GENERALIZATION OF MUSSELMAN’S THEOREM. SOME PROPERTIES OF ISOGONAL CONJUGATE POINTS

NGO QUANG DUONG

ABSTRACT. In this article, we generalize of Musselman’s theorem and study on some properties of isogonal conjugate points with angle chasing mainly.

1. INTRODUCTION

Theorem 1. (Musselman, [1]) \( \triangle ABC \), \( D, E, F \) are reflections of \( A, B, C \) in \( BC, CA, AB \), respectively. Let \( O \) be circumcenter of \( \triangle ABC \). \((AOD), (BOE), (COF)\) are coaxial and the intersection other than \( O \) is the inverse of Kosnita point with respect to \( (O) \).

The inverse of Kosnita point \((X_{54})\) with respect to \((O)\) is \( X_{1157} \) in Encyclopedia of Triangle Centers, see [2]. \( X_{1157} \) lies on Neuberg cubic and it is the tangential of \( O \) on the Neuberg cubic.

2010 Mathematics Subject Classification. 51M04.
Key words and phrases. Triangle geometry, isogonal conjugate, circumcircle, concyclic, coaxial circle, angle chasing, collinear, concurrent.
Theorem 2. (Yiu, [3]) \((AEF), (BFD), (CDE)\) pass through the inverse of Kosnita point with respect to \((O)\).

\[\text{Figure 2}\]

Theorem 3. (Gibert, [4]) \(X, Y, Z\) are reflections of \(X_{1157}\) in \(BC, CA, AB\). \(AX, BY, CZ\) are concurrent at a point on \((O)\).

Neuberg cubic is locus of \(P\) such that reflections of \(P\) in \(BC, CA, AB\) form a triangle that perspective with \(\triangle ABC\), locus of the perspectors is a cubic [5]. When \(P\) coincides with \(X_{1157}\), we obtain \(X_{1141}\), the only perspector lies on circumcircle other than \(A, B, C\).
2. **Generalization of Musselman’s Theorem and Some Properties Around Its Configuration**

2.1. **Generalization theorem.**

**Theorem 4.** (Generalization of Musselman’s theorem, [6]) Let $P, Q$ be isogonal conjugate points with respect to $\triangle ABC$.

$PA, PB, PC$ intersects $(PBC), (PCA), (PAB)$ at $D, E, F \neq P$, respectively. Then $(AQD), (BQE), (CQF)$ are coaxial.

![Figure 4. Generalization of Musselman’s theorem](image)

If $P$ coincides with orthocenter of $\triangle ABC$, we have Musselman’s theorem.

**Proof.** Let $QA, QB, QC$ intersect $(QBC), (QCA), (QAB)$ at $X, Y, Z \neq Q$.

First, we need some lemmas.

**Lemma 5.** $PQ$ is parallel to $DX, EY, FZ$.

**Proof.** Since $P, Q$ are isogonal conjugate:

$(AB, AP) = (AQ, AC) = (AX, AC)$

$(XA, XC) = (XQ, XC) = (BQ, BC) = (BA, BP)$

Therefore $\triangle ABP$ and $\triangle AXC$ are directly similar (angle-angle).

Thus $AB.AC = AP.AX$. Similarly, $AB.AC = AQ.AD$.

$AP.AX = AQ.AD \iff \frac{AP}{AD} = \frac{AQ}{AX}$

$\Rightarrow PQ \parallel DX$. Similarly, we can prove $PQ \parallel EY, FZ$. \hfill \square

**Lemma 6.** $D$ and $X, E$ and $Y, F$ and $Z$ are isogonal conjugate points with respect to $\triangle ABC$.

$BF, CE$ pass through $X$; $CD, AF$ pass through $Y$; $AE, BD$ pass through $Z$.

$BZ, CY$ pass through $D$; $CX, AZ$ pass through $E$; $AY, BX$ pass through $F$. 

