivalent to commutativity of next diagram :



Theorem 3.1. *There exists injective functor $\mathcal{F}$ between category* **Plane** *and category of algebraical varieties with rational morphisms between them, where to objects from* **Plane**, *functor $\mathcal{F}$ correspond it's irreducible algebraical varieties of parameters.*

Proof. We prove this theorem in particular case, general argument will be the same. Take for example morphism $\pi : \triangle \to Quad$, from first example of morphisms. Note that if we take parameters for each quadrangle on $\mathbb{C}^2$ then we get irreducible algebraical variety $X := \mathcal{F}(Quad) \cong \mathbb{C}^8$ of dimension 8, also parameters of triangles form irreducible algebraical variety $Y := \mathcal{F}(\triangle) \cong \mathbb{C}^6$ of dimension 6. Consider subvariety

$$Z := \{(x, y) \in X \times Y | \text{ quadrangle with coordinates } (x)$$

$$\text{is set of incenter and excenters for triangle with coordinates } (y)\} \subset X \times Y.$$

So easy to see that $\mathcal{F}(\pi)$ is projection morphism $Z \to X$, which is rational morphism of algebraical varieties. □

About every triangle we know from Kimberling Encyclopedia of triangle centers (ETC) (see [7]), that every such triangle has a lot of triangle centers X_i (Kimberling centers), also we know that every triangle has a lot of triangle curves. So we can define source $(\triangle, f_i)_I$, where morphisms f_i corresponds to considerations of finite set of triangle centers and curves $(X_{i_1}, X_{i_2}, \ldots, X_{i_N}, \mathcal{K}_{j_1}, \ldots, \mathcal{K}_{j_M})$ for triangles $ABC \in \triangle$ (which fits with definition of morphism). From theorem 3 we get that to every object from category **Plane** we can correspond it's dimension as dimension of functor $\mathcal{F}$ image i.e $\dim \triangle := \dim \mathcal{F}(\triangle) = 6$. Also we say that morphism ϕ between objects in **Plane** is dominant iff $\mathcal{F}(\phi)$ is dominant. So we get next table :

Plane geometry	Category **Plane**
Finite sets of points and curves of fixed type	Object
Triangles on $\mathbb{C}^2$	Object $\triangle$
Quadrangles on $\mathbb{C}^2$	Object Quad
Complete quadrangles on $\mathbb{C}^2$	Object $CQuad$
Consideration of related set of points and curves	Morphism
Triangle with it's centers and curves	Source $(\triangle, f_i)_I$
Circumcircle of triangle	Morphism f_ω
Theorem in plane geometry	Commutative diagram
Parameter space of plane figure	Functor $\mathcal{F}$ image

Also we have another table if we consider categories **Plane** and category **Var** of algebraical varieties with rational morphisms between them:

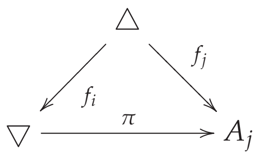
Plane geometry	Category **Plane**
Category **Plane**	Category **Var**
Object	Irreducible algebraical variety
Dimension of object	Dimension of functor $\mathcal{F}$ image
Morphism f between objects is dominant	Morphism $\mathcal{F}(f)$ is dominant

In the next section we will show how from object $(\triangle, f_i)_I$ and from **EF** we can construct example of **TG**. In fact we will see that object $(\triangle, f_i)_I$ can be seen as "unifying object", and many theorems in plane geometry have their "natural" places in it.

