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THE SQUARE OF THE ARBELOS

HIROSHI OKUMURA

Abstract. A square with two points on each of the sides are naturally constructed from an arbelos. It has relationships to the other squares associated with the arbelos and special type of generalized arbelos, and yields several dozens of Archimedean circles.

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1. Introduction

Let us consider an arbelos (α, β, γ) consisting of three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB in the plane. Let a and b be the radii of α and β, respectively. Circles of radius $r_{\mathrm{A}} = ab/(a+b)$ form a special class of circles and are called Archimedean circles of the arbelos. In this paper we consider a square with two points on each of the sides. We show that it has relationships to the squares associated with the arbelos considered in [7, 8] and to a special type of a generalized arbelos called the arbelos with overhang [9] and yields several dozens of Archimedean circles. Special cases where $b/a = (\sqrt{n^2+4} \pm n)/2$ for $n = 1, 2$ are investigated.

2. The square

In this section we construct a square from the arbelos. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that all the semicircles α, β and γ are constructed in the region $y \geq 0$. Let P_1 and S_1 be the centers of α and β, respectively, and let C be the farthest point on the circle with a diameter P_1S_1 from the line AB in the region $y > 0$. Let σ be the rotation about C through 90°. The center of the semicircle γ is denoted by S_2 and let $S_3 = O$ (see Figure 1). Then we get $P_1 = S_1^{\sigma}$, and let $P_2 = S_2^{\sigma}$, $P_3 = S_3^{\sigma}$ and $Q_i = P_i^{\sigma}$, $R_i = Q_i^{\sigma}$ for $i = 1, 2, 3$. Then $P_iQ_iR_iS_i$ is a square with center C, and $|S_1S_2| = a$, $|S_2P_1| = b$ and $|S_2S_3| = |a-b|$.

We call $P_1Q_1R_1S_1$ and C the square and the center of (α, β, γ). Notice that the lines P_2R_3, P_3R_2 and Q_1R_1 are the tangents of α, β and γ parallel to AB with points of tangency P_2, R_2 and Q_3, and the lines S_1R_1, S_2Q_3, S_3Q_2 and P_1Q_1 are the perpendiculars from R_1, Q_3, Q_2 and Q_1 to AB. The points P_1, P_2 and P_3 (resp. P_3 and P_2) lie on P_1Q_1 in this order if $a < b$ (resp. $a > b$).

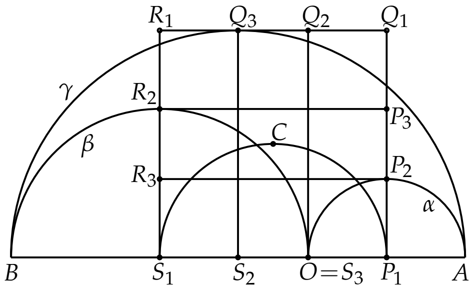

Figure 1.

3. Squares

In this section we consider relationships between the squares $P_iQ_iR_iS_i$ $(i = 1, 2, 3)$ and the squares considered in [7, 8]. Let the line P_2R_2 meet α and β at points P and R again, respectively (see Figure 2). Then the lines AP and BR meet in a point Q lying on γ and $OPQR$ is the inscribed square of (α, β, γ) [8]. The points P, Q and R have coordinates

$$(2jb, 2ja), \ (-2jd, 2je) \text{ and } (-2ja, 2jb), \qquad (1)$$

respectively, where $d = a - b$, $e = a + b$ and $j = ab/(a^2 + b^2)$. Let O' be the point of intersection of OQ_2 and γ. Let $\tilde{Q}$ and $\tilde{R}$ be the points on the segments AO' and BO', respectively such that $\tilde{P}\tilde{Q}\tilde{R}\tilde{S}$ is a square, where $\tilde{P}$ and $\tilde{S}$ are the feet of perpendiculars from $\tilde{Q}$ and $\tilde{R}$ to AB, respectively. Then $\tilde{Q}$ and $\tilde{R}$ have coordinates $(2af/g, 2ef/g)$ and $(-2bf/g, 2ef/g)$, respectively, where $f = \sqrt{ab}$ and $g = e + f$ [7].

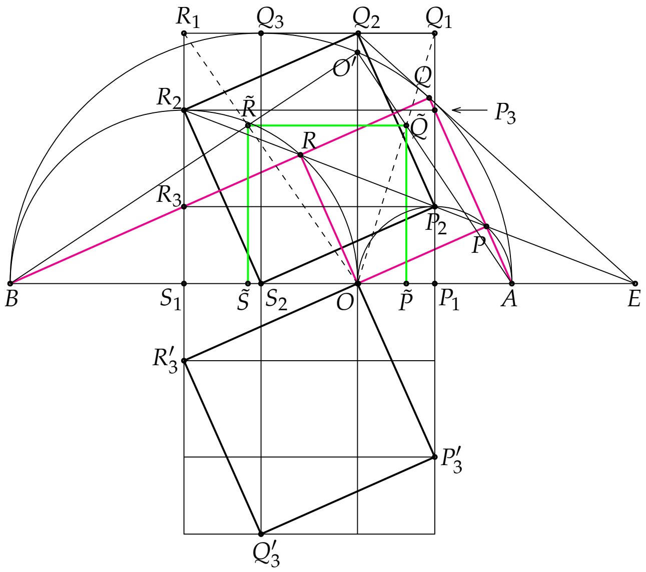

Figure 2.