17
Proof. (See figure 6) Similar to the proof of lemma 5, we have \( \triangle APC \) and \( \triangle ABX \) are directly similar, \( \triangle APB \) and \( \triangle ACX \) are directly similar.

\[
(BC, BD) = (PC, PD) = (PC, PA) = (BX, BA)
\]
\[
(CB, CD) = (PB, PD) = (PB, PA) = (CX, CA)
\]

So \( D, X \) are isogonal conjugate.

\[
(BX, BF) = (BX, BA) + (BA, BF) = (PC, PA) + (PA, PF) = 0
\]

Hence \( BF \) passes through \( X \).
**Lemma 7.** \((ABC), (APX), (AQP)\) are coaxial.

**Proof.** Considering the inversion \(I(A, AB, AC)\):

\[
B, C, P, Q, D, X \mapsto B', C', P', Q', D', X'
\]

\((ABC), (APX), (AQP) \rightarrow B'C', P'X', Q'D'\)

Since \(AP.AX = AQ.AD = AB.AC\), these pairs of points: \((B, C'), (C, B'), (P, X'), (Q, D'), (D, Q'), (X, P')\) are symmetrically through bisector of \(\angle BAC\).

Hence, instead of prove \(B'C', P'X', Q'D'\) are concurrent, we prove \(BC, PX, QD\) are concurrent.

Considering \(\triangle BPD\) and \(\triangle CXQ:\)

According to lemma 6:

- \(BD\) intersects \(CQ\) at \(Z\), \(BP\) intersects \(CX\) at \(E\), \(PD\) intersects \(QX\) at \(A\) and \(Z\), \(A\), \(E\) are collinear.

Then by Desargues’s theorem, \(BC, PX, QD\) are concurrent.

**Back to the main proof.**

From lemma 7: \((ABC), (APX), (AQP)\) have two common points \(A, A'\)

\((ABC), (BYP), (BQE)\) have two common points \(B, B'\)

\((ABC), (CPZ), (CQF)\) have two common points \(C, C'\)

Let \(N\) be midpoint of \(PQ\).

\[
(DA', DA) = (DA', DA) = (QA', QA) = (QX, Q)
\]

\[
(PA', PD) = (PA', PA) = (XA', XA) = (XQ, X)
\]
Hence, $\triangle A'DP$ and $\triangle A'QX$ are similar.

$$\Rightarrow \frac{AP}{AQ} = \frac{PD}{QX} = \frac{d(A', AP)}{d(A', AQ)}$$

( Note that $d(M, \ell)$ is distance from $M$ to the line $\ell$ ).
This means distances from $A'$ to $AP, AQ$ are proportional to $AP, AQ$.
So $AA'$ is the symmedian of $\triangle APQ$ then $AN, AA'$ are isogonal lines with respect to $\angle BAC$. Similarly, $BB', BN$ are isogonal lines with respect to $\angle ABC; CC', CN$ are isogonal
Generalization of Musselman’s theorem. Some properties of isogonal conjugate points

lines with respect to $\angle ACB$.

So $AA', BB', CC'$ are concurrent at $N'$ - isogonal conjugate of $N$ with respect to $\triangle ABC$ (when $P, Q$ coincide with orthocenter and circumcenter, $N'$ become Kosnita point). Let $P', Q'$ be two points on $PN', QN'$ such that:

$$N'P'N' = N'Q'N' = P_{N/(ABC)}$$

Then $(AQD), (BQE), (CQF)$ pass through $Q'$ and $(APX), (BPY), (CPZ)$ pass through $P$.

$\implies (AQD), (BQE), (CQF)$ are coaxial, $(APX), (BPY), (CPZ)$ are coaxial. 

**Theorem 8.** The circles $(AEF), (BFD), (CDE)$ pass through $Q'$. 

![Figure 10](image)

**Proof.**

$$(Q'E, Q'F) = (Q'E, Q'Q) + (Q'Q, Q')$$

$$= (BE, BQ) + (CQ, CF) \quad (B, Q, E, Q' \text{ are concyclic and } C, Q, F, Q' \text{ are concyclic})$$

$$= (BE, BA) + (BA, BQ) + (CQ, CA) + (CA, CF)$$

$$= (BP, BA) + (BP, BC) + (CB, CP) + (CA, CP) \quad (P, Q \text{ are isogonal conjugate})$$

$$= (BP, BA) + (AB, AC) + (CA, CP) + (AC, AB) + (PB, PC)$$

$$= (AC, AB) + 2(PB, PC)$$

$$= (AC, AB) + (PB, PF) + (PE, PC)$$

$$= (AC, AB) + (AB, AF) + (AE, AC)$$

$$= (AE, AF)$$

$\implies Q'$ lies on $(AEF)$.

