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A PROOF OF DAO'S GENERALIZATION OF THE SIMSON LINE THEOREM

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ABSTRACT. Using the complex number, we give a proof of Dao's generalization of the Simson line theorem.

1. INTRODUCTION

The famous Simson line theorem states that: *Given a triangle ABC and a point P on its circumcircle, the three closest points to P on lines AB, AC, and BC are collinear* [1]. More properties of the Simson line we can see in [2][3][4] and [5] and many another texts. In 2014, O. T. Dao posed a very nice generalization of the Simson line theorem [6]. The Dao theorem as follows:

Theorem 1.1 (Dao-[6]). *Let ABC be a triangle, let a point P on its circumcircle and let a line l through its circumecenter. Let AP, BP, CP meet l at A_P, B_P, C_P. Let A_0, B_0, C_0 be the projections of A_P, B_P, C_P to BC, CA, AB respectively. Then A_0, B_0, C_0 are collinear, and the line $\overline{A_0B_0C_0}$ bisect the orthocenter and P.*

- When l through P, the Dao theorem is the Simson line theorem.

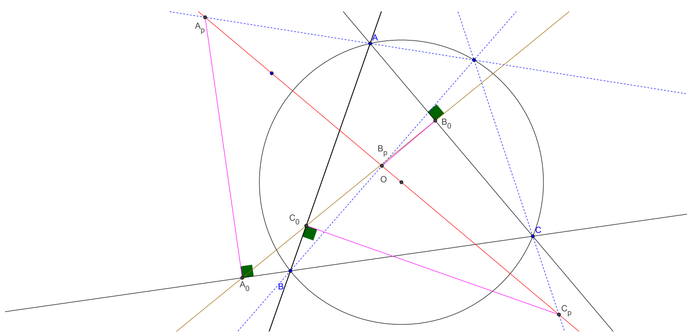

FIGURE 1

In this article we give a proof of this theorem by complex number.

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2. A PROOF OF THEOREM 1

Lemma 2.1. *Let z be a complex number, such that $|z| = 1$, then $\overline{z} = \frac{1}{z}$*

Proof. Let $z = x + yi$, where x,y be two real numbers, then $z\overline{z} = (x - yi)(x + yi) = x^2 + y^2 = 1$, so $\overline{z} = \frac{1}{z}$. □

Now back to main theorem:

Proof. We choose the circle C with $|z| = 1$, and the line l, such that imaginary part of $z = 0$. Let $A = a$,$B = b$,$C = c$, $P = p$, where $|a| = |b| = |c| = |p| = 1$.
Consider $A_p = x \in R$, then $\frac{x-a}{p-a} \in R \Rightarrow \frac{x-a}{p-a} = \overline{(\frac{x-a}{p-a})} \Rightarrow \frac{x-a}{p-a} = \frac{x-\frac{1}{a}}{\frac{1}{p}-\frac{1}{a}}$
$\Leftrightarrow x - a = p(1 - ax) \Leftrightarrow A_p = \frac{a+p}{1+ap}$, similarly we get that $B_p = \frac{b+p}{1+bp}$ and $C_p = \frac{c+p}{1+cp}$.
Let A_1 be the reflection of A_p in BC. Then the triangles A_pBC and A_1BC are congruent and inverse ortientated, so $\frac{A_1-B}{C-B} = \overline{(\frac{A_p-B}{C-B})}$. Let $A_1 = z \Rightarrow$

$$\frac{z-b}{c-b} = \left(\frac{\frac{1}{a}+\frac{1}{p}}{1+\frac{1}{ap}} - \frac{1}{b}\right)\left(\frac{1}{\frac{1}{c}-\frac{1}{b}}\right)$$

$\Leftrightarrow$

$$z - b = \frac{c(1 + ap - ab - bp)}{1 + ap}$$

$\Leftrightarrow$

$$z = \frac{b + c - abc + p(ab + ac - bc)}{1 + ap}$$

On the other hand A_0 is the midpoint of A_1A_p, so

$$A_0 = \frac{a + b + c - abc + p(1 + ab + ac - bc)}{2(1 + ap)}$$

Similarly we get that:

$$B_0 = \frac{a + b + c - abc + p(1 + ab + bc - ac)}{2(1 + bp)}$$

$$C_0 = \frac{a + b + c - abc + p(1 + ac + bc - ba)}{2(1 + cp)}$$

For simplify the calculations, denote that: $a + b + c - abc + p(1 + ab + ac + bc) = k$ then

$$B_0 - A_0 = \frac{k - 2pac}{2(1 + bp)} - \frac{k - 2pbc}{2(1 + ap)} = \frac{p(a - b)(k - 2c - 2acp - 2bcp)}{2(1 + ap)(1 + bp)}$$
$$= \frac{p(a - b)[a + b - c - abc + p(1 + ab - ac - bc)]}{2(1 + ap)(1 + bp)}$$