4. Constructing of $\mathbf{TG}_\triangle$ related to triangle geometry

Informal definition of $\mathbf{TG}_\triangle$ is next : consider any three different Kimberling centers X_i, X_j, X_k, then for most cases of (i, j, k) we can say that for any three points $X, Y, Z \in \mathbb{C}^2$ there exists some complex triangle ABC, such that $X = X_i(ABC), Y = X_j(ABC), Z = X_k(ABC)$, so we can try to find some theorems about arbitrary points X, Y, Z by using information that they can be seen as "good" triangle centers.
Now we give more formal definition of $\mathbf{TG}_\triangle$:

(1) First this $\mathbf{TG}_\triangle$ considers source $(\triangle, f_i)_I$ and chose one of morphisms f_i, which is morphism between objects $\triangle$ and $\bigtriangledown := A_i$, where $\dim \bigtriangledown = 6$.
(2) Second it uses **EF** and says that f_i is dominating (and it is true in most of cases of general position of f_i).
(3) Next it tries to construct morphism $f_j : \triangle \to A_j$, where f_j is some morphism from source $(\triangle, f_i)_I$, such that there exists morphism $\pi : \bigtriangledown \to A_j$, where next diagram is commutative :



(4) At this last part $\mathbf{TG}_\triangle$ uncovers theorem from arrow π and existence of commutative diagram, where f_i is dominating.

5. Examples of using $\mathbf{TG}_\triangle$

Here we will show in examples of how to use $\mathbf{TG}_\triangle$ to generate new theorems.

Problem 1. *Let given triangle ABC. Let circle ω goes through point C and has center at point A, let point D is intersection point of circles ω and circle (ABC), let circle ω_1 goes through points C, B and is tangent to line AB at point B. Let point E is intersection point of circle ω_1 with circle ω, let point F is reflection of point E wrt line AB, let line FC intersect circle (ABC) at point G. We prove that line AG is perpendicular to line FD.*

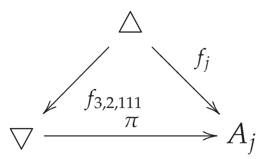
To get problem 1 :

(1) First consider object $\bigtriangledown (= \triangle)$ which is equivalent to sets of triangles on plane $\mathbb{C}^2$. Denote morphism $f_{3,2,111} : \triangle \to \bigtriangledown$, $f_{3,2,111}$ is morphism of consideration of points X_3, X_2, X_{111} — circumcenter, centroid and Parry point (Kimberling center X_{111}) of triangle $ABC \in \triangle$.
(2) One can check that $f_{3,2,111}$ is dominating (in general position).
Next we consider well known facts from triangle geometry to construct morphism f_j (see definition of $\mathbf{TG}_\triangle$). If we consider for every triangle ABC it's Kimberling centers $X_2, X_3, X_6, X_{23}, X_{111}, X_{691}, X_{476}$, so we get morphism

$$f_{2,3,6,23,111,691,476} : \triangle \to \mathcal{A}.$$

From [5, page 47] and [4, pages 199-200] we know that points X_2, X_3, X_6, X_{111} lie on same circle, points X_{476}, X_{111} lie on same circle (ABC) with center at X_3, points X_{23}, X_2, X_3 lie on same line, circle $(X_2X_{476}X_{111})$ is tangent to X_2X_3, reflection E of point X_{691} wrt line X_2X_3 is equivalent to reflection of point X_{691} wrt line X_3X_6 and is equivalent to intersection of circles $(X_2X_{23}X_{111})$ and (ABC).

(3) From previous easy to see that if we consider for triangle ABC it's lines X_3X_6 and EX_{691}, so we get morphism $f_j : \triangle \to A_j$, where A_j is object which corresponds to sets of two lines. From formulation of problem 1 we can see that consideration of lines AG and FD (for starting triangle ABC) can be seen as morphism $\pi : \bigtriangledown \to A_j$, where A_j is object of sets of two lines. From construction easy to check that next diagram is commutative :



(4) So from domination of $f_{3,2,111}$ we get that $\mathrm{Im}\pi \subseteq \mathrm{Im}f_j$, but $\mathrm{Im}f_j$ is sets of two orthogonal lines, so is $\mathrm{Im}\pi$. So information about π and commutativity of diagram gives to us that problem 1 is correct.

