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EVALUATION OF INTEGRALS ASSOCIATED WITH MULTIPLE (MULTIINDEX) MITTAG-LEFFLER FUNCTION

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ABSTRACT. The aim of the present paper is to investigate some interesting integrals involving the product of multiple (multiindex) Mittag-Leffler function $E_{(\frac{1}{\rho_i}),(\mu_i)}(z)$ with Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, which are expressed in terms of Generalized hypergeometric function. Some interesting integrals involving the product of multiple (multiindex) Mittag-Leffler function with Legendre polynomials, Legendre function, Bessel Maitland function, Hermite polynomial, Hypergeometric function and Generalized hypergeometric function are also derived as a special case of our main results.

1. INTRODUCTION

As a result of researchers and scientists increasing interest in pure as well as applied mathematics in nonconventional models, particularly those using fractional calculus, Mittag-Leffler functions have recently caught the interest of the scientific community. The Mittag-Leffler function was introduced by [7] in connection with his method of summation of some divergent series. Its importance is realised during the last two decades due to its involvement in the problems of physics, chemistry, biology, engineering and applied sciences. Mittag-Leffler function naturally occurs as the solution of fractional order differential equation or fractional order integral equations. The Mittag-Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of noninteger order differential equations. The Mittag-Leffler function appears also in the solution of the fractional master equation. Such an equation characterizes the renewal processes with rewards modeling by the random walk model known as continuous time random walk. The question arises for the solution of differential equations with the fractional derivatives of the Riemann-Liouville type of arbitrary order α, $Re\ (\alpha > 0)$ and n boundary conditions $(n = Re\alpha + 1)$ in the form of the values at the initial point (these initial condition are so called Cauchy-type conditions). Its was observe that the suitable class of functions, the solution is unique and is represented though Mittag-Leffler function. Although some generalization of Mittag-Leffler function have been given by a number of authors, (see [1],[9],[10] and [11]). The Mittag-Leffler function plays an important role in pure as well as applied mathematics which were studied by many authors for many

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different point of view. In this paper we make a contribution to the subject by showing some geometrical properties including univalency, starlikeness, spirallikeness, convexity and close-to-convexity in the open unit disk. The Mittag-Leffler functions have certain applications in treating various situation and process in viscoelasticity, hydrodynamics, diffusion and wave phenomena as well as stochastics.

Recently, Singh and Rawat [3] have established some integrals involving the product of generalized Mittag-Leffler function with Jacobi polynomial, which are expressed in terms of generalized hypergeometric function. Very recently, Choudhary and Choudhary [2] further established some more integrals associated with generalized Mittag-Leffler functions. Motivated by the above mentioned work, in the present paper we establish some new integrals associated with multiple (multiindex) Mittag-Leffler function.

Kiryakova [13] defined the multiple (multiindex) Mittag-Leffler function as follows: Let $m > 1$ be an integer, $\rho_1, \cdots \rho_m > 0$ and $\mu_1, \cdots, \mu_m$ be arbitrary real numbers. By means of "multiindices" $(\rho_i)(\mu_i)$ we introduce the so-called multiindex(m-tuple,multiple) Mittag-Leffler functions.

$$E_{(\frac{1}{\rho_i}),(\mu_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \tag{1}$$

2. Integrals with Jacobi polynomials

The Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ is defined by (see [4],[8]) :

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!} \, {}_2F_1 \left[\begin{array}{cc} -n, \quad 1+\alpha+\beta+n; & \frac{1-z}{z} \\ 1+\alpha; & \end{array} \right], \tag{2}$$

or equivalently,

$$P_n^{(\alpha,\beta)}(z) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1+\alpha+\beta)_{n+k}}{k! \, (n-k)! \, (1+\alpha)_k (1+\alpha+\beta)_n} \left(\frac{z-1}{z} \right)^k. \tag{3}$$

From (2) and (3) it follows that $P_n^{(\alpha,\beta)}(z)$ is a polynomial of degree precisely n and that

$$P_n^{(\alpha,\beta)}(1) = \frac{(1+\alpha)_n}{n!} \tag{4}$$

In dealing with Jacobi polynomials, it is natural to make much use of our knowledge of the ${}_2F_1$ function ([4], p. 45)

$$I_1 = \int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} \, P_n^{(\alpha,\beta)}(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1+x)^h] \, dx$$

$$= \int_{-1}^{1} x^{\lambda}(1-x)^{\alpha}(1+x)^{\delta} \, P_n^{(\alpha,\beta)}(x) \, \sum_{k=0}^{\infty} \frac{[z(1+x)^h]^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \, dx$$