Similarly, $Q'$ lies on $(BFD), (CDE)$. 

**Theorem 9.** (Generalization of Gibert point) Let the lines that pass through $Q'$ and parallel to $PA, PB, PC$ intersects $(AQD), (BQE), (CQF)$ at $A_Q, B_Q, C_Q \neq Q$. $AA_Q, BB_Q, CC_Q$ are concurrent at a point on $(ABC)$. 

21
Proof. Let $G$ be intersection of $AA_Q$ and $BB_Q$. We show that $G$ lies on $(ABC)$.

$$(GA, GB) = (AA_Q, BB_Q)$$

$$= (AA_Q, Q'A_Q) + (Q'A_Q, Q'B_Q) + (Q'B_Q, BB_Q)$$

$$= (QA, QQ') + (PA, PB) + (QQ', QB)$$

$$= (PA, PB) + (QA, QB)$$

$$= (AP, AB) + (BA, BP) + (AQ, AB) + (BA, BQ)$$

$$= (AP, AB) + (BA, BP) + (AC, AP) + (BP, BC)$$  \hspace{0.5cm} (P, Q \text{ are isogonal conjugate})

$$= (CA, CB)$$

Similarly, the intersections of $BB_Q$, $CC_Q$ lies on $(ABC)$, therefore $AA_Q$, $BB_Q$, $CC_Q$ are concurrent at a point on $(ABC)$.

\[\square\]

2.2. Some properties.

**Proposition 10.** The following sets of 4 points are concyclic:

(B, C, F, Y), (B, C, E, Z).

(C, A, D, Z), (C, A, F, X).

(A, B, E, X), (A, B, D, Y).

Proof.

$$(FB, FC) = (FB, FP) = (AB, AP)$$

Since $P, Q$ are isogonal conjugate

$$(AB, AP) = (AQ, AC) = (YQ, YC) = (YB, YC)$$

Hence, $B, C, F, Y$ are concyclic.
Proposition 11. \( EF, YZ, BC \) are concurrent.

Proof. From lemma 6, \( FY \) intersect \( EZ \) at \( A \). \( BF \) intersects \( CE \) at \( X \). \( BY \) intersects \( CZ \) at \( Q \). Since \( A, Q, X \) are collinear then by Desargues’s theorem, \( EF, YZ, BC \) are concurrent. \( \square \)

Proposition 12.

\[(DYZ), (EZX), (FXY), (PDX), (PEY), (PFZ) \text{ have a common point.}\]
\[(XEF), (YFD), (ZDE), (QDX), (QEY), (QFZ) \text{ have a common point.}\]

Proof. From lemma 6, \( D, Y, C \) are collinear and \( D, Z, B \) are collinear, then:
\[(DY, DZ) = (DC, DB) = (PC, PB)\]
Similarly:
\[(EZ, EX) = (EA, EC) = (PA, PC)\]
\[(FX, FY) = (FB, FA) = (PB, PA)\]
\[\Rightarrow (DY, DZ) + (EZ, EX) + (FX, FY) = 0. \text{ Hence } (DYZ), (EZX), (FXY) \text{ have a common point.}\]

23
point $S$. Now from symmetry we only need to prove that $S$ lies on $(PDX)$.

$$(SD, SX) = (SD, SY) + (SY, SX)$$

$$= (ZD, ZY) + (FY, FX) \quad (S, D, Y, Z \text{ are concyclic, } S, X, Y, F \text{ are concyclic})$$

$$= (ZB, ZY) + (FA, FB) \quad (Z, D, B \text{ are collinear})$$

$$= (ZB, ZY) + (PA, PB) \quad (F, A, B, P \text{ are concyclic})$$

$$(PD, PX) = (PA, PX)$$

$$= (P'A, P'X) \quad (A, P, X, P' \text{ are concyclic})$$

$$= (P'A, P'Z) + (P'Z, P'X)$$

$$= (AY, YZ) + (BZ, BX) \quad (A, P', Y, Z \text{ are concyclic, } B, Z, X, P' \text{ are concyclic})$$