Problem 2. *Let given line l and two points X, Y on l, let lines x and y goes through points X, Y and are perpendicular to l, let conic $\mathcal{G}$ is tangent to l, x, y (at points L_1, X_1, Y_1), let circle ω with diameter XY intersect conic $\mathcal{G}$ at two points P, Q. Prove that tangents to conic $\mathcal{G}$ from points P and Q intersects on circle ω.*

To get problem 2 :

(1) Consider object $\bigtriangledown$ as sets of pairs: conic **c** and some point P on it. Easy to see that $\dim \bigtriangledown = 6$. Next if we consider for any triangle ABC it's De Longchamps conic $\mathcal{K}$ (see [6]), and tangent point of $\mathcal{K}$ with side AB then we get morphism $f_i : \triangle \to \bigtriangledown$.

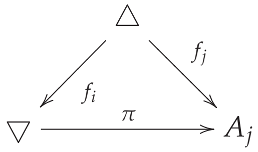
(2) One can check that f_i is dominating (in general position).

Next we consider well known facts from triangle geometry to construct morphism f_j (see definition of $\mathbf{TG}_\triangle$). From well known properties of

De Longchamps conic we know that for any triangle ABC if W — midpoint of AB, H — orthocenter of ABC, then circle **w** with center W and radius WH intersect line AB at two points P_1, P_2, where lines through points P_1, P_2, which are perpendicular to line AB are tangent to De Longchamps conic — $\mathcal{G}$. Also well known that $\mathcal{G}$ is tangent to sides of triangle ABC. Let $\mathcal{G} \cap AB = T$, $\mathcal{G} \cap \mathbf{w} = \{P, Q\}$, let AA', BB' be two altitudes of ABC. From well known properties of De Longchamps conic we get that P lies on AA', Q lies on BB' and $|AP| = |HA'|$, $|BQ| = |HB'|$. So lines through P and Q perpendicular to AA' and BB' respectively, meet on circle **w** at point V. From [6] we get that points P and Q are symmetric to tangent points of conic $\mathcal{G}$ with sides BC, CA wrt center of conic $\mathcal{G}$ respectively. So lines VP and VQ are tangent to conic $\mathcal{G}$. Also from [6] we know that line VT is perpendicular to line AB.

(3) From previous easy to see that if we consider for any triangle ABC circle **w** and point V, then we get morphism $f_j : \triangle \to A_j$, where A_j is object which corresponds to sets of circle and point on plane $\mathbb{C}^2$. In formulation of problem 2 we can see that

consideration of circle ω and intersecting point of tangent lines to conic $\mathcal{G}$ through points P and Q (for starting conic $\mathcal{G}$ and point L_1 on it) can be seen as morphism $\pi : \bigtriangledown \to A_j$. From previous constructions we can easily see that next diagram is commutative :

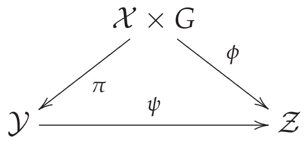


(4) From density of f_i we get that problem 2 is correct, because from commutativity of diagram we can see that $\mathrm{Im}\phi \subseteq \mathrm{Im}f_j$ and $\mathrm{Im}f_j$ are circles and points on them.

In the next examples of using $\mathbf{TG}_\triangle$ we will omit numeration of steps 1, 2, 3, 4 and write solutions in standard way (one can easily reformulate solutions of this problems in steps 1-4 language and see how to find this problems by using $\mathbf{TG}_\triangle$).