Interchanging the order of summation and integration, we can write above expression as

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \int_{-1}^{1} x^{\lambda} (1-x)^{\alpha} (1+x)^{\delta+kh} \, P_n^{(\alpha,\beta)}(x) \, dx \tag{5}$$

But we have the formula ([12] p. 52)

$$\int_{-1}^{1} x^{\lambda} (1-x)^{\alpha} (1+x)^{\delta} \, P_n^{(\alpha,\beta)}(x) \, dx = \frac{(-1)^n 2^{\alpha+\delta+1}}{n!} \frac{\Gamma(\delta+1)\Gamma(\alpha+n+1)\Gamma(\delta+\beta+1)}{\Gamma(\delta+\beta+n+1)\Gamma(\delta+\alpha+n+2)}$$

$$\times \, {}_3F_2 \left(\begin{array}{c} -\lambda, \ \ \delta+\beta+1, \ \ \delta+1; \\ \\ \delta+\beta+n+1, \delta+\alpha+n+2; \end{array} \ 1 \right), \tag{6}$$

Provided $\alpha > -1$ and $\beta > -1$.

Now, by using (5) and (6), we get

$$I_1 = \frac{(-1)^n 2^{\alpha+\delta+1}}{n!} \frac{\Gamma(\delta+kh+1)\Gamma(\alpha+n+1)\Gamma(\delta+kh+\beta+1)}{\Gamma(\delta+kh+\beta+n+1)\Gamma(\delta+kh+\alpha+n+2)}$$

$$\times E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^h) \, {}_3F_2 \left(\begin{array}{c} -\lambda, \ \ \delta+kh+\beta+1, \ \ \delta+kh+1; \\ \\ \delta+kh+\beta+n+1, \delta+kh+\alpha+n+2; \end{array} \ 1 \right). \tag{7}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0, \ \Re(\mu_i) > 0$.
(ii) $\Re(\lambda) > -1, \ \alpha > -1$ and $\beta > -1$.

$$I_2 = \int_{-1}^{1} (1-x)^{\delta} (1+x)^{\beta} \, P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\sigma)}(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h] \, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \int_{-1}^{1} (1-x)^{\delta+kh} (1+x)^{\beta} \, P_n^{(\alpha,\beta)}(x) P_m^{(\rho,\sigma)}(x) \, dx \tag{8}$$

Now using (2) in above expression we get

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \frac{(1+\rho)_m}{m!} \sum_{k=0}^{\infty} \frac{(-m)_k (1+\rho+\sigma+m)_k}{(1+\rho)_k 2^k k!}$$

$$\times \int_{-1}^{1} (1-x)^{\delta+kh+k} (1+x)^{\beta} \, P_n^{(\alpha,\beta)}(x) dx \tag{9}$$

Again using (2) in (9), we get

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \frac{\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m! n!}$$

$$\times \sum_{k=0}^{\infty} \frac{(-n)_k (-m)_k (1+\rho+\sigma+m)_k (1+\alpha+\beta+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\alpha+k) 2^{2k} (k!)^2} \int_{-1}^{1} (1-x)^{\delta+kh+2k} (1+x)^{\beta} \, dx \tag{10}$$

But by the formula

$$\int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{n+\beta}\,dx = 2^{2n+\alpha+\beta+1}\,B(1+\alpha+n,1+\beta+n). \tag{11}$$

Hence (10) becomes,

$$I_2 = \frac{2^{\delta+\beta+1}\Gamma(1+\rho+m)\Gamma(1+\alpha+n)}{m!n!}$$

$$\times\sum_{k=0}^{\infty}\frac{(-n)_k(-m)_k(1+\rho+\sigma+m)_k(1+\alpha+\beta+n)_k}{\Gamma(1+\rho+k)\Gamma(1+\alpha+k)2^{2k}(k!)^2}\,E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^h)B(1+\delta+kh+2k,1+\beta). \tag{12}$$