Theorem 3.1. *The following statements hold.*
(i) *If P'_3, Q'_3 and R'_3 are the reflections of the points P_3, Q_3 and R_3 in the line AB,respectively, then the squares $OPQR$ and $OR'_3Q'_3P'_3$ are homothetic with center O and ratio $2ab : (a^2 + b^2)$.*
(ii) *If $a \neq b$ and E is the point of intersection of the lines AB and P_2R_2, the squares $OPQR$ and $S_2P_2Q_2R_2$ are homothetic with center E and ratio $2ab : (a^2 + b^2)$.*
(iii) *The squares $P_1Q_1R_1S_1$ and $\tilde{P}\tilde{Q}\tilde{R}\tilde{S}$ are homothetic with center O and ratio $g : 2f$.*

Proof. If $s = 2ab/e^2$ and $t = (a^2 + b^2)/e^2$, we have $O = sP_3' + tR = sR_3' + tP$. This proves (i). The part (ii) follows from (i) and the fact E being the homothety center of the triangles OPR and $S_2P_2R_2$. The part (iii) follows from $O = (g/(g-2f))\tilde{Q} + (-2f/(g-2f))Q_1 = (g/(g-2f))\tilde{R} + (-2f/(g-2f))R_1$. □

The part (ii) shows that the point Q_2 lies on the line EQ, which is the tangent of γ with point of tangency Q [8]. Let $\Phi(n) = \frac{1}{2}\left(\sqrt{n^2+4} + n\right)$ for a real number n. It is called a metallic mean if n is a natural number [11]. The first part of the following proposition follows from (1), and the last part is obvious.

Proposition 3.1. *The slopes of the lines OP and OR equal a/b and $-b/a$, respectively. If n is a non-negative real number, then $|d| = nr_{\mathrm{A}}$ if and only if $b/a = \Phi(n)^{\pm 1}$.*

Since $\Phi(n)$ is a monotonically increasing function of n and $\Phi(0) = 1$, there exists a non-negative real number n such that $b/a = \Phi(n)$ if and only if $b \geq a$.

Theorem 3.2. *The following statements are equivalent for a non-negative real number n.*
(i) $b/a = \Phi(n)$. (ii) *The point R_3 divides the segment QR in the ratio $(2+n) : n$ externally.*
(iii) *The point P_3 divides the segment PQ in the ratio $n : |2-n|$ internally if $n < 2$ externally if $2 < n$, and coincides with Q if $n = 2$.*

Proof. The part (i) is equivalent to to each of the equations $R_3 = ((2+n)/2)R + (-n/2)Q$ and $P_3 = ((2-n)/2)P + (n/2)Q$ being true. □

4. An arbelos with overhang

In [9] we considered a generalization of the arbelos, called an arbelos with overhang. In this section we show that the square of (α, β, γ) is closely related to a special type of arbelos with overhang. Let A_h and B_h be point on the line AB with x-coordinates a' and $-b'$, respectively such that $a' - a = b' - b = h > -\min(a, b)$. Let α_h and β_h be the semicircles with diameters A_hO and B_hO, respectively erected on the same side of AB as γ. The configuration of the three semicircles α_h, β_h and γ is called an arbelos with overhang h and is denoted by $(\alpha_h, \beta_h, \gamma)$ [9]. If $h > 0$, let $\varepsilon_{\alpha,1}$ (resp. $\varepsilon_{\beta,1}$) be the circle touching γ and α_h (resp. β_h) internally and α (resp. β) externally, also let $\varepsilon_{\alpha,2}$ (resp. $\varepsilon_{\beta,2}$) be the circle touching γ and α_h (resp. β_h) externally and the axis from the side opposite to B (resp. A). We use the next proposition.