Problem 3. *Let given triangle ABC, let conic $\mathcal{G}$ is inscribed in ABC and conic $\mathcal{K}$ tangent to lines AB, AC at points B, C. Let conics $\mathcal{G}$ and $\mathcal{K}$ intersects at two points P, Q. We prove that tangents to conic $\mathcal{G}$ from points P, Q intersects on conic $\mathcal{K}$.*

Proof. Consider algebraical group $G := \mathbf{PGL}(\mathbb{C}^2)$ as group of projective transformations of plane $\mathbb{C}^2$. Consider object $\mathcal{X}$ as sets of conic $\mathcal{K}$ and lines l, x, y which are tangent to it and where $x||y$, x is perpendicular to l. Consider morphism $\pi : \mathcal{X} \times G \to \mathcal{Y}$, where $\mathcal{Y}$ is sets : conic and three lines which are tangent to it, $\pi((\mathcal{K}, l, x, y), g) := (g(\mathcal{K}), g(l), g(x), g(y)) \in \mathcal{Y}$. Easy to check that π is dominating. If for any set $(\mathcal{K}, l_1, l_2, l_3) \in \mathcal{Y}$ we construct conic $\mathcal{C}$ which is tangent to lines l_2, l_3 at points $l_1 \cap l_2, l_1 \cap l_3$, and consider intersection point of tangents from points $P, Q := \mathcal{K} \cap \mathcal{C}$ to conic $\mathcal{K}$, and also consider conic $\mathcal{C}$, then we get morphism $\psi : \mathcal{Y} \to \mathcal{Z}$, where $\mathcal{Z}$ is sets : point on plane and conic. Easy to check that if $\phi := \pi \circ (\psi, e)$, then next diagram is commutative :



From density of π we get that $\mathrm{Im}\psi \subseteq \mathrm{Im}\phi$ and from problem 2 we know that $\mathrm{Im}(\phi)$ is conics with points on them. ☐

Problem 4. *Let given two triangles ABC and $A_1B_1C_1$ such that ABC is similar to $A_1B_1C_1$ but has opposite orientation. Let given that ABC and $A_1B_1C_1$ have same orthocenters. Then we can prove that lines AA_1, BB_1, CC_1 intersects at same point.*

Proof. Denote objects $\triangle, \bullet \in$ **Plane** as set of triangles and set of points on $\mathbb{C}^2$. Consider object $\mathcal{A}$ as set of all similar triangles with opposite orientation and with same orthocenters. We can construct two morphisms $\pi_1, \pi_2 : \mathcal{A} \to \bullet$, where for any two triangles ABC and $A_1B_1C_1$ with same orthocenterss, π_1 is consideration of intersection point of lines AA_1, BB_1, like the same π_2 is consideration of intersection point of lines AA_1, CC_1.
For any element $(ABC, P) \in \triangle \times \bullet$, we can construct points $P', P'' \in \mathbb{C}^2$ such that triangles $APB, BP'C, CP''A$ are similar (and same oriented), also we can construct similar triangles $AQB, BQ'C, CQ''A$, where Q is symmetric to P wrt line AB, Q' is symmetric to P' wrt line BC, Q'' is symmetric to P'' wrt line CA. Let points $C^*, A^*, B^*, C^{**}, A^{**}, B^{**}$ be

circumcenters of triangles $APB, BP'C, CP''A$, $AQB, BQ'C, CQ''A$. Consider points $C_0 = AB \cap PC^*, A_0 = BC \cap P'A^*, B_0 = CA \cap P''B^*$. Easy to check that lines AA_0, BB_0, CC_0 intersects at same point X and that $|AX|/|XA_0| = |P'A^*|/|A^*A_0|$, so we get that $XA^*||AP'$. Like the same we can get that $XB^*||BP''$, $XC^*||CP$. Well known that line AP' — radical line for circles $(AQB), (CQ''A)$, so lines AP' and B^*C^* are orthogonal and so line XA^* is orthogonal to B^*C^*. So we get that point X is orthocenter for triangle $A^*B^*C^*$, like the same point X is orthocenter for triangle $A^{**}B^{**}C^{**}$. Also easy to check that triangles $A^*B^*C^*, A^{**}B^{**}C^{**}$ are similar and opposite oriented, so we can construct morphism $\psi : \triangle \times \bullet \to \mathcal{A}$ if we consider two triangles $A^*B^*C^*, A^{**}B^{**}C^{**}$ related to base triangle ABC. One can easily check that ψ is dominating.
Denote morphism $\pi_O : \triangle \to \bullet$, which takes for every triangle it's circumcenter. Note that from construction of ψ we can easily get that next diagrams are commutative :

