THE SQUARE OF THE ARBELOS

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Abstract. A square with two points on each of the sides are naturally constructed from an arbelos. It has relationships to the other squares associated with the arbelos and special type of generalized arbelos, and yields several dozens of Archimedean circles.

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1. INTRODUCTION

Let us consider an arbelos \((a, b, g)\) consisting of three semicircles \(a, b\) and \(g\) with diameters \(AO, BO\) and \(AB\), respectively for a point \(O\) on the segment \(AB\) in the plane. Let \(a\) and \(b\) be the radii of \(a\) and \(b\), respectively. Circles of radius \(r_A = \frac{ab}{a+b}\) form a special class of circles and are called Archimedean circles of the arbelos. In this paper we consider a square with two points on each of the sides. We show that it has relationships to the squares associated with the arbelos considered in \([7, 8]\) and to a special type of a generalized arbelos called the arbelos with overhang \([9]\) and yields several dozens of Archimedean circles. Special cases where \(b/a = (\sqrt{n^2 + 4} \pm n)/2\) for \(n = 1, 2\) are investigated.

2. THE SQUARE

In this section we construct a square from the arbelos. We use a rectangular coordinate system with origin \(O\) such that the points \(A\) and \(B\) have coordinates \((2a, 0)\) and \((-2b, 0)\), respectively, where we assume that all the semicircles \(a, b\) and \(g\) are constructed in the region \(y \geq 0\). Let \(P_1\) and \(S_1\) be the centers of \(a\) and \(b\), respectively, and let \(C\) be the farthest point on the circle with a diameter \(P_1S_1\) from the line \(AB\) in the region \(y > 0\). Let \(\sigma\) be the rotation about \(C\) through 90°. The center of the semicircle \(\gamma\) is denoted by \(S_2\) and let \(S_3 = O\) (see Figure 1). Then we get \(P_1 = S_1^\sigma\), and let \(P_2 = S_2^\sigma\), \(P_3 = S_3^\sigma\) and \(Q_i = P_i^\sigma, R_i = Q_i^\sigma\) for \(i = 1, 2, 3\). Then \(P_iQ_iR_iS_i\) is a square with center \(C\), and \(|S_1S_2| = a, |S_2P_1| = b\) and \(|S_2S_3| = |a - b|\).

We call \(P_1Q_1R_1S_1\) and \(C\) the square and the center of \((a, b, \gamma)\). Notice that the lines \(P_2R_3, P_3R_2\) and \(Q_1R_1\) are the tangents of \(a, b\) and \(\gamma\) parallel to \(AB\) with points of tangency \(P_2, R_2\) and \(Q_3, S_3Q_2, S_3Q_2\) and \(P_1Q_1\) are the perpendiculars from \(R_1, Q_3, Q_2\) and \(Q_1\) to \(AB\). The points \(P_1, P_2\) and \(P_3\) (resp. \(P_3\) and \(P_2\)) lie on \(P_1Q_1\) in this order if \(a < b\) (resp. \(a > b\)).
In this section we consider relationships between the squares $P_iQ_iR_iS_i$ ($i = 1, 2, 3$) and the squares considered in [7, 8]. Let the line $P_2R_2$ meet $a$ and $b$ at points $P$ and $R$ again, respectively (see Figure 2). Then the lines $AP$ and $BR$ meet in a point $Q$ lying on $\gamma$ and $OPQR$ is the inscribed square of $(a, b, \gamma)$ [8]. The points $P, Q$ and $R$ have coordinates 

$$
(2jb, 2ja), (-2jd, 2je) \text{ and } (-2ja, 2jb),
$$

respectively, where $d = a - b, e = a + b$ and $j = ab/(a^2 + b^2)$. Let $O'$ be the point of intersection of $OQ_2$ and $\gamma$. Let $Q$ and $R$ be the points on the segments $AO'$ and $BO'$, respectively such that $PQRS$ is a square, where $P$ and $S$ are the feet of perpendiculars from $Q$ and $R$ to $AB$, respectively. Then $Q$ and $R$ have coordinates $(2af/g, 2ef/g)$ and $(-2bf/g, 2ef/g)$, respectively, where $f = \sqrt{ab}$ and $g = e + f$ [7].