Proposition 4.1 ([9]). *If $h > 0$, the following statements are true for $(\alpha_h, \beta_h, \gamma)$.*
(i) *The circles $\varepsilon_{\alpha,1}$ and $\varepsilon_{\beta,1}$ are congruent and have radius $(1/r_{\mathrm{A}} + 1/h)^{-1}$.*
(ii) *The circles $\varepsilon_{\alpha,2}$ and $\varepsilon_{\beta,2}$ are congruent and have radius ab/h.*

Let us consider an arbelos with overhang $(\alpha_h, \beta_h, \gamma)$ with $h = r_{\mathrm{A}}$. The radius of the circle touching α externally and γ internally is proportional to the distance between the center of this circle and the radical axis of α and γ [3, p. 108]. Therefore the circle ε touching α externally γ internally and P_1Q_1 from the side opposite to B has radius $r_{\mathrm{A}}/2$. While the circle $\varepsilon_{\alpha,1}$ has radius $r_{\mathrm{A}}/2$ by (i) of the proposition. Therefore the circles ε and $\varepsilon_{\alpha,1}$ coincide, i.e., $\varepsilon_{\alpha,1}$ touches P_1Q_1 from the side opposite to B. Similarly the circle $\varepsilon_{\beta,1}$ touches R_1S_1 from the side opposite to A (see Figure 3). The circles $\varepsilon_{\alpha,2}$ and $\varepsilon_{\beta,2}$ have radius e by (ii) of the proposition, i.e., they are congruent to γ. Since the centers of γ and $\varepsilon_{\alpha,2}$ have x-coordinates d and e, their point of tangency, which is the midpoint of the segment joining their centers, has x-coordinate a, i.e., the point lies on P_1Q_1. Similarly the point of tangency of γ and $\varepsilon_{\beta,2}$ lies on R_1S_1. The perpendicular to AB at the center of α_h (resp, β_h) also touches $\varepsilon_{\alpha,1}$ (resp, $\varepsilon_{\beta,1}$). If we erect a square on the same side of AB as γ with a side joining the centers of α_h and β_h, the distance between Q_1R_1 and the side of this square parallel to AB equals $2r_A$. Therefore circles touching the two sides are Archimedean circles of (α, β, γ).

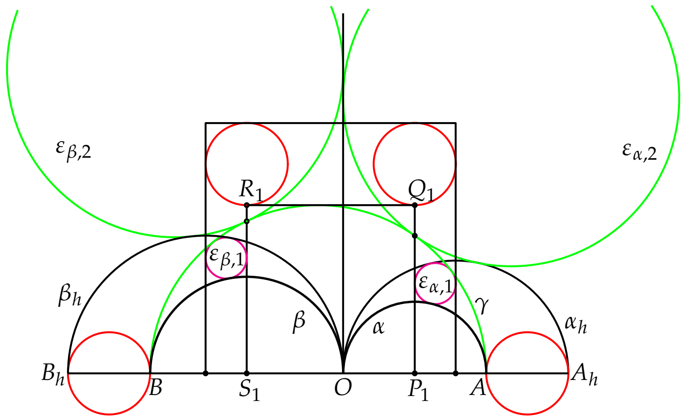

Figure 3.

5. Quadruplets and octuplets of Archimedean circles

In this section we construct quadruplets and octuplets of Archimedean circles. Let $T = OQ_2 \cap P_2R_3$ (which denotes the point of intersection of the lines OQ_2 and P_2R_3) and $U = P_1Q_2 \cap P_2R_3$. Then $|TU| = r_A$ by the similar triangles Q_2OP_1 and Q_2TU, (see Figure 4). Similarly the distances from T to the points $OQ_2 \cap P_2R_1$, $OQ_2 \cap P_1R_3$ and $OR_1 \cap P_2R_3$ equal r_A. Hence the circle with center T and passing through those points of intersection is Archimedean and is denoted by $\mathcal{C}_1$. Similarly we get Archimedean circles $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ with centers $P_3R_2 \cap Q_2O$, $Q_3S_2 \cap R_2P_3$ and $R_3P_2 \cap S_2Q_3$, which also coincide with $\mathcal{C}_1^{\sigma}$, $\mathcal{C}_1^{\sigma^2}$ and $\mathcal{C}_1^{\sigma^3}$, respectively (see Figure 5). The result that the distances from the points $P_3R_2 \cap OQ_1$ and $OR_1 \cap P_2R_3$ to OQ_2 being equal to r_A can be seen in [6].

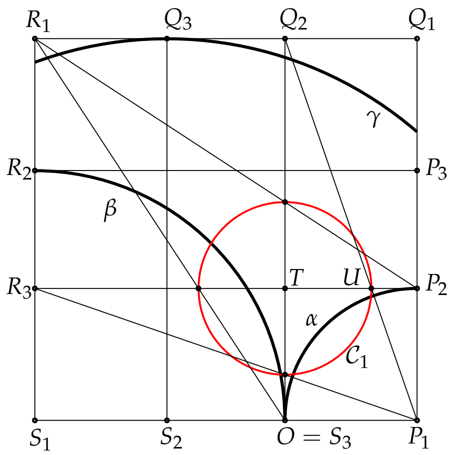

Figure 4.