Provided
(i) $\Re(\frac{1}{\rho_i})>0,\ \Re(\mu_i)>0$.
(ii)$\Re(\beta)>-1$, h and δ are positive numbers.

$$I_3 = \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h(1+x)^t]\,dx$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\int_{-1}^{1}(1-x)^{\rho+kh}(1+x)^{\sigma+kt}\,P_n^{(\alpha,\beta)}(x)dx \tag{13}$$

Now, by using (2) in (13), we obtain

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\frac{(1+\alpha)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k2^kk!}\int_{-1}^{1}(1-x)^{\rho+kh+k}(1+x)^{\sigma+kt}\,dx \tag{14}$$

Using (11) in (14), we get

$$I_3=\frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k\,k!}\,E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{h+t})B(1+\rho+kh+k,1+\sigma+tk). \tag{15}$$

Provided
(i) $\Re(\frac{1}{\rho_i})>0,\ \Re(\mu_i)>0$.
(ii) $\Re(\alpha)>-1$ and $\Re(\beta)>-1$.

$$I_4 = \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}\,P_n^{(\alpha,\beta)}(x)E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h(1+x)^{-t}]\,dx$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\int_{-1}^{1}(1-x)^{\rho+kh}(1+x)^{\sigma-kt}\,P_n^{(\alpha,\beta)}(x)dx \tag{16}$$

Now, by using (2) in (16), we obtain

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\frac{(1+\alpha)_n}{n!}\sum_{k=0}^{\infty}\frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k2^kk!}\int_{-1}^{1}(1-x)^{\rho+kh+k}(1+x)^{\sigma-kt}\,dx \tag{17}$$

Using (11) in (17), we get

$$I_4 = \frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k\, k!} E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{h-t})B(1+\rho+kh+k, 1+\sigma-tk). \tag{18}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) > -1$ and $\Re(\beta) > -1$.

$$I_5 = \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, P_n^{(\alpha,\beta)}(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1+x)^{-h}]\, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})} \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma-kh}\, P_n^{(\alpha,\beta)}(x)dx \tag{19}$$

Now, by using (2) in (19), we obtain

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})} \frac{(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k 2^k k!} \int_{-1}^{1} (1-x)^{\rho+k}(1+x)^{\sigma-kh}\, dx \tag{20}$$

Using (11) in (20), we get

$$I_5 = \frac{2^{\rho+\sigma+1}(1+\alpha)_n}{n!} \sum_{k=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{(1+\alpha)_k\, k!} E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{-h})B(1+\rho+k, 1+\sigma-kh). \tag{21}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) > -1$ and $\Re(\beta) > -1$.

3. Special Cases

(i) If we replace δ by $\lambda - 1$ and put $\alpha = \beta = \rho = \sigma = 0$ then the integral I_2 transforms into the following integral involving Legendre polynomial (see [4],[8]),

$$I_6 = \int_{-1}^{1} (1-x)^{\lambda-1}\, P_n(x)P_m(x)E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h]\, dx$$

$$= 2^{\lambda} \sum_{k=0}^{\infty} \frac{(-n)_k(-m)_k(1+m)_k(1+n)_k}{(k!)^2(k!)^2} E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^h)B(\lambda+kh+2k, 1). \tag{22}$$

(ii) If $\alpha = \beta = 0$ and ρ is replaced by $\rho - 1$ and σ by $\sigma - 1$, then I_3 transforms into the following integral involving Legendre polynomial (see [4],[8]),

$$I_7 = \int_{-1}^{1} (1-x)^{\rho-1}(1+x)^{\sigma-1}\, P_n(x)E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h(1+x)^t]\, dx$$

$$= 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k}{(k!)^2} E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{h+t})B(\rho+kh+k, \sigma+tk). \tag{23}$$

(iii) If $\alpha = \beta = 0$ and ρ is replaced by $\rho - 1$ and σ by $\sigma - 1$, then I_4 transforms into the following integral involving Legendre polynomial (see [4],[8]),

$$I_8 = \int_{-1}^{1} (1-x)^{\rho-1}(1+x)^{\sigma-1} \, P_n(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[z(1-x)^h(1+x)^{-t}] \, dx$$

$$= 2^{\rho+\sigma-1} \sum_{k=0}^{\infty} \frac{(-n)_k(1+n)_k}{(k!)^2} \, E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{h-t}) B(\rho + kh + k, \sigma - tk). \tag{24}$$