We know that ψ is dominating, so $\pi_1 = \pi_2$ and this finishes the proof. □

Definition 3. *For points any points A, B, C, D, denote point $\mathcal{M}(AB, CD)$ as Miquel point of lines AC, DA, CB, DB.*

Problem 5. *Let given parabola $\mathcal{G}$ and six points A, B, C, D, E, F on it. Consider points $R = CE \cap BF$, $Q = CD \cap AF$, $P = BD \cap EA$. Prove that circle*

$$(\mathcal{M}(DR, QE)\mathcal{M}(QE, FP)\mathcal{M}(DR, FP))$$

is tangent to line RQP.

Proof. Consider object $\triangle$ as triangles on plane. Consider morphism $f : \triangle \times \triangle \to \triangle \times \triangle$, where

$$f(\{\text{triangle } ABC, \text{and points on plane } P, Q, R\}) := (P, P', Q, Q', R, R') \in \triangle \times \triangle,$$

where points P', Q', R' are isogonal to points P, Q, R wrt triangle ABC, then one can check that f — dominating morphism. So we can say that for some triangle XYZ on complex plane we have that pairs of points D, R; E, Q and F, P are isogonal wrt triangle XYZ. From [2, lemma 1] we get that circle

$$(\mathcal{M}(DR, QE)\mathcal{M}(QE, FP)\mathcal{M}(DR, FP))$$

is circumcircle of triangle XYZ. Note that isogonal conjuation of line RQP wrt XYZ is conic which goes through six points A, B, C, D, E, F, so it is equivalent to conic $\mathcal{G}$. As we know $\mathcal{G}$ — parabola, so line PQR is tangent to circle

$$(XYZ) = (\mathcal{M}(DR, QE)\mathcal{M}(QE, FP)\mathcal{M}(DR, FP)).$$

□

Problem 6. *Let given three segments AA', BB', CC'. Prove that next conditions are equivalent:*
a) midpoints of segments AA', BB', CC' lie on same line.
b) points $\mathcal{M}(AA', BB')$, $\mathcal{M}(BB', CC')$, $\mathcal{M}(AA', CC')$ lie on same line.

Proof. From same arguments as in problem 5 we can say that for some triangle XYZ on complex plane we have that pairs of points A, A'; B, B' and C, C' are isogonal wrt triangle XYZ. Name midpoints of segments AA', BB', CC' as M_A, M_B, M_C.
b) ⇒ a). Let given that points $\mathcal{M}(AA', BB')$, $\mathcal{M}(BB', CC')$, $\mathcal{M}(AA', CC')$ lie on same line l. From [2, lemma 1] we get that circle

$$(\mathcal{M}(AA', BB')\mathcal{M}(AA', CC')\mathcal{M}(BB', CC'))$$

is equivalent to line l and is circumsircle of triangle XYZ, so one of the point X, Y or Z should be infinite, let it be X. So from isogonality of pairs of points A, A'; B, B' and C, C' easy to see that then midpoints M_A, M_B, M_C lie on same line which is equal distant from lines XY, XZ.
a) ⇒ b). Let given that M_A, M_B, M_C lie on same line l. Consider point X_∞ — infinite point on line l. Then if we construct reflection of line $X_\infty Y$ wrt l and intersect it with reflection of line $X_\infty Y$ wrt angle bisector of $\angle AYA'$, then we get intersection point Z^*. And from isogonal conjuation theorem we get that pairs of points A, A'; B, B' and C, C' are isogonal wrt triangle $X_\infty YZ^*$. Circum circle of this triangle is equivalent to line YZ^*, because point X_∞ is infinite point. So from [2, lemma 1] we get that circle