Theorem 3.1. The following statements hold.

(i) If $P_3', Q_3'$ and $R_3'$ are the reflections of the points $P_3, Q_3$ and $R_3$ in the line $AB$, respectively, then the squares $OPQR$ and $O'R_3'Q_3'P_3'$ are homothetic with center $O$ and ratio $2ab : (a^2 + b^2)$.

(ii) If $a \neq b$ and $E$ is the point of intersection of the lines $AB$ and $P_2R_2$, the squares $OPQR$ and $S_2P_2Q_2R_2$ are homothetic with center $E$ and ratio $2ab : (a^2 + b^2)$.

(iii) The squares $P_1Q_1R_1S_1$ and $P'Q'R'S$ are homothetic with center $O$ and ratio $g : 2f$. 
The following statements are equivalent for a non-negative real number $n$.

(i) The part (ii) follows from (i) and the fact $E$ being the homothety center of the triangles $OPR$ and $S_2P_3R_2$. The part (iii) follows from $O = (g/(g-2f))Q + (-2f/(g-2f))Q_1 = (g/(g-2f))R + (-2f/(g-2f))R_1$. \hfill \square

The part (ii) shows that the point $Q_2$ lies on the line $EQ$, which is the tangent of $\gamma$ with point of tangency $Q$ [8]. Let $\Phi(n) = \frac{1}{2} \left( \sqrt{n^2 + 4} + n \right)$ for a real number $n$. It is called a metallic mean if $n$ is a natural number [11]. The first part of the following proposition follows from (1), and the last part is obvious.

**Proposition 3.1.** The slopes of the lines $OP$ and $OR$ equal $a/b$ and $-b/a$, respectively. If $n$ is a non-negative real number, then $|d| = nr_A$ if and only if $b/a = \Phi(n)^{\pm 1}$.

Since $\Phi(n)$ is a monotonically increasing function of $n$ and $\Phi(0) = 1$, there exists a non-negative real number $n$ such that $b/a = \Phi(n)$ if and only if $b \geq a$.

**Theorem 3.2.** The following statements are equivalent for a non-negative real number $n$.

(i) $b/a = \Phi(n)$.
(ii) The point $R_3$ divides the segment $QR$ in the ratio $(2 + n) : n$ externally.
(iii) The point $P_3$ divides the segment $PQ$ in the ratio $n : |2 - n|$ internally if $n < 2$ externally if $2 < n$, and coincides with $Q$ if $n = 2$.

**Proof.** The part (i) is equivalent to each of the equations $R_3 = ((2 + n)/2)R + (-n/2)Q$ and $P_3 = ((2 - n)/2)P + (n/2)Q$ being true. \hfill \square

4. AN ARBELOS WITH OVERHANG

In [9] we considered a generalization of the arbelos, called an arbelos with overhang. In this section we show that the square of $(a, b, \gamma)$ is closely related to a special type of arbelos with overhang. Let $A_h$ and $B_h$ be on the line $AB$ with $x$-coordinates $a'$ and $-b'$, respectively such that $a' - a = b' - b = h > -\min(a, b)$. Let $a_h$ and $b_h$ be the semicircles with diameters $A_hO$ and $B_hO$, respectively erected on the same side of $AB$ as $\gamma$. The configuration of the three semicircles $a_h, b_h$ and $\gamma$ is called an arbelos with overhang $h$ and is denoted by $(a_h, b_h, \gamma)$ [9]. If $h > 0$, let $\varepsilon_{a,1}$ (resp. $\varepsilon_{b,1}$) be the circle touching $\gamma$ and $a_h$ (resp. $b_h$) internally and $\alpha$ (resp. $\beta$) externally, also let $\varepsilon_{a,2}$ (resp. $\varepsilon_{b,2}$) be the circle touching $\gamma$ and $a_h$ (resp. $b_h$) externally and the axis from the side opposite to $B$ (resp. $A$). We use the next proposition.