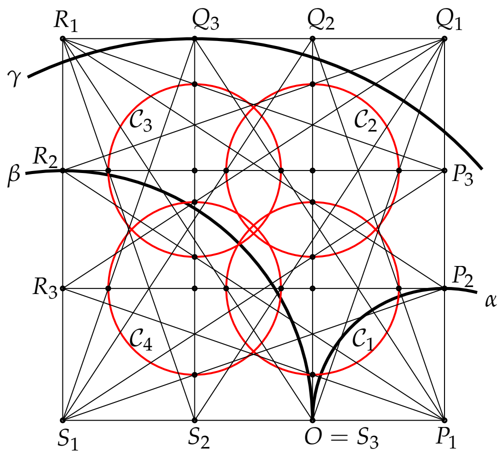

Figure 5.

The segments OQ_1, P_1Q_3 and P_2R_3 meet in a point V and the distances from V to the points P_2 and $OQ_3 \cap P_2R_3$ equal r_A (see Figure 6). The facts are also proved by similar triangles. Therefore the circle with center V passing through P_2 and $OQ_3 \cap P_2R_3$ is Archimedean, and is denoted by $\mathcal{D}_1$. The endpoints of the diameter of $\mathcal{D}_1$ parallel to P_1Q_1 lie on the lines OP_2 and P_2Q_3 as shown by the dotted lines in Figure 6. Similarly we get an Archimedean circle $\mathcal{E}_1$ touching P_1Q_1 at P_3 from the side opposite to A. Also we get Archimedean circles $\mathcal{D}_2$, $\mathcal{E}_2$; $\mathcal{D}_3$, $\mathcal{E}_3$ and $\mathcal{D}_4$, $\mathcal{E}_4$ touching Q_1R_1, R_1S_1 and S_1P_1, respectively, where $\mathcal{D}_i = \mathcal{D}_1^{\sigma^{i-1}}$ and $\mathcal{E}_i = \mathcal{E}_1^{\sigma^{i-1}}$ for $i = 2, 3, 4$. We now get octuplets of Archimedean circles (see Figure 7). The circle $\mathcal{E}_4$ is the Bankoff triplet circle [2]. Since the circles $\mathcal{D}_4$ and $\mathcal{E}_4$ touch the circle with a diameter S_1P_1 [5], the pairs of circles $\mathcal{D}_1$, $\mathcal{E}_1$; $\mathcal{D}_2$, $\mathcal{E}_2$; $\mathcal{D}_3$, $\mathcal{E}_3$ also touch the circles with diameters P_1Q_1, Q_1R_1 and R_1S_1, respectively. The circles $\mathcal{D}_1$ and $\mathcal{D}_3$ can be seen in [6].

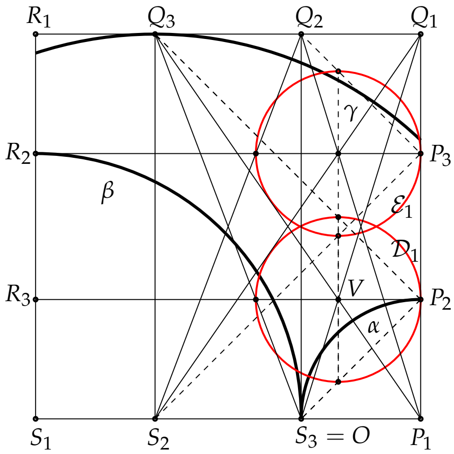

Figure 6.

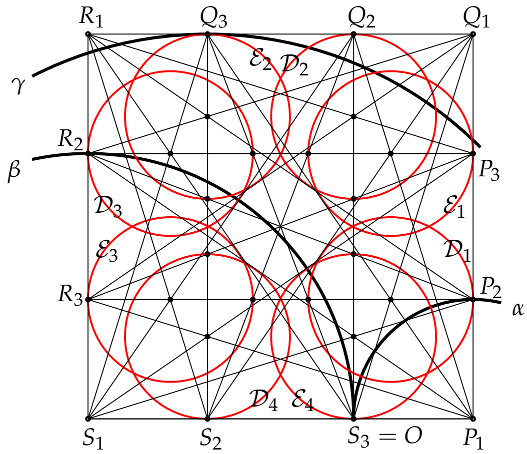

Figure 7.

6. A CONFIGURATION OF TWELVE CONGRUENT CIRCLES

In this section we assume that δ_1 and δ_2 are congruent circles with centers F and G, respectively and have a point H in common, and ε is a circle passing through H intersecting δ_1 and δ_2 again at points I and J, respectively. We show that δ_1 and δ_2 yield a configuration of twelve congruent circles if $FIJG$ is a parallelogram.