$$(\mathcal{M}(AA', BB')\mathcal{M}(AA', CC')\mathcal{M}(BB', CC'))$$

is equivalent to line YZ^*. □

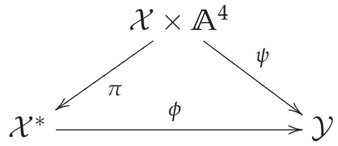
Problem 7. *Let given* 8 *conics on plane* $\mathbb{C}^2$*, name them as* $\mathcal{K}_1, \ldots, \mathcal{K}_8$*. Let given that for any numbers* $(i, j) \neq (4, 8)$*, quadrangles* $\mathcal{K}_i \cap \mathcal{K}_j$ *are all cyclic (their vertexes lie on circles). Then we can prove that quadrangle* $\mathcal{K}_4 \cap \mathcal{K}_8$ *is also cyclic.*

To get this problem we can take theorem from example 4 in part 2 of this article and intersect it with $2D$ planes P in $\mathbb{C}^3$, so we will get some pictures on planes P. So if we use dimension counting + **EF** then we will get plane geometry theorem $\mathbb{T}$ with same formulation as in problem 7.

Problem 8. *Let given conic* $\mathcal{K}$ *and points* A, B, C, D, E, F *on it, such that points* A, B, C, D *lie on same circle and points* A, B, E, F *lie on same circle. We prove that then* $CD||EF$*.*

Proof. Consider object $\mathcal{X}$ as pairs: sphere S and quadratic cone $\mathcal{C}$ with vertex on S in space $\mathbb{C}^3$. Easy to see that for any such pair $(S, \mathcal{C}) \in \mathcal{X}$ we can consider quadratic polynomes $F_S, F_\mathcal{C} \in \mathbb{C}[x, y, z]$, such that $S = \mathbb{V}(F_S), \mathcal{C} = \mathbb{V}(F_\mathcal{C})$. So for any quadratic polynomial $F_\lambda := F_S + \lambda F_\mathcal{C}, \lambda \in \mathbb{C}$, we get that $S \cap \mathcal{C} = \mathbb{V}(F_\lambda) \cap S$. Note that $\mathcal{C}_\lambda := \mathbb{V}(F_\lambda)$ is quadratic surface which is tangent to sphere S at vertex V of cone $\mathcal{C}$. Also we know that any such quadratic surface $\mathcal{C}_\lambda$ has two lines bundles on it, denote them as $\mathcal{L}_{\lambda,1}, \mathcal{L}_{\lambda,2}$.
For any pair $(S, \mathcal{C}) \in \mathcal{X}$, construct plane P which goes through center of S and is parallel to tangent plane from vertex V of cone $\mathcal{C}$ to sphere S, also construct intersection conic $\mathcal{K}$ of plane P with cone $\mathcal{C}$. Denote ρ — inversion wrt sphere with center at point V, which sends sphere S to plane P. Chose any parameters $\lambda, t, s, v \in \mathbb{C}$ and three points X, Y, Z on conic $\mathcal{K}$ with "coordinates" t, s, v. Also construct points X_1, Y', Z', where $\rho(X_1) \in S \cap \mathcal{C} = \rho(\mathcal{K})$ is second intersection point of line from $\mathcal{L}_{\lambda,1}$, which goes through point $\rho(X)$, with curve $\rho(\mathcal{K})$ and $\rho(Y'), \rho(Z')$ are second intersection points of lines from $\mathcal{L}_{\lambda,2}$, which goes through points $\rho(Y), \rho(Z)$, with curve $\rho(\mathcal{K})$. From construction of points X, X^*, Y, Y', Z, Z' we know that $\rho(X)\rho(X^*) \in \mathcal{L}_{\lambda,1}, \rho(Y)\rho(Y') \in \mathcal{L}_{\lambda,2}$, $\rho(Z)\rho(Z') \in \mathcal{L}_{\lambda,2}$, so lines $\rho(Y)\rho(Y')$, $\rho(X)\rho(X')$ has intersection point, so points Y, Y', X, X^* lie on same