**Proposition 4.1** ([9]). If $h > 0$, the following statements are true for $(a_h, b_h, \gamma)$.

(i) The circles $\varepsilon_{a,1}$ and $\varepsilon_{b,1}$ are congruent and have radius $(1/r_A + 1/h)^{-1}$.
(ii) The circles $\varepsilon_{a,2}$ and $\varepsilon_{b,2}$ are congruent and have radius $ab/h$.

Let us consider an arbelos with overhang $(a_h, b_h, \gamma)$ with $h = r_A$. The radius of the circle touching $a$ externally and $\gamma$ internally is proportional to the distance between the center of this circle and the radical axis of $a$ and $\gamma$ [3, p. 108]. Therefore the circle $\varepsilon$ touching $a$ externally $\gamma$ internally and $P_1Q_1$ from the side opposite to $B$ has radius $r_A/2$. While the circle $\varepsilon_{a,1}$ has radius $r_A/2$ by (i) of the proposition. Therefore the circles $\varepsilon$ and $\varepsilon_{a,1}$ coincide, i.e., $\varepsilon_{a,1}$ touches $P_1Q_1$ from the side opposite to $B$. Similarly the circle $\varepsilon_{b,1}$ touches $R_1S_1$ from the side opposite to $A$ (see Figure 3). The circles $\varepsilon_{a,2}$ and $\varepsilon_{b,2}$ have radius $e$ by (ii) of the proposition, i.e., they are congruent to $\gamma$. Since the centers of $\gamma$ and $\varepsilon_{a,2}$ have $x$-coordinates $d$ and $e$, their point of tangency, which is the midpoint of the
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segment joining their centers, has x-coordinate \( a \), i.e., the point lies on \( P_1Q_1 \). Similarly

the point of tangency of \( \gamma \) and \( \varepsilon_{a,2} \) lies on \( R_1S_1 \). The perpendicular to \( AB \) at the center

of \( \alpha_h \) (resp, \( \beta_h \)) also touches \( \varepsilon_{a,1} \) (resp, \( \varepsilon_{b,1} \)). If we erect a square on the same side of \( AB \)

as \( \gamma \) with a side joining the centers of \( \alpha_h \) and \( \beta_h \), the distance between \( Q_1R_1 \) and the

side of this square parallel to \( AB \) equals \( 2r_A \). Therefore circles touching the two sides are

Archimedean circles of \((a, \beta, \gamma)\).

Figure 3.

5. QUADRUPLETS AND OCTUPLETS OF ARCHIMEDEAN CIRCLES

In this section we construct quadruplets and octuplets of Archimedean circles. Let \( T = OQ_2 \cap P_2R_3 \) (which denotes the point of intersection of the lines \( OQ_2 \) and \( P_2R_3 \)) and

\( U = P_1Q_2 \cap P_2R_3 \). Then \( |TU| = r_A \) by the similar triangles \( Q_2OP_1 \) and \( Q_2TU \), (see Figure

4). Similarly the distances from \( T \) to the points \( OQ_2 \cap P_1R_3 \) and \( OR_1 \cap P_2R_3 \) equal \( r_A \). Hence the circle with center \( T \) and passing through those points of intersection is Archimedean and is denoted by \( C_1 \). Similarly we get Archimedean circles \( C_2, C_3 \) and

\( C_4 \) with centers \( P_3R_2 \cap OQ_3 \), \( Q_3S_2 \cap R_3P_3 \) and \( R_3P_2 \cap S_2Q_3 \), which also coincide with \( C_1^\gamma \),

\( C_1^\alpha \) and \( C_1^\beta \), respectively (see Figure 5). The result that the distances from the points

\( P_3R_2 \cap OQ_1 \) and \( OR_1 \cap P_2R_3 \) to \( OQ_2 \) being equal to \( r_A \) can be seen in [6].