4. Integral with Bessel Maitland function

The special case of the Wright function ([15], vol. 3, section 18.1) and ([5] and [6]) in the form

$$\phi(B, b; z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(Bk+b)} \frac{z^k}{k!} \tag{25}$$

with complex z, $b \in C$ and real $B \in R$. When $B = \delta, b = \nu + 1$ and z is replaced by -z, the function $\phi(\delta, \nu+1; -z)$ is defined by $J_\nu^\delta(z)$:

$$J_\nu^\delta(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\delta n + \nu + 1)} \frac{(-z)^n}{n!} \tag{26}$$

and such a function is known as the Bessel Maitland function, or the Wright generalized Bessel function.

$$I_9 = \int_0^\infty x^\rho J_\nu^\delta(x) \, E_{(\frac{1}{\rho_i}),(\mu_i)}(zx^\alpha) \, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \int_0^\infty x^{\rho+\alpha k} J_\nu^\delta(x) \, dx \tag{27}$$

By the well known formula, ([12], p. 55)

$$\int_0^\infty x^\rho J_\nu^\delta(x) \, dx = \frac{\Gamma(\rho+1)}{\Gamma(1+\nu-\delta-\delta\rho)} \,, \tag{28}$$

Provided $\mathrm{Re}(\rho) > -1, \ 0 < \delta < 1$.

Now using (28) in (27), we get

$$I_9 = \sum_{k=0}^{\infty} \frac{\Gamma(\rho + kh + 1)}{\Gamma(1+\nu-\delta-\delta(\rho+\alpha k))} \, E_{(\frac{1}{\rho_i}),(\mu_i)}(z). \tag{29}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0, \ \Re(\mu_i) > 0$.
(ii) $\alpha - \delta\alpha > -1$ and $\alpha > 0$.
(iii) $0 < \delta < -1$ and $\Re(\rho+1) > 0$.

5. INTEGRALS WITH LEGENDRE FUNCTIONS

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F\left[-\nu, \nu+1; 1-\mu; \frac{1}{2} - \frac{z}{2}\right], \ |1-z| < 2. \tag{30}$$

The function $P_\nu^\mu(z)$ is known as the Legendre function of first kind ([14], vol. 1). It is one of valued and regular in z-plane supposed cut along the real axis from 1 to $-\infty$.

$$I_{10} = \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} \, P_\nu^\delta(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^\alpha] \, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \int_0^1 x^{\sigma+\alpha k-1}(1-x^2)^{\frac{\delta}{2}} \, P_\nu^\delta(x) \, dx \tag{31}$$

The above integral (31) can be solved by using the formula ([14], section 3.12 vol.1).

$$\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{\delta}{2}} \, P_\nu^\delta(x) \, dx = \frac{(-1)^\delta (2)^{-\sigma-\delta} \, \pi^{\frac{1}{2}} \Gamma(\sigma)\Gamma(1+\delta+\nu)}{\Gamma(\frac{1}{2} + \frac{\sigma}{2} + \frac{\delta}{2} - \frac{\nu}{2}) \, \Gamma(\frac{1}{2} + \frac{(\sigma}{2} + \frac{\delta}{2} + \frac{\nu}{2}) \, \Gamma(1-\delta+\nu)}, \tag{32}$$

Provided $\Re(\sigma) > 0, \ \delta = 1, 2, 3, ...$

Now (31) becomes,

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \frac{(-1)^\delta (2)^{-\sigma-\alpha k-\delta} \, \pi^{\frac{1}{2}} \Gamma(\sigma+\alpha k)\Gamma(1+\delta+\nu)}{\Gamma(\frac{1}{2} + \frac{(\sigma+\alpha k)}{2} + \frac{\delta}{2} - \frac{\nu}{2}) \, \Gamma(\frac{1}{2} + \frac{(\sigma+\alpha k)}{2} + \frac{\delta}{2} + \frac{\nu}{2}) \, \Gamma(1-\delta+\nu)}$$

$$I_{10} = \frac{(-1)^\delta (2)^{-\sigma-\delta} \, \pi^{\frac{1}{2}} \Gamma(1+\delta+\nu)}{\Gamma(1-\delta+\nu)}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\sigma+\alpha k)}{\Gamma(\frac{1}{2} + \frac{(\sigma+\alpha k)}{2} + \frac{\delta}{2} - \frac{\nu}{2}) \, \Gamma(\frac{1}{2} + \frac{(\sigma+\alpha k)}{2} + \frac{\delta}{2} + \frac{\nu}{2})} \, E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{-\alpha}) \, . \tag{33}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0, \ \Re(\mu_i) > 0$.
(ii) $\sigma > 0$ and δ is a non negative integer .