circle (because ρ transform circles on sphere S to circles on plane P). Like the same points X, X^*, Z, Z' lie on same circle. We know that quadric $\mathcal{C}_\lambda$ is tangent to sphere S, so lines l_1, l_2 from $\mathcal{L}_{\lambda,1}, \mathcal{L}_{\lambda,2}$, which goes through point P are tangent to sphere S. We know that line $\rho(Y)\rho(Y')$ intersect line l_1, so $YY'||l_1$, like the same $ZZ'||l_1$, so $YY'||ZZ'$.
Consideration for pair $(S, \mathcal{C}) \in \mathcal{X}$ and any set $(\lambda, t, s, v) \in \mathbb{A}^4$, of conic $\mathcal{K}$ and four points X, X^*, Y, Z (see previous construction) gives to us morphism $\pi : \mathcal{X} \times \mathbb{A}^4 \to \mathcal{X}^*$, where $\mathcal{X}^*$ is object : conic and four points on it. Also we can consider for any element from $\mathcal{X} \times \mathbb{A}^4$, conic $\mathcal{K}$ and 4 points on it Y, Y', Z, Z' (see definition of this objects in previous construction) and get another morphism $\psi : \mathcal{X} \times \mathbb{A}^4 \to \mathcal{Y}$, where $\mathcal{Y}$ is object : conic and 4 points on it. Consider for any conic $\mathcal{U}$ and four points X_1, X_2, X_3, X_4 on it, set of 4 points $\{X_1, X_2, X_3', X_4'\} \subseteq \mathcal{U}$, such that X_1, X_2, X_3, X_3' lie on same circle and X_1, X_2, X_4, X_4' lie on same circle, also consider conic $\mathcal{K} := \mathcal{U}$, so we get morphism $\phi : \mathcal{X}^* \to \mathcal{Y}$. From constructions we get that next diagram is commutative :



Easy to check that π is dominating (because $\dim \mathcal{X} \times \mathbb{A}^4 = \dim \mathcal{X}^*$ + **EF**), so it finishes the proof, because $\mathrm{Im}(\phi) \subseteq \mathrm{Im}(\psi)$ is conics with points C, D, E, F on it, where $CD||EF$. □

Consequence 1. *Note that if we add projective transformations to problem 8, then from* **EF** *we will get three conics theorem (see formulation here [9, page 116]).*

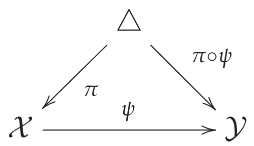
Problem 9. *In notions of problem 7 prove that center of conic $\mathcal{K}$, midpoint of segment AB and point $\mathcal{M}(CD, EF)$ lie on same line.*

Proof. Solution comes from next observation : from existence of some dominating morphism $\pi : \triangle \times \ldots \to \ldots$ (easy to find it), we can say that for some triangle XYZ (where X is infinite point) on $\mathbb{C}^2$, conic $\mathcal{K}$ with infinite line is isogonal cubic for triangle ABC, point C is isogonal to point D, point E is isogonal to point F, so $\mathcal{M}(CD, EF) \in YZ$. Also well known from triangle geometry that midpoint of segment AB lies on line YZ and center of conic $\mathcal{K}$ lies on line YZ. So this finishes the proof. □