Figure 4.

Figure 5.
The circle $OQ_1$, $P_1Q_3$ and $P_2R_3$ meet in a point $V$ and the distances from $V$ to the points $P_2$ and $OQ_3 \cap P_2R_3$ equal $r_A$ (see Figure 6). The facts are also proved by similar triangles. Therefore the circle with center $V$ passing through $P_2$ and $OQ_3 \cap P_2R_3$ is Archimedean, and is denoted by $D_1$. The endpoints of the diameter of $D_1$ parallel to $P_1Q_1$ lie on the lines $OP_2$ and $P_2Q_3$ as shown by the dotted lines in Figure 6. Similarly we get an Archimedean circle $E_1$ touching $P_1Q_1$ at $P_3$ from the side opposite to $A$. Also we get Archimedean circles $D_2$, $E_2'$; $D_3$, $E_3$ and $D_4$, $E_4$ touching $Q_1R_1$, $R_1S_1$ and $S_1P_1$, respectively, where $D_i = D_i^{i-1}$ and $E_i = E_i^{i-1}$ for $i = 2, 3, 4$. We now get octuplets of Archimedean circles (see Figure 7). The circle $E_4$ is the Bankoff triplet circle [2]. Since the circles $D_4$ and $E_4$ touch the circle with a diameter $S_1P_1$ [5], the pairs of circles $D_1$, $E_1$; $D_2$, $E_2$; $D_3$, $E_3$ also touch the circles with diameters $P_1Q_1$, $Q_1R_1$ and $R_1S_1$, respectively. The circles $D_1$ and $D_3$ can be seen in [6].

6. A CONFIGURATION OF TWELVE CONGRUENT CIRCLES

In this section we assume that $\delta_1$ and $\delta_2$ are congruent circles with centers $F$ and $G$, respectively and have a point $H$ in common, and $\epsilon$ is a circle passing through $H$ intersecting $\delta_1$ and $\delta_2$ again at points $I$ and $J$, respectively. We show that $\delta_1$ and $\delta_2$ yield a configuration of twelve congruent circles if $FIJG$ is a parallelogram.

**Theorem 6.1.** The circle $\epsilon$ is congruent to $\delta_1$ if and only if $FIJG$ is a parallelogram.

**Proof.** Let $K$ be the center of $\epsilon$ (see Figure 8). If $\epsilon$ and $\delta_1$ are congruent, $FIKH$ and $GJKH$ are rhombuses with side $HK$ in common, i.e., $FIJG$ is a parallelogram. Conversely if $FIJG$ is a parallelogram, let $K'$ be the point such that $HK' = F\bar{T}$. Then $FIK'H$ and $GJK'H$ are rhombuses and $|K'H| = |K'I| = |K'J|$. Hence $K' = K$ and $\epsilon$ is congruent to $\delta_1$. \hfill $\Box$

Let $H'$ be the remaining point of intersection of $\delta_1$ and $\delta_2$, and let $\epsilon'$ be the circle passing through $I$, $J$ and $H'$. If $\delta_1$, $\delta_2$ and $\epsilon$ are congruent, $FIJG$ is a parallelogram. Therefore $\epsilon'$ is also congruent to $\delta_1$. Hence we get Johnson’s circle theorem [4]:

**Corollary 6.1.** If three congruent circles meet in a point, the circle passing through their other three points of intersection is also congruent to the three.