Similarly we get that:

$$C_0 - A_0 = \frac{p(a - c)[a + c - b - abc + p(1 + ac - ab - bc)]}{2(1 + ap)(1 + cp)}$$

So we get that:

$$\frac{B_0 - A_0}{C_0 - A_0} = (\frac{a-b}{a-c}).(\frac{1+cp}{1+bp}).\left[\frac{a+b-c-abc+p(1+ab-ac-bc)}{a+c-b-abc+p(1+ac-ab-bc)}\right]$$

Now by $|z| = 1$ then $\overline{z} = \frac{1}{z} = 1$ we get that:

$$\overline{\frac{B_0 - A_0}{C_0 - B_0}} = \overline{\left(\frac{a-b}{a-c}\right) \cdot \left(\frac{1+cp}{1+bp}\right) \cdot \left[\frac{a+b-c-abc+p(1+ab-ac-bc)}{a-b+c-abc+p(1-ab+ac-bc)}\right]}$$

$$= \frac{\frac{b-a}{ab}}{\frac{c-a}{ac}} . \frac{\frac{1+cp}{cp}}{\frac{1+cp}{cp}} . \frac{\frac{bc+ac-ab-1}{abc} + \frac{1}{p}\left(\frac{abc+c-a-b}{abc}\right)}{\frac{bc-ac+ab-1}{abc} + \frac{1}{p}\left(\frac{abc+b-a-c}{abc}\right)}$$

$$= \left(\frac{a-b}{a-c}\right) \cdot \left(\frac{1+cp}{1+bp}\right) \cdot \left[\frac{a+b-c-abc+p(1+ab-ac-bc)}{a-b+c-abc+p(1-ab+ac-bc)}\right] = \frac{B_0 - A_0}{C_0 - B_0}$$

Hence, A_0, B_0, C_0 are collinear.
By Sylvester's theorem we get thet the orthocenter of triangle ABC is
$H = a + b + c \Rightarrow M = \frac{a+b+c+p}{2}$ is the midpoint of PH, therefor:

$$M - A_0 = \frac{abc + p(a^2 + ap + bc)}{2(1+ap)} \text{ and } M - B_0 = \frac{abc + p(b^2 + bp + ac)}{2(1+bp)}$$

So we get that:

$$\overline{\left(\frac{M - A_0}{M - B_0}\right)} = \frac{\frac{1+bp}{bp}}{\frac{1+ap}{ap}} . \frac{\frac{1}{abc} + \frac{1}{p}\left(\frac{bcp+abc+a^2p}{a^2bcp}\right)}{\frac{1}{abc} + \frac{1}{p}\left(\frac{bcp+abc+b^2p}{b^2acp}\right)} = \left(\frac{1+bp}{1+ap}\right)\left[\frac{abc+p(a^2+ap+bc)}{abc+p(b^2+bp+ac)}\right]$$

$$= \frac{M - A_0}{M - B_0}$$

Thus, A_0, B_0, M are collinear. This complete the proof of Theorem 1.1. □

References

[1] H. S. M. Coxeter and S.L. Greitzer, *Geometry revisited*, Math. Assoc. America, **1967**: p.41.
[2] Ramler, O. J. *The Orthopole Loci of Some One-Parameter Systems of Lines Referred to a Fixed Triangle.* Amer. Math. Monthly 37, 130-136, **1930.**
[3] Butchart, J. H. *The Deltoid Regarded as the Envelope of Simson Lines.* Amer. Math. Monthly 46, 85-86, **1939.**
[4] van Horn, C. E. *The Simson Quartic of a Triangle.* Amer. Math. Monthly 45, 434-437, **1938**.
[5] Gallatly, W. *The Simson Line. Ch. 4 in The Modern Geometry of the Triangle*, 2nd ed. London: Hodgson, pp. 24-36, **1913.**
[6] Personal communication with the author, **06/May/2014**

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