Theorem 6.1. *The circle ε is congruent to δ_1 if and only if $FIJG$ is a parallelogram.*

Proof. Let K be the center of ε (see Figure 8). If ε and δ_1 are congruent, $FIKH$ and $GJKH$ are rhombuses with side HK in common, i.e., $FIJG$ is a parallelogram. Conversely if $FIJG$ is a parallelogram, let K' be the point such that $\overrightarrow{HK'} = \overrightarrow{FI}$. Then $FIK'H$ and $GJK'H$ are rhombuses and $|K'H| = |K'I| = |K'J|$. Hence $K' = K$ and ε is congruent to δ_1. □

Let H' be the remaining point of intersection of δ_1 and δ_2, and let ε' be the circle passing through I, J and H'. If δ_1, δ_2 and ε are congruent, $FIJG$ is a parallelogram. Therefore ε' is also congruent to δ_1. Hence we get Johnson's circle theorem [4]:

Corollary 6.1. *If three congruent circles meet in a point, the circle passing through their other three points of intersection is also congruent to the three.*

We now assume ε is congruent to δ_1. Since the three congruent circles δ_1, ε and ε' meet in I, the circle with center I passing through the centers of the three circles is congruent to the three. Similarly, we can get circles congruent to δ_1 with centers J, H and H' (see Figure 9). Now we get a configuration of eight congruent circles. Since each of ε and the circle with center I passes through the center of the other, the circle with center at one of the points of intersection of the two circles and passing through their centers is also congruent to the two. But such a circle is trivial, and is excluded to consider.

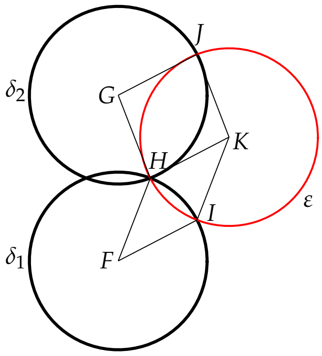

Figure 8.

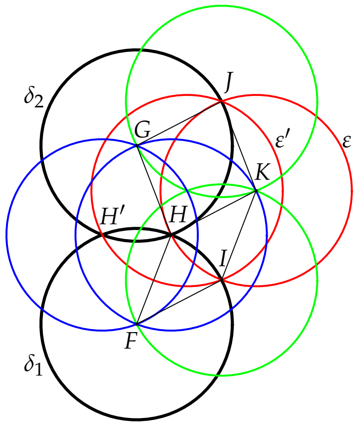

Figure 9.

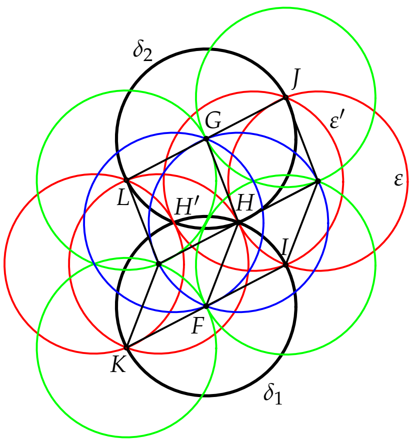

Figure 10.

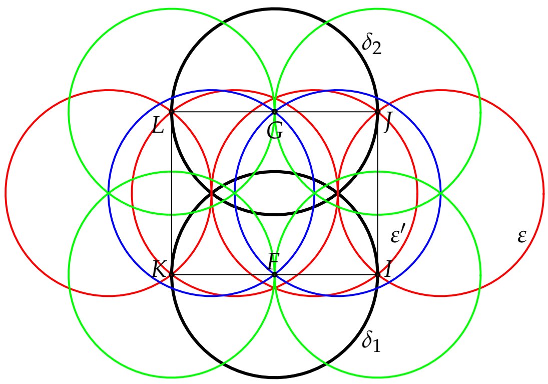

Figure 11: (δ_1, δ_2)

Let K and L be the points such that IK and JL are diameters of of δ_1 and δ_2. Then we get four more circles congruent to δ_1: the two circles passing through K, L and H, and K, L and H', and the two circles with centers K and L. Hence we get a configuration of twelve congruent circles (see Figure 10). If $FIJG$ is a rectangle, we denote the configuration by (δ_1, δ_2) (see Figure 11). Let us assume that F, I and G have coordinates (p, s), $(p + r, s)$ $(r > 0)$ and (p, t), respectively in this case. Then J, K and L have coordinates $(p + r, t)$, $(p - r, s)$ and $(p - r, t)$, respectively. And the three pairs of circles passing through F, G; I, J; K, L have centers with coordinates $(p \pm v, u)$, $(p + r \pm v, u)$ and $(p - r \pm v, u)$, respectively, where $u = (s + t)/2$ and $v = \sqrt{r^2 - (s - u)^2}$.

7. Configurations of Archimedean circles

In this section, we show that the quadruplets and the octuplets of Archimedean circles in section 5 yield configurations of several dozens of Archimedean circles if the two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ ($\mathcal{D}_1$ and $\mathcal{E}_1$) intersect. Hence we assume $|d| < 2r_A$, where recall $d = a - b$.