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We now assume $\varepsilon$ is congruent to $\delta_1$. Since the three congruent circles $\delta_1$, $\varepsilon$ and $\varepsilon'$ meet in $I$, the circle with center $I$ passing through the centers of the three circles is congruent to the three. Similarly, we can get circles congruent to $\delta_1$ with centers $J$, $H$ and $H'$ (see Figure 9). Now we get a configuration of eight congruent circles. Since each of $\varepsilon$ and the circle with center $I$ passes through the center of the other, the circle with center at one of the points of intersection of the two circles and passing through their centers is also congruent to the two. But such a circle is trivial, and is excluded to consider.

Let $K$ and $L$ be the points such that $IK$ and $JL$ are diameters of of $\delta_1$ and $\delta_2$. Then we get four more circles congruent to $\delta_1$: the two circles passing through $K$, $L$ and $H$, and $K$, $L$ and $H'$, and the two circles with centers $K$ and $L$. Hence we get a configuration of twelve congruent circles (see Figure 10). If $FIJG$ is a rectangle, we denote the configuration by $(\delta_1, \delta_2)$ (see Figure 11). Let us assume that $F$, $I$ and $G$ have coordinates $(p, s)$, $(p + r, s)$ ($r > 0$) and $(p, t)$, respectively in this case. Then $J$, $K$ and $L$ have coordinates $(p + r, t)$, $(p - r, s)$ and $(p - r, t)$, respectively. And the three pairs of circles passing through $F$, $G$; $I$, $J$; $K$, $L$ have centers with coordinates $(p \pm v, u)$, $(p + r \pm v, u)$ and $(p - r \pm v, u)$, respectively, where $u = (s + t)/2$ and $v = \sqrt{r^2 - (s - u)^2}$.
7. Configurations of Archimedean Circles

In this section, we show that the quadruplets and the octuplets of Archimedean circles in section 5 yield configurations of several dozens of Archimedean circles if the two circles $C_1$ and $C_2$ ($D_1$ and $E_1$) intersect. Hence we assume $|d| < 2r_A$, where recall $d = a - b$.

\[
\begin{array}{|c|c|}
\hline
(C_1, C_2) & (0, a), (\pm r_A, a); (0, b), (\pm r_A, b); (\pm v, e/2), (\pm r_A \pm v, e/2) \\
(C_3, C_4) & (d, b), (d \pm r_A, b); (d, a), (d \pm r_A, a); (d \pm v, e/2), (d \pm r_A \pm v, e/2) \\
(C_4, C_1) & (d, a), (d, a \pm r_A); (0, a), (0, a \pm r_A); (d/2, a \pm v), (d/2, a \pm r_A \pm v) \\
\hline
\end{array}
\]

Table 1.

We consider the configuration of twelve Archimedean circles ($C_1, C_2$). Recall $e = a + b$, and let $v = \sqrt{r_A^2 - d^2/4}$. The coordinates of the centers of the circles in this configuration are indicated in the first row of Table 1. The second row and the third row indicate the coordinates of the centers of the circles in ($C_3, C_4$) and ($C_4, C_1$), respectively. The table shows that each of the centers of the circles in the configuration $(C_1, C_2) \cup (C_2, C_3) \cup (C_3, C_4) \cup (C_4, C_1)$ except $C_1, C_2, C_3$ and $C_4$ is different from the others in general. Therefore it consists of forty-four distinct Archimedean circles, which we denote by $C_{44}$ (see Figure 12). The circles $C_1, C_2, C_3$ and $C_4$ are indicated in black and the circles with centers on the perpendicular bisectors of $P_2P_3$ and $Q_2Q_3$ are indicated in red, and the remainings are indicated in green.

\[
\begin{array}{|c|c|}
\hline
(D_1, E_1) & (a - r_A, a), (a - r_A \pm r_A, a); (a - r_A, b), (a - r_A \pm r_A, b); \\
& (a - r_A \pm v, e/2), (a - r_A \pm r_A \pm v, e/2) \\
(D_3, E_3) & (-b + r_A, b), (-b + r_A \pm r_A, b); (-b + r_A, a), (-b + r_A \pm r_A, a); \\
& (-b + r_A \pm v, e/2), (-b + r_A \pm r_A \pm v, e/2) \\
(D_4, E_4) & (d, r_A), (d, r_A \pm r_A); (0, r_A), (0, r_A \pm r_A); (d/2, r_A \pm v), (d/2, r_A \pm r_A \pm v) \\
\hline
\end{array}
\]

Table 2.