$$I_{11} = \int_0^1 x^{\sigma-1}(1-x^2)^{\frac{-\delta}{2}} \, P_\nu^\delta(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^\alpha] \, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} \int_0^1 x^{\sigma+\alpha k-1}(1-x^2)^{\frac{-\delta}{2}} \, P_\nu^\delta(x) \, dx \tag{34}$$

Now we have the formula ([14], section 3.12, vol. 1)

$$\int_0^1 x^{\sigma-1}(1-x^2)^{\frac{-\delta}{2}}\ P_\nu^\delta(x)\ dx = \frac{(2)^{\delta-\sigma}\ \pi^{\frac{1}{2}}\Gamma(\sigma)}{\Gamma(\frac{1}{2}+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2})\ \Gamma(1+\frac{\sigma}{2}-\frac{\delta}{2}-\frac{\nu}{2})}\ , \tag{35}$$

where $\Re(\sigma) > 0,\ \delta = 1, 2, 3....$

Finally, by using (35) in (34), we get

$$I_{11} = (2)^{\delta-\sigma}\ \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma(\sigma+\alpha k)}{\Gamma(\frac{1}{2}+\frac{(\sigma+\alpha k)}{2}-\frac{\delta}{2}-\frac{\nu}{2})\ \Gamma(1+\frac{(\sigma+\alpha k)}{2}-\frac{\delta}{2}-\frac{\nu}{2})}\ E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{-\alpha})\ . \tag{36}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0,\ \Re(\mu_i) > 0$.
(ii) $\Re(\sigma) > 0$ and $\Re(\delta) > 1$.

6. INTEGRALS WITH HERMITE POLYNOMIALS

Hermite polynomial $H_n(x)$ ([4], p. 187) may be defined by means of the relation

$$exp(2xt - t^2) = \sum_{n=0}^{\infty} \frac{H_n(x)t^n}{n!} \tag{37}$$

valid for all finite x and t. Since

$$\begin{aligned} exp(2xt - t^2) &= exp(2xt)exp(-t^2) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\frac{n}{2}} \frac{(-1)^k(2x)^{n-2k}t^n}{k!(n-2k)!} \end{aligned}$$

It follows from (37) that

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!} \tag{38}$$

Examination of equation (38) shows that $H_n(x)$ is a polynomial of degree precisely n in x and that

$$H_n(x) = 2^n x^n + \pi_{n-2}(x), \tag{39}$$

in which $\pi_{n-2}(x)$ is a polynomial of degree (n-2) in x.

$$\begin{aligned} I_{12} &= \int_{-\infty}^{\infty} x^{2\rho}\ e^{-x^2}\ H_{2\nu}(x)E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^{-2h}]\ dx \\ &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})} \int_{-\infty}^{\infty} x^{2\rho-2kh}\ e^{-x^2}\ H_{2\nu}(x)\ dx \end{aligned} \tag{40}$$

Now by the formula ([12], p. 59)

$$\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2}\, H_{2\nu}(x)\, dx = \pi^{\frac{1}{2}} 2^{2(\nu-\rho)} \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)} \tag{41}$$

Hence (40) can be written as

$$I_{12} = (2)^{2(\nu-\rho)}\, \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma(2\rho-2kh+1)}{\Gamma(\rho-kh-\nu+1)}\, E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{2h}). \tag{42}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0,\ \Re(\mu_i) > 0$.
(ii) $h > 0$ and $\rho = 0, 1, 2, \ldots$.

$$I_{13} = \int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2}\, H_{2\nu}(x) E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^{2h}]\, dx$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})} \int_{-\infty}^{\infty} x^{2\rho+2kh}\, e^{-x^2}\, H_{2\nu}(x)\, dx \tag{43}$$