$(\mathcal{C}_1, \mathcal{C}_2)$	$(0, a)$, $(\pm r_A, a)$; $(0, b)$, $(\pm r_A, b)$; $(\pm v, e/2)$, $(\pm r_A \pm v, e/2)$
$(\mathcal{C}_3, \mathcal{C}_4)$	(d, b), $(d \pm r_A, b)$; (d, a), $(d \pm r_A, a)$; $(d \pm v, e/2)$, $(d \pm r_A \pm v, e/2)$
$(\mathcal{C}_4, \mathcal{C}_1)$	(d, a), $(d, a \pm r_A)$; $(0, a)$, $(0, a \pm r_A)$; $(d/2, a \pm v)$, $(d/2, a \pm r_A \pm v)$

Table 1.

We consider the configuration of twelve Archimedean circles $(\mathcal{C}_1, \mathcal{C}_2)$. Recall $e = a + b$, and let $v = \sqrt{r_A^2 - d^2/4}$. The coordinates of the centers of the circles in this configuration are indicated in the first row of Table 1. The second row and the third row indicate the coordinates of the centers of the circles in $(\mathcal{C}_3, \mathcal{C}_4)$ and $(\mathcal{C}_4, \mathcal{C}_1)$, respectively. The table shows that each of the centers of the circles in the configuration $(\mathcal{C}_1, \mathcal{C}_2) \cup (\mathcal{C}_2, \mathcal{C}_3) \cup (\mathcal{C}_3, \mathcal{C}_4) \cup (\mathcal{C}_4, \mathcal{C}_1)$ except $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ is different from the others in general. Therefore it consists of forty-four distinct Archimedean circles, which we denote by C_{44} (see Figure 12). The circles C_1, C_2, C_3 and C_4 are indicated in black and the circles with centers on the perpendicular bisectors of P_2P_3 and Q_2Q_3 are indicated in red, and the remainings are indicated in green.

$(\mathcal{D}_1, \mathcal{E}_1)$	$(a - r_A, a)$, $(a - r_A \pm r_A, a)$; $(a - r_A, b)$, $(a - r_A \pm r_A, b)$; $(a - r_A \pm v, e/2)$, $(a - r_A \pm r_A \pm v, e/2)$
$(\mathcal{D}_3, \mathcal{E}_3)$	$(-b + r_A, b)$, $(-b + r_A \pm r_A, b)$; $(-b + r_A, a)$, $(-b + r_A \pm r_A, a)$; $(-b + r_A \pm v, e/2)$, $(-b + r_A \pm r_A \pm v, e/2)$
$(\mathcal{D}_4, \mathcal{E}_4)$	(d, r_A), $(d, r_A \pm r_A)$; $(0, r_A)$, $(0, r_A \pm r_A)$; $(d/2, r_A \pm v)$, $(d/2, r_A \pm r_A \pm v)$

Table 2.

The coordinates of the centers of the circles in $(\mathcal{D}_1, \mathcal{E}_1)$ are indicated in the first row of Table 2. The second row and the third row indicate the coordinates of the centers of the circles in $(\mathcal{D}_3, \mathcal{E}_3)$ and $(\mathcal{D}_4, \mathcal{E}_4)$. Since those centers are different in general, the configuration $\cup_{i=1}^{4} (\mathcal{D}_i, \mathcal{E}_i)$ consists of forty-eight distinct Archimedean circles, which we denote by C_{48} and indicate in Figure 13 in a similar way to as Figure 12.

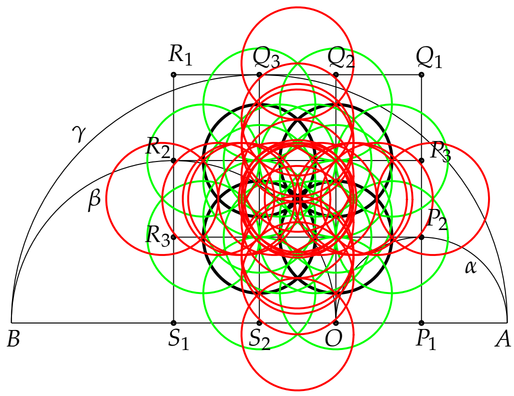

Figure 12: C_{44}

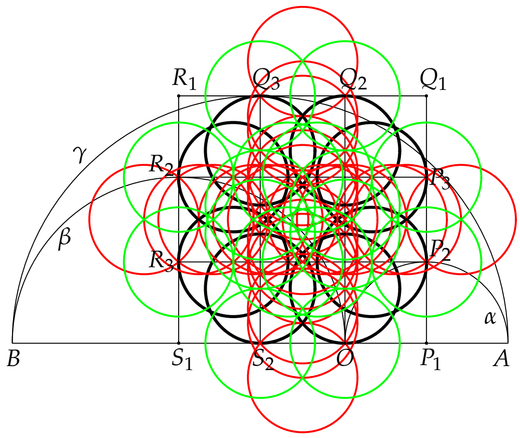

Figure 13: C_{48}

8. GOLDEN ARBELOS

We consider the case in which the Archimedean circle $\mathcal{C}_1$ touches the line Q_3S_2. It is equivalent to $a/b = \Phi(1)^{\pm 1}$ by Proposition 3.1. In this case (α, β, γ) is called a golden arbelos [1]. The equivalence of (i), (ii) and (iii) of the next theorem follows from Theorem 3.2 (see Figure 14). The proof of the equivalence of (i), (iv) and (v) is straightforward and is omitted.