$$(SD, SX) - (PD, PX) = (PA, PB) + (BZ, AY) + (BX, BZ)$$

$$= (PA, PB) + (BZ, BA) + (AB, AY) + (BX, BC) + (BC, BZ)$$

$$= (PA, PB) + (BC, BF) + (AE, AC) + (BA, BD) + (BF, BA)$$

$$= (PA, PB) + (BC, BD) + (AE, AC)$$

$$= (PA, PB) + (PC, PD) + (PE, PC)$$

$$= (PA, PB) + (PC, PA) + (PB, PC)$$

$$= 0$$

Therefore, $S$ lies on $(PDX)$. \qed
**Proposition 13.** \((ADX), (AEY), (AFZ), (APQ)\) are tangent at \(A\).
\((BDX), (BEY), (BFZ), (BPQ)\) are tangent at \(B\).
\((CDX), (CEY), (CFZ), (CPQ)\) are tangent at \(C\).

*Figure 15*

*Proof.* Since \(EY \parallel FZ\) and \(EZ, FY\) pass through \(A\), \((AEY)\) and \((AFZ)\) are tangent at \(A\).
\(DX \parallel PQ, PD, QX\) pass through \(A\) so \((APQ), (ADX)\) are tangent at \(A\).
Let \(AM, AN\) be tangent lines of \((APQ), (AEY)\) at \(A\).

\[
(AM, AN) = (AM, AP) + (AP, AE) + (AE, AN)
= (QA, QP) + (AP, AE) + (YE, YA)\quad (AN \text{ is tangent line of } (AEY))
\]

Since \(PQ \parallel EY:\)

\[
(AM, AN) = (AQ, AY) + (AP, AE)
= (AQ, AC) + (AC, AY) + (AP, AB) + (AB, AE)
\]

Because \(P, Q\) and \(E, Y\) are isogonal conjugate with respect to \(\triangle ABC:\)

\[
(AQ, AC) + (AP, AB) = 0\quad (AC, AY) + (AB, AE) = 0
\]

\(\Rightarrow (AM, AN) = 0,\) then \(A, M, N\) are collinear.

Hence, \((ADX), (AEY), (AFZ), (APQ)\) are tangent at \(A\). \(\square\)

**Proposition 14.** Suppose that:
\(\ell_a\) is radical axis of \((ADX), (AEY), (AFZ), (APQ)\)
\(\ell_b\) is radical axis of \((BDX), (BEY), (BFZ), (BPQ)\)
\(\ell_c\) is radical axis of \((CDX), (CEY), (CFZ), (CPQ)\)
Then \(\ell_a, \ell_b, \ell_c\) are concurrent at a point on \((ABC)\).
Proof. $\ell_a, \ell_b, \ell_c$ are tangent lines at $A, B, C$ of $(APQ), (BPQ), (CPQ)$. Tangent line at $A$ of $(APQ)$ is isogonal line of the line that passes through $A$ and parallel to $PQ$ with respect to $\angle BAC$. Therefore, $\ell_a$ passes through isogonal conjugate of infinity point on $PQ$, which lies on $(ABC)$. Hence $\ell_a, \ell_b, \ell_c$ are concurrent at a point on $(ABC)$. □

**Figure 16**

**Proposition 15.** The following sets of 4 points are concyclic:

$$(Q', D, X, P'), (Q', E, Y, P'), (Q', F, Z, P'), (Q', P, Q, P')$$

**Figure 17**
Proof. Let $N_a, N_b, N_c$ be midpoints of $DX, EY, FZ$ and $N'_a, N'_b, N'_c$ be isogonal conjugate of $N_a, N_b, N_c$ with respect to $\triangle ABC$. In the proof of theorem 4, we had:

$$NP \cdot N'P = NQ \cdot N'Q = P'N/(ABC)$$

So $P, Q, P', Q'$ are concyclic.