Now by the formula ([12], p. 59)

$$\int_{-\infty}^{\infty} x^{2\rho}\, e^{-x^2}\, H_{2\nu}(x)\, dx = \pi^{\frac{1}{2}} 2^{2(\nu-\rho)} \frac{\Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)} \tag{44}$$

Hence (43) can be written as

$$I_{12} = (2)^{2(\nu-\rho)}\, \pi^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{\Gamma(2\rho+2kh+1)}{\Gamma(\rho+kh-\nu+1)}\, E_{(\frac{1}{\rho_i}),(\mu_i)}(z2^{-2h}). \tag{45}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0,\ \Re(\mu_i) > 0$.
(ii) $h > 0$ and $\rho = 0, 1, 2, \ldots$.

7. INTEGRAL WITH HYPERGEOMETRIC FUNCTION

In the study of second order linear differential equations with three singular points, there arise a function

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \tag{46}$$

for c is neither zero nor a negative integer in (46) the notation

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)(\alpha+3)\cdots(\alpha+n-1),\ n \geq 1, \tag{47}$$

$$(\alpha)_0 = 1,\ \alpha \neq 0$$

is called the factorial notation and the function in () is called the hypergeometric function ([4], p. 45).

$$I_{14}=\int_1^{\infty}x^{-\rho}(x-1)^{\sigma-1}\,{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho;\\ \\ \sigma;\end{array}(1-x)\right]E_{(\frac{1}{\rho_i}),(\mu_i)}(zx)\,dx$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\int_1^{\infty}x^{k-\rho}(x-1)^{\sigma-1}\,{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho;\\ \\ \sigma;\end{array}(1-x)\right]dx$$

Let $x=t+1$, then

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\int_1^{\infty}(t+1)^{k-\rho}\,t^{\sigma-1}\,{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho;\\ \\ \sigma;\end{array}-t\right]dt,$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}\sum_{k=0}^{\infty}\frac{(-1)^k(\nu+\sigma-\rho)_k(\lambda+\sigma-\rho)_k}{(\sigma)_k k!}\int_1^{\infty}(t+1)^{k-\rho}\,t^{k+\sigma-1}\,dt$$

$$I_{14}=E_{(\frac{1}{\rho_i}),(\mu_i)}(z)\,{}_2F_1\left[\begin{array}{c}\nu+\sigma-\rho,\lambda+\sigma-\rho;\\ \\ \sigma;\end{array}-1\right]B(\sigma+k,\ \rho-2k-\sigma).\qquad(48)$$

Provided $\Re(\frac{1}{\rho_i})>0$ and $\Re(\mu_i)>0$.

8. INTEGRALS WITH GENERALIZED HYPERGEOMETRIC FUNCTION

A generalized hypergeometric function ([4], p. 73) is defined by

$${}_pF_q\left[\begin{array}{cccc}(\alpha_1), & (\alpha_2), & \cdots & (\alpha_p);\\ \\ (\beta_1), & (\beta_2), & \cdots & (\beta_q);\end{array}z\right]=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!}={}_pF_q(\alpha_1,\cdots\alpha_p;\beta_1,\cdots\beta_q;z)\qquad(49)$$

where no denominator parameter β_j is allowed to be zero or negative integer.

$$I_{15}=\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\,{}_pF_q[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}]E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^u(t-x)^v]\,dx$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}t^{vk+\sigma-1}\int_0^t x^{uk+\rho-1}\left(1-\frac{x}{t}\right)^{vk+\sigma-1}{}_pF_q[(g_p);(h_q);ax^{\alpha}(t-x)^{\beta}]\,dx$$

Let $x=st$ and $dx=tds$, then we get

$$=\sum_{k=0}^{\infty}\frac{zt^{(u+v)_k}}{\Gamma(\mu_1+\frac{k}{\rho_1})\cdots\Gamma(\mu_m+\frac{k}{\rho_m})}t^{\rho+\sigma-1}\int_0^1 s^{uk+\rho-1}\,(1-s)^{vk+\sigma-1}\,{}_pF_q[(g_p);(h_q);as^{\alpha}t^{\alpha+\beta}(1-s)^{\beta}]\,ds$$