Problem 10. *Let given conic $\mathcal{G}$ with center at point O and with two diameters AA', BB' on it. Let ω_a, ω_b be two circles with diameters AA', BB' respectively. Let circle ω is tangent to circles ω_a, ω_b at points A_1, B_1. Consider conic $\mathcal{K}$ which has one of it's focuses at point O and which is tangent to circle ω at points A_1, B_1. Then one of the common tangent lines of two conics $\mathcal{G}, \mathcal{K}$ goes through one of the intersection points of circle ω with conic $\mathcal{G}$.*

Proof. From triangle geometry we knows that M'Cay cubic $\mathbf{Z} := \mathbf{Z}(X_3)$ is a self-isogonal cubic given by the locus of all points whose pedal circle touches the nine-point circle, or equivalently, the locus of all points P for which P, the isogonal conjugate P' of P, and the circumcenter X_3 of a reference triangle ABC are collinear. Well known that center of De Longchamps conic $\mathcal{Q}$ is equivalent to circumcenter X_3 of triangle ABC. So if we use this cubic to De Longchamps conic $\mathcal{Q}$ of triangle ABC, then we get that all 4 focuses of conic $\mathcal{Q}$ (2 real and two complex in case when conic $\mathcal{Q}$ is real) lie on M'Cay cubic $\mathbf{Z}$. So two pedal circles ω_1, ω_2 of this 4 focuses has diameters same as two diameters of De

Longchamps conic $\mathcal{Q}$ of triangle ABC and are tangent to nine-point circle of triangle ABC. Denote circle ω as reflection of nine-point circle wrt perpendicular bisector of segment AB. So easy to check that circle ω is also tangent to circles ω_1, ω_2. Let CC' be C – altitude of triangle ABC. From [6] we know that tangent point P of conic $\mathcal{Q}$ with line AB is symmetric to point C' wrt midpoint M of segment AB. So $P \in \omega$ and tangent line from point P to conic $\mathcal{G}$ is equivalent to line AB and goes through point $M \in \omega$. Point X_3 is circumcenter of triangle ABC, so line X_3M is perpendicular to line PM.
Consider for any triangle ABC it's De Longchamps conic $\mathcal{G}$, circles $\omega_1, \omega_2, \omega$ and three points P, M, X_3 (see previous construction). So we get morphism $\pi : \triangle \to \mathcal{X}$, where $\mathcal{X}$ is next set of curves : $(\mathcal{G}, \omega_1, \omega_2, \omega_3, A, B, C)$, where $\omega_1, \omega_2, \omega$ is three circles, $\mathcal{G}$ is conic which is tangent to circles ω_1, ω_2 at it's diameters, circle ω_3 is tangent to circles ω_1, ω_2 at points X, Y, point C is center of circles ω_1, ω_2 and is center of conic $\mathcal{G}$, point A lie on conic $\mathcal{G}$, point B lie on circle ω_3, tangent line through point A to conic $\mathcal{G}$ goes through point B and is perpendicular to line BC. Well known from properties of conics that conic $\mathcal{C}$ with focus at point C and which is tangent to circle ω at points X, Y also is tangent to line AB. Consider for any element $(\mathcal{G}, \omega_1, \omega_2, \omega_3, A, B, C) \in \mathcal{X}$ set of curves $(\mathcal{G}, \omega_1, \omega_2, \omega_3)$, so we get morphism $\psi : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{Y}$ is sets of two circles ω_1, ω_2 with same centers, of the conic which is tangent to circles ω_1, ω_2 at their two diameters and of third circle ω_3, which is tangent to circles ω_1, ω_2. From **EF** we can get that morphism $\pi \circ \psi$ is dominant. So we get next commutative diagram :



And one can easily check that from domination of morphism $\pi \circ \psi$ and from existence of morphism ψ we get end of the proof. □

From proof of problem 8 one can see that we can construct **TG** similar to $\mathbf{TG}_\triangle$, which can be applied to geometry of quadratic surfaces. Also easy to generalize $\mathbf{TG}_\triangle$ to **TG** which deals with different other constructions from basic algebraical geometry.

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