A PROOF OF DAO’S GENERALIZATION OF THE SIMSON LINE THEOREM

LEONARD MIHAI GIUGIUC

Abstract. Using the complex number, we give a proof of Dao’s generalization of the Simson line theorem.

1. INTRODUCTION

The famous Simson line theorem states that: *Given a triangle ABC and a point P on its circumcircle, the three closest points to P on lines AB, AC, and BC are collinear* [1]. More properties of the Simson line we can see in [2][3][4] and [5] and many another texts. In 2014, O. T. Dao posed a very nice generalization of the Simson line theorem [6]. The Dao theorem as follows:

**Theorem 1.1** (Dao-[6]). *Let ABC be a triangle, let a point P on its circumcircle and let a line l through its circumcenter. Let AP, BP, CP meet l at A_P, B_P, C_P. Let A_0, B_0, C_0 be the projections of A_P, B_P, C_P to BC, CA, AB respectively. Then A_0, B_0, C_0 are collinear, and the line \( \overline{A_0B_0C_0} \) bisect the orthocenter and P.*

- When \( l \) through \( P \), the Dao theorem is the Simson line theorem.

![Figure 1](image)

In this article we give a proof of this theorem by complex number.

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2. A PROOF OF THEOREM 1

Lemma 2.1. Let \( z \) be a complex number, such that \( |z| = 1 \), then \( \overline{z} = \frac{1}{z} \).

Proof. Let \( z = x + yi \), where \( x, y \) be two real numbers, then \( z\overline{z} = (x - yi)(x + yi) = x^2 + y^2 = 1 \), so \( \overline{z} = \frac{1}{z} \).

Now back to main theorem:

Proof. We choose the circle \( C \) with \( |z| = 1 \), and the line \( l \), such that imaginary part of \( z = 0 \). Let \( A = a, B = b, C = c, P = p \), where \( |a| = |b| = |c| = |p| = 1 \).
Consider \( A_p = x \in R \), then \( \frac{x - a}{p - a} \in R \Rightarrow \frac{x - a}{p - a} = (\overline{\frac{x - a}{p - a}}) \Rightarrow \frac{x - a}{p - a} = \frac{x - \overline{a}}{p - \overline{a}} \).
\( \Leftrightarrow x - a = p(1 - ax) \Leftrightarrow A_p = \frac{a + p}{1 + ap} \), similarly we get that \( B_p = \frac{b + p}{1 + cp} \) and \( C_p = \frac{c + p}{1 + cp} \).
Let \( A_1 \) be the reflection of \( A_p \) in \( BC \). Then the triangles \( A_pBC \) and \( A_1BC \) are congruent and inverse orientated, so \( \frac{A_1 - B}{C - B} = (\frac{A_p - B}{C - B}) \). Let \( A_1 = z \Rightarrow \)

\[
\frac{z - b}{c - b} = \left( \frac{1}{a} + \frac{1}{p} \right) - \frac{1}{b} \left( \frac{1}{c} - \frac{1}{b} \right)
\]

\( \Leftrightarrow \)

\[
z - b = \frac{c(1 + ap - ab - bp)}{1 + ap}
\]

\( \Leftrightarrow \)

\[
z = \frac{b + c - abc + p(ab + ac - bc)}{1 + ap}
\]

On the other hand \( A_0 \) is the midpoint of \( A_1A_p \), so

\[
A_0 = \frac{a + b + c - abc + p(1 + ab + ac - bc)}{2(1 + ap)}
\]

Similarly we get that:

\[
B_0 = \frac{a + b + c - abc + p(1 + ab + bc - ac)}{2(1 + bp)}
\]

\[
C_0 = \frac{a + b + c - abc + p(1 + ac + bc - ba)}{2(1 + cp)}
\]

For simplify the calculations, denote that: \( a + b + c - abc + p(1 + ab + ac + bc) = k \) then

\[
B_0 - A_0 = \frac{k - 2pac}{2(1 + bp)} = \frac{k - 2pbc}{2(1 + bp)} = \frac{p(a - b)(k - 2c - 2acp - 2bcp)}{2(1 + ap)(1 + bp)} = \frac{p(a - b)[a + b - c - abc + p(1 + ab - ac - bc)]}{2(1 + ap)(1 + bp)}
\]

Similarly we get that:

\[
C_0 - A_0 = \frac{p(a - c)[a + c - b - abc + p(1 + ac - ab - bc)]}{2(1 + ap)(1 + cp)}
\]

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So we get that:

\[
\frac{B_0 - A_0}{C_0 - A_0} = \left( \frac{a - b}{a - c} \right) \cdot \left( 1 + \frac{1 + cp}{1 + bp} \right) \cdot \left[ \frac{a + b - c - abc + p(1 + ab - ac - bc)}{a + c - b - abc + p(1 + ac - ab - bc)} \right]
\]

Now by \(|z| = 1\) then \(z = \frac{1}{z} = 1\) we get that:

\[
\frac{B_0 - A_0}{C_0 - B_0} = \left( \frac{a - b}{a - c} \right) \cdot \left( 1 + \frac{1 + cp}{1 + bp} \right) \cdot \left[ \frac{a + b - c - abc + p(1 + ab - ac - bc)}{a - b + c - abc + p(1 - ab + ac - bc)} \right]
\]

\[
= \left( \frac{b - a}{ac} \right) \left( \frac{1 + cp}{1 + bp} \right) \left[ \frac{a + b - c - abc + p(1 + ab - ac - bc)}{a - b + c - abc + p(1 - ab + ac - bc)} \right]
\]

Hence, \(A_0, B_0, C_0\) are collinear.

By Sylvester’s theorem we get that the orthocenter of triangle ABC is

\(H = a + b + c \Rightarrow M = \frac{a + b + c + p}{2}\) is the midpoint of \(PH\), therefor:

\[
M - A_0 = \frac{abc + p(a^2 + ap + bc)}{2(1 + ap)} \quad \text{and} \quad M - B_0 = \frac{abc + p(b^2 + bp + ac)}{2(1 + bp)}
\]

So we get that:

\[
\frac{M - A_0}{M - B_0} = \left( \frac{1 + bp}{2bp} \right) \frac{1}{abc} + \frac{1}{p} \left( \frac{bcp + abc + a^2p}{a^2bc} \right) \left[ \frac{1 + bp}{1 + ap} \right]
\]

\[
= \frac{M - A_0}{M - B_0}
\]

Thus, \(A_0, B_0, M\) are collinear. This complete the proof of Theorem 1.1.

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\square
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References

[6] Personal communication with the author, 06/May/2014

Bulevardul Carol I No 6, Colegiul National Traian, Drobeta Turnu-Severin, Romania
E-mail address: leonardgiugiuc@yahoo.com