Since $D, X$ are isogonal conjugate with respect to $\triangle ABC$ and $DA, DB, DC$ intersect $(DBC), (DCA), (DAB)$ at $P, Z, Y$. Then by theorem 4, $(AXP), (BXZ), (CXY)$ are coaxial and from theorem 5, $(AXP), (BXZ), (CXY)$ pass through $X$ and $P'$. Similarly, $(ADQ), (BDF), (CDE)$ pass through $D$ and $Q'$, so $DN'_a, XN'_a$ pass through $Q', P'$, respectively, and:

$$N'd, N'dQ' = N'aX.N'aP = P'N/(ABC)$$

Hence, $D, X, P', Q'$ are concyclic.

Proposition 16. The following sets of lines are concurrent:

$(NN', NA_N, BC), (NN', NB_N, CA), (NN', NC_N, AB)$,

$(N_bN'_b, N_cN'_c, BC), (N_cN'_c, N_aN'_a, CA), (N_aN'_a, N_bN'_b, AB)$.

\[\text{Figure 18}\]

Proof. From lemma 5 and lemma 7, $PQXD$ is a trapezoid, the intersection $L_a$ of $PX, QD$ lies on $BC$. Then $A, N, L_a, N_a$ are collinear and $(AL_aNN_a) = -1$ so $B(AL_aNN_a) = -1$.

Since $BA, BC, BN, BN_a$ are reflections of $BC, BA, BN', BN'_a$ in bisector of $\angle ABC$:

$$\Rightarrow B(CAN'_a) = B(AL_aNN_a) = -1$$

$AN', AN'_a$ are isogonal lines of $AN, AN_a$ with respect to $\angle BAC$ so $A, N', N'_a$ are collinear. Let $AN'$ intersects $BC$ at $K_a$.

$$\Rightarrow B(CAN'_a) = (K_aAN'_a) = (AK_aNN'_a) = -1 = (AL_aNN_a)$$

So $BC, NN', N_aN'_a$ are concurrent.
**Proposition 17.** Suppose that $P$ is inside $\triangle ABC$. Let $R$, $R_a$, $R_b$, $R_c$ be radii of pedal circles of $P$, $D$, $E$, $F$ with respect to $\triangle ABC$. Then:

\[
\frac{1}{R} = \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}
\]

![Figure 19](image)

**Proof.** $H_a$, $J_a$ are orthogonal projections of $Q$, $D$ on $BC$. It is well-known that $N$ is center of pedal circle of $P$ with respect to $\triangle ABC$ and $H_a$ lies on it. So $R = NH_a$. Similarly, $R_a = N_a J_a$. By Thales’s theorem:

\[
\frac{L_a H_a}{L_a J_a} = \frac{L_a Q}{L_a D} = \frac{L_a N}{L_a N_a}
\]

Hence, $NH_a \parallel N_a J_a$ and $\frac{NH_a}{N_a J_a} = \frac{L_a N}{L_a N_a} = \frac{AN}{AN_a} = \frac{AP}{AD}$

From the proof of lemma 5:

\[
\frac{AP}{AD} = \frac{AP \cdot AQ}{AQ \cdot AD} = \frac{AP \cdot AQ}{AB \cdot AC}
\]

Therefore,

\[
\frac{R}{R_a} = \frac{AP \cdot AQ}{AB \cdot AC}
\]

According to IMO Shortlist 1998, geometric problem 4(see [7]):

\[
\frac{AP \cdot AQ}{AB \cdot AC} + \frac{BP \cdot BQ}{BC \cdot BA} + \frac{CP \cdot CQ}{CA \cdot CB} = 1
\]

\[
\Rightarrow \frac{R}{R_a} + \frac{R}{R_b} + \frac{R}{R_c} = 1 \implies \frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} = \frac{1}{R}
\]

$\square$
REFERENCES

   http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
   https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/topics/4533
   https://groups.yahoo.com/neo/groups/Hyacinthos/conversations/topics/1498
   https://groups.yahoo.com/neo/groups/Anopolis/conversations/topics/2648

HIGH SCHOOL FOR GIFTED STUDENT, HANOI UNIVERSITY OF SCIENCE, VIETNAM NATIONAL UNIVERSITY,
HANOI, VIETNAM

E-mail address: tenminhduong@gmail.com