Theorem 8.1. *The following statements are equivalent.*
(i) $b/a = \Phi(1)$. (ii) *The point* R_3 *divides* QR *in the ratio* 3 : 1 *externally.*
(iii) *The point* P_3 *is the midpoint of* PQ*.* (iv) *The points* R_1 *and* $R_3P_2 \cap Q_3S_2$ *lie on* OR*.*
(v) *The points* Q_1 *and* $Q_3S_2 \cap R_2P_3$ *lie on* BQ*.*

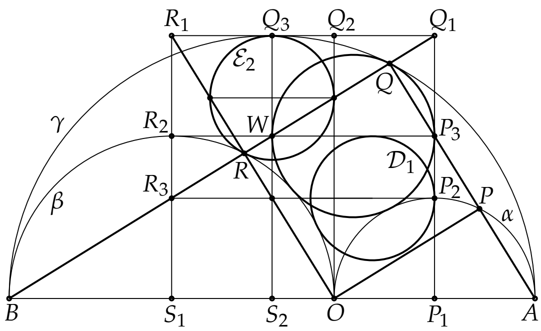

Figure 14.

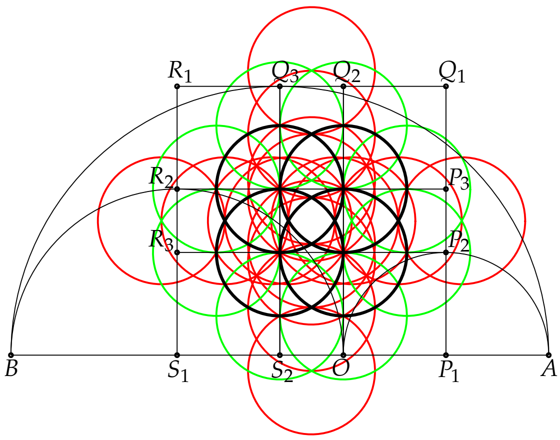

Figure 15: C_{44} in a golden arbelos

We now assume $b/a = \Phi(1)$. The circle touching γ internally β externally and OQ_2 from the side opposite to A is well-known as being Archimedean and the point of tangency of this circle and β has coordinates $(-2ab/k, 2b\sqrt{ae}/k)$, where $k = 2a + b$ in general case. Therefore the circle $\mathcal{E}_2$ coincides with this circle and touches β (see Figure 14), and the point R has coordinates $\left(-2a/\sqrt{5}, 2b/\sqrt{5}\right)$ and coincides with the point of tangency of β and $\mathcal{E}_2$. Since R is the homothetic center of β and $\mathcal{E}_2$, the endpoints of the diameter of $\mathcal{E}_2$ parallel to AB lie on the lines BR and OR. Let $W = Q_3S_2 \cap R_2P_3$. Since Q is the homothetic center of the triangles P_3QW and AQB, the circumcircle of P_3QW touches γ at Q. While the inscribed circle of the arbelos has radius $abe/(a^2 + ab + b^2)$ [10], and touches γ at Q [8] in general case. Hence it has radius $b/2$ in this case, and coincides with the circumcircle of P_3QW, because $|P_3W| = b$. Therefore the inscribed circle of the arbelos touches the segments P_1Q_1 and Q_3S_2 at P_3 and W. The circle $\mathcal{D}_1$ also touches β. Several golden rectangles appear in the figure, one of which is $P_1P_3R_2S_1$.

Each of the two circles passing through F, G and K, L in $(\mathcal{C}_1, \mathcal{C}_2)$ are also members of $(\mathcal{C}_3, \mathcal{C}_4)$ (see Figure 11). Hence there are eight distinct circles with centers on the perpendicular bisector of P_2P_3 in $(\mathcal{C}_1, \mathcal{C}_2) \cup (\mathcal{C}_3, \mathcal{C}_4)$. Also there are four distinct circles with centers on each of P_2R_3 and P_3R_2 in $(\mathcal{C}_1, \mathcal{C}_2) \cup (\mathcal{C}_3, \mathcal{C}_4)$. Hence $(\mathcal{C}_1, \mathcal{C}_2) \cup (\mathcal{C}_3, \mathcal{C}_4)$ consists of sixteen distinct circles, four of which are $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$. Therefore C_{44} consists of twenty-eight distinct Archimedean circles (see Figure 15).