$$= \sum_{k=0}^{\infty} \frac{zt^{(u+v)_k}}{\Gamma(\mu_1 + \frac{k}{\rho_1}) \cdots \Gamma(\mu_m + \frac{k}{\rho_m})} t^{\rho+\sigma-1} \int_0^1 s^{uk+\alpha k+\rho-1} (1-s)^{vk+\beta k+\sigma-1} \sum_{k=0}^{\infty} \frac{(g_p)_k}{(h_q)_k} \frac{t^{(\alpha+\beta)_k} a^k}{k!} \, ds,$$

where

$$f(k) = \sum_{k=0}^{\infty} \frac{(g_p)_k}{(h_q)_k} \frac{a^k}{k!} = \frac{(g_1)_k \cdots (g_p)_k}{(h_1)_k \cdots (h_q)_k} \frac{a^k}{k!} \tag{50}$$

where α, β are non negative integer such that $\alpha + \beta \geq 1$.

$$I_{15} = t^{\sigma+\rho-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)_k} E_{(\frac{1}{\rho_i}),(\mu_i)}[zt^{u+v}] B(\rho + uk + \alpha k, \sigma + vk + \beta k). \tag{51}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) \geq 0$ and $\Re(\nu) \geq 0$ (both are not zero simultaneously).
(iii) α and β are non negative integer such that $\alpha + \beta \geq 1$.

$$\begin{aligned} I_{16} &= \int_0^t x^{\rho-1}(t-x)^{\sigma-1} \, {}_pF_q[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}] E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^{-u}(t-x)^{-v}] \, dx \\ &= t^{\sigma+\rho-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)_k} E_{(\frac{1}{\rho_i}),(\mu_i)}[zt^{-u-v}] B(\rho - uk + \alpha k, \sigma - k + \beta k). \end{aligned} \tag{52}$$

where f(k) is defined in (50).

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) \geq 0$ and $\Re(\nu) \geq 0$ (both are not zero simultaneously).
(iii) α and β are non negative integer such that $\alpha + \beta \geq 1$.

$$\begin{aligned} I_{17} &= \int_0^t x^{\rho-1}(t-x)^{\sigma-1} \, {}_pF_q[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}] E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^{u}(t-x)^{-v}] \, dx \\ &= t^{\sigma+\rho-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)_k} E_{(\frac{1}{\rho_i}),(\mu_i)}[zt^{u-v}] B(\rho + uk + \alpha k, \sigma - k + \beta k). \end{aligned} \tag{53}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) \geq 0$ and $\Re(\nu) \geq 0$ (both are not zero simultaneously).
(iii) α and β are non negative integer such that $\alpha + \beta \geq 1$.

$$I_{18} = \int_0^t x^{\rho-1}(t-x)^{\sigma-1} \, {}_pF_q[(g_p);(h_q); ax^{\alpha}(t-x)^{\beta}] E_{(\frac{1}{\rho_i}),(\mu_i)}[zx^{-u}(t-x)^{v}] \, dx$$

$$= t^{\sigma+\rho-1} \sum_{k=0}^{\infty} f(k) t^{(\alpha+\beta)_k} E_{(\frac{1}{\rho_i}),(\mu_i)}[zt^{-u+v}] B(\rho - uk + \alpha k, \sigma + k + \beta k). \tag{54}$$

Provided
(i) $\Re(\frac{1}{\rho_i}) > 0$, $\Re(\mu_i) > 0$.
(ii) $\Re(\alpha) \geq 0$ and $\Re(\nu) \geq 0$ (both are not zero simultaneously).
(iii) α and β are non negative integer such that $\alpha + \beta \geq 1$.

9. Conclusions

In this paper, we briefly touch on a number of results related to the multiple (multiindex) Mittag-Leffler function involving Jacobi polynomial, Bessel Maitland function, Legendre function, Hermite polynomial, Hypergeometric function and Generalized Hypergeometric function. The results associated with this paper involving multiple (multiindex) Mittag-Leffler function applied to many different engineering discipline including signal processing, control engineering and many other field such as biology and neuro sciences. Fractional operators are the generalization of differentiation and integration of integer order calculus that allow us to present more accurate discription of real systems which includes a combination of multi-disciplinary fields of geometry. A fractional generalization of the Poisson probability distribution (which is called Mittag-Leffler distribution) was introduced using the complete monotonicity of the Mittag-Leffler function.

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