The coordinates of the centers of the circles in ($D_1, E_1$) are indicated in the first row of Table 2. The second row and the third row indicate the coordinates of the centers of the circles in ($D_3, E_3$) and ($D_4, E_4$). Since those centers are different in general, the configuration $\bigcup_{i=1}^4 (D_i, E_i)$ consists of forty-eight distinct Archimedean circles, which we denote by $C_{48}$ and indicate in Figure 13 in a similar way to as Figure 12.
8. Golden arbelos

We consider the case in which the Archimedean circle \( C_1 \) touches the line \( Q_3S_2 \). It is equivalent to \( a/b = \Phi(1) \pm 1 \) by Proposition 3.1. In this case \( (a, \beta, \gamma) \) is called a golden arbelos [1]. The equivalence of (i), (ii) and (iii) of the next theorem follows from Theorem 3.2 (see Figure 14). The proof of the equivalence of (i), (iv) and (v) is straightforward and is omitted.

**Theorem 8.1.** The following statements are equivalent.

(i) \( b/a = \Phi(1) \).

(ii) The point \( R_3 \) divides \( QR \) in the ratio \( 3 : 1 \) externally.

(iii) The point \( P_3 \) is the midpoint of \( PQ \).

(iv) The points \( R_1 \) and \( R_3P_2 \cap Q_3S_2 \) lie on \( OR \).

(v) The points \( Q_1 \) and \( Q_3S_2 \cap R_2P_3 \) lie on \( BQ \).

![Figure 14](image1)

![Figure 15: C_{44} in a golden arbelos](image2)

We now assume \( b/a = \Phi(1) \). The circle touching \( \gamma \) internally \( \beta \) externally and \( OQ_2 \) from the side opposite to \( A \) is well-known as being Archimedean and the point of tangency of this circle and \( \beta \) has coordinates \((-2ab/k, 2b\sqrt{k}/k)\), where \( k = 2a + b \) in general case. Therefore the circle \( E_2 \) coincides with this circle and touches \( \beta \) (see Figure 14), and the point \( R \) has coordinates \((-2a/\sqrt{5}, 2b/\sqrt{5})\) and coincides with the point of tangency of \( \beta \) and \( E_2 \). Since \( R \) is the homothetic center of \( \beta \) and \( E_2 \), the endpoints of the diameter of \( E_2 \) parallel to \( AB \) lie on the lines \( BR \) and \( OR \). Let \( W = Q_3S_2 \cap R_2P_3 \). Since \( Q \) is the homothetic center of the triangles \( P_3QW \) and \( AQB \), the circumcircle of \( P_3QW \) touches \( \gamma \) at \( Q \). While the inscribed circle of the arbelos has radius \( abe/(a^2 + ab + b^2) \) [10], and touches \( \gamma \) at \( Q \) [8] in general case. Hence it has radius \( b/2 \) in this case, and coincides with the circumcircle of \( P_3QW \), because \(|P_3W| = b\). Therefore the inscribed circle of the arbelos touches the segments \( P_3Q_1 \) and \( Q_3S_2 \) at \( P_3 \) and \( W \). The circle \( D_1 \) also touches \( \beta \). Several golden rectangles appear in the figure, one of which is \( P_1P_3R_2S_1 \).