9. Silver arbelos

We consider the case where the Archimedean circles $\mathcal{C}_1$ and $\mathcal{C}_2$ touch. It is equivalent to $b/a = \Phi(2)^{\pm 1}$. Following to $\Phi(2)$ being called the silver mean [11], we call (α, β, γ) a silver arbelos in this case. This case is also considered in [8].

Theorem 9.1. *The following statements are equivalent.*
(i) $b/a = \Phi(2)$. (ii) *The point P_3 coincides with Q.*
(iii) *The point R coincides with the center of the arbelos.*

Proof. Let us assume that the part (iii) holds. Then P_3, R_3 and R are collinear. Since R and R_3 lie on BQ by (ii) of Theorem 3.2, P_3 lies on BQ. While P_3 also lies on AQ by (iii) of the same theorem. Therefore $P_3 = Q$, i.e., (iii) implies (ii). The rest of the theorem follows from Theorem 3.2. □

We now assume $b/a = \Phi(2)$. Since the slopes of the line OR equals $-\Phi(2)$ by Proposition 3.1, the point R is the midpoints of the arc OR_2 of the semicircles β. Also P and Q are the midpoints of the arcs AP_2 and AQ_3 of α and γ by the similarity. The circle with center P touching AB and P_1Q_1 is Archimedean. Several rectangles with aspect ratio $1 : \sqrt{2}$, which are sometimes called silver rectangles, appear in the figure, one of which is $P_1P_3R_2S_1$.

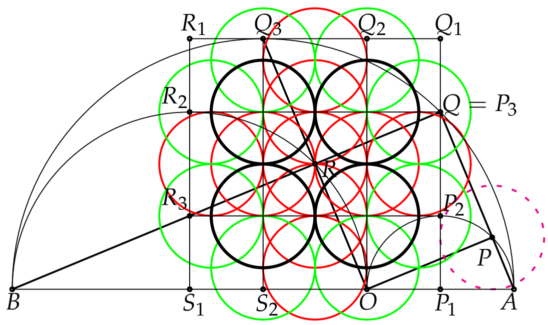

Figure 16: C_{44} in a silver arbelos

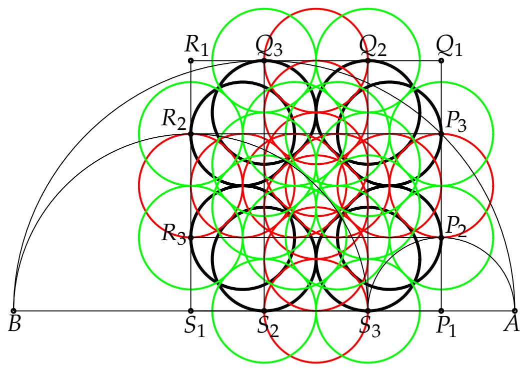

Figure 17: C_{48} in a silver arbelos

If the circles δ_1 and δ_2 touch, the circles in each pairs of the two circles passing through I and J, F and G, and K and L in (δ_1, δ_2) coincide (see Figure 11). Therefore (δ_1, δ_2) consists of nine congruent circles. Especially if $\delta_1 = \mathcal{C}_1$ and $\delta_2 = \mathcal{C}_2$, the center of the circle passing through K and L coincides with the center of the arbelos (see Figure 16). Hence $(\mathcal{C}_1, \mathcal{C}_2) \cup (\mathcal{C}_3, \mathcal{C}_4)$ consists of fifteen circles, nine of which are also members of $(\mathcal{C}_2, \mathcal{C}_3) \cup (\mathcal{C}_4, \mathcal{C}_1)$. Therefore the configuration C_{44} consists of twenty-one distinct Archimedean circles. On the other hand, each of the nine circles in $(\mathcal{D}_1, \mathcal{E}_1)$ are different from the circles in $(\mathcal{D}_i, \mathcal{E}_i)$ for $i = 2, 3, 4$ by Table 2 (notice $v = 0$). Therefore the configuration C_{48} consists of thirty-six distinct Archimedean circles (see Figure 17). The Archimedean circle with center on OQ_2 and passing through R in Figure 16 and the Archimedean circle with a diameter P_2P_3 in Figure 17 can be seen in [8].

We conclude our paper by pointing out one more special case $b/a = \Phi\left(\sqrt{2}\right)^{\pm 1}$. In this case the two Archimedean circles $\mathcal{C}_1$ and $\mathcal{C}_3$ touch at the center of the arbelos, also $\mathcal{D}_1$ and $\mathcal{D}_2$ touch.

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Department of Mathematics
Yamato University
2-5-1 Katayama Suita Osaka 564-0082, Japan
E-mail address: okumura.hiroshi@yamato-u.ac.jp