Each of the two circles passing through \( F \), \( G \) and \( K \), \( L \) in \( (C_1, C_2) \) are also members of \( (C_3, C_4) \) (see Figure 11). Hence there are eight distinct circles with centers on the perpendicular bisector of \( P_2P_3 \) in \( (C_1, C_2) \cup (C_3, C_4) \). Also there are four distinct circles with centers on each of \( P_2R_3 \) and \( P_3R_2 \) in \( (C_1, C_2) \cup (C_3, C_4) \). Hence \( (C_1, C_2) \cup (C_3, C_4) \) consists of sixteen distinct circles, four of which are \( C_1, C_2, C_3, C_4 \). Therefore \( C_{44} \) consists of twenty-eight distinct Archimedean circles (see Figure 15).
9. **Silver Arbelos**

We consider the case where the Archimedean circles $C_1$ and $C_2$ touch. It is equivalent to $b/a = \Phi(2)^{\pm 1}$. Following to $\Phi(2)$ being called the silver mean [11], we call $(a, b, \gamma)$ a silver arbelos in this case. This case is also considered in [8].

**Theorem 9.1.** The following statements are equivalent.

(i) $b/a = \Phi(2)$.

(ii) The point $P_3$ coincides with $Q$.

(iii) The point $R$ coincides with the center of the arbelos.

**Proof.** Let us assume that the part (iii) holds. Then $P_3, R_3$ and $R$ are collinear. Since $R$ and $R_3$ lie on $BQ$ by (ii) of Theorem 3.2, $P_3$ lies on $BQ$. While $P_3$ also lies on $AQ$ by (iii) of the same theorem. Therefore $P_3 = Q$, i.e., (iii) implies (ii). The rest of the theorem follows from Theorem 3.2. □

We now assume $b/a = \Phi(2)$. Since the slopes of the line $OR$ equals $-\Phi(2)$ by Proposition 3.1, the point $R$ is the midpoints of the arc $OR$ of the semicircles $\beta$. Also $P$ and $Q$ are the midpoints of the arcs $AP_2$ and $AQ_3$ of $\alpha$ and $\gamma$ by the similarity. The circle with center $P$ touching $AB$ and $P_1Q_1$ is Archimedean. Several rectangles with aspect ratio $1 : \sqrt{2}$, which are sometimes called silver rectangles, appear in the figure, one of which is $P_1P_3R_2S_1$.

![Figure 16: $C_{44}$ in a silver arbelos](image1)

![Figure 17: $C_{48}$ in a silver arbelos](image2)

If the circles $\delta_1$ and $\delta_2$ touch, the circles in each pairs of the two circles passing through $I$ and $J$, $F$ and $G$, and $K$ and $L$ in $(\delta_1, \delta_2)$ coincide (see Figure 11). Therefore $(\delta_1, \delta_2)$ consists of nine congruent circles. Especially if $\delta_1 = C_1$ and $\delta_2 = C_2$, the center of the circle passing through $K$ and $L$ coincides with the center of the arbelos (see Figure 16). Hence $(C_1, C_2) \cup (C_3, C_4)$ consists of fifteen circles, nine of which are also members of $(C_2, C_3) \cup (C_4, C_1)$. Therefore the configuration $C_{44}$ consists of twenty-one distinct Archimedean circles. On the other hand, each of the nine circles in $(D_i, E_i)$ are different from the circles in $(D_i, E_i)$ for $i = 2, 3, 4$ by Table 2 (notice $v = 0$). Therefore the configuration $C_{48}$ consists of thirty-six distinct Archimedean circles (see Figure 17). The Archimedean circle with center on $OQ_2$ and passing through $R$ in Figure 16 and the Archimedean circle with a diameter $P_2P_3$ in Figure 17 can be seen in [8].
We conclude our paper by pointing out one more special case \( b/a = \Phi \left( \sqrt{2} \right)^{\pm 1} \). In this case the two Archimedean circles \( C_1 \) and \( C_3 \) touch at the center of the arbelos, also \( D_1 \) and \( D_2 \) touch.

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