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GENERALIZATION OF MUSSELMAN'S THEOREM. SOME PROPERTIES OF ISOGONAL CONJUGATE POINTS

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ABSTRACT. In this article, we generalize of Musselman's theorem and study on some properties of isogonal conjugate points with angle chasing mainly.

1. INTRODUCTION

Theorem 1. *(Musselman, [1]) $\triangle ABC$, D, E, F are reflections of A, B, C in BC, CA, AB, respectively. Let O be circumcenter of $\triangle ABC$. (AOD), (BOE), (COF) are coaxial and the intersection other than O is the inverse of Kosnita point with respect to (O).*

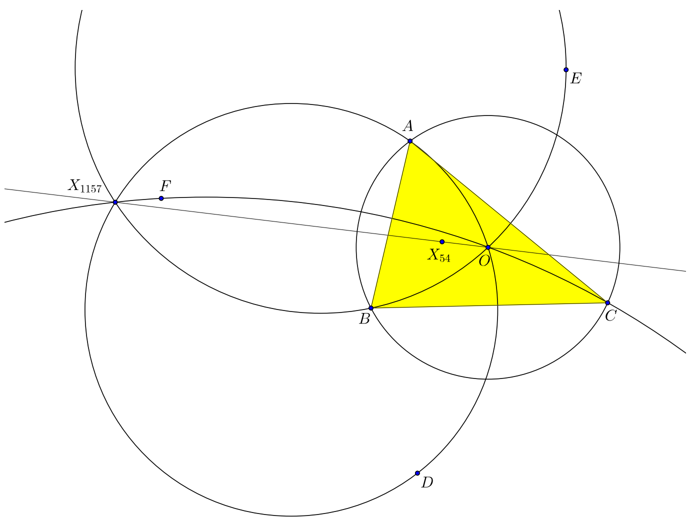

Figure 1. Musselman's theorem

The inverse of Kosnita point (X_{54}) with respect to (O) is X_{1157} in Encyclopedia of Triangle Centers, see [2]. X_{1157} lies on Neuberg cubic and it is the tangential of O on the Neuberg cubic.

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Key words and phrases. Triangle geometry, isogonal conjugate, circumcircle, concyclic, coaxial circle, angle chasing, collinear, concurrent.

Theorem 2. *(Yiu, [3])* (AEF), (BFD), (CDE) *pass through the inverse of Kosnita point with respect to* (O).

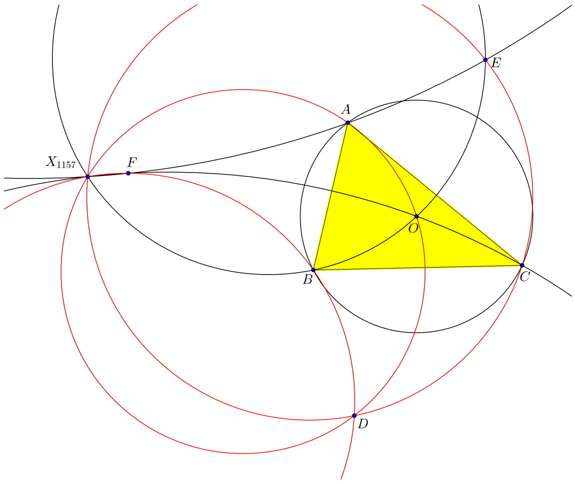

Figure 2

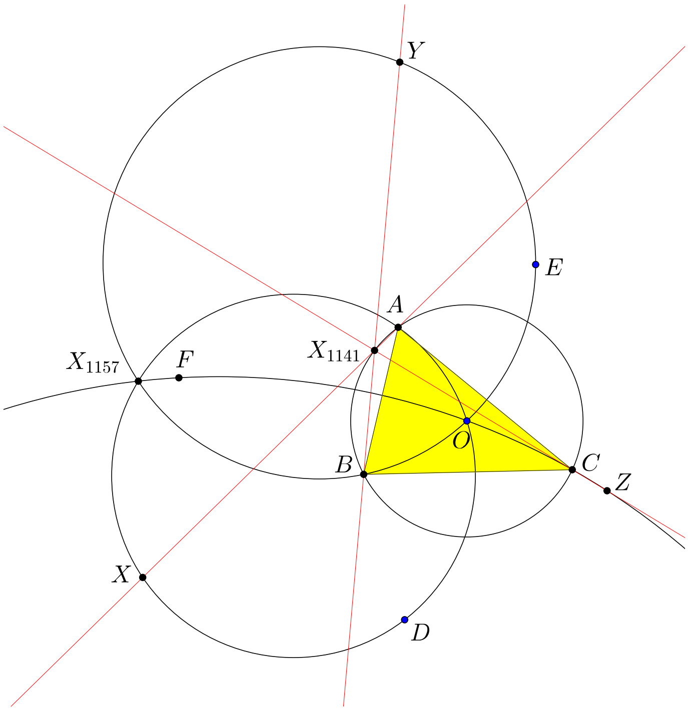

Figure 3. Gibert point

Theorem 3. *(Gibert, [4])* X, Y, Z *are reflections of* X_{1157} *in* BC, CA, AB.
AX, BY, CZ *are concurrent at a point on* (O).

Neuberg cubic is locus of P such that reflections of P in BC, CA, AB form a triangle that perspective with $\triangle ABC$, locus of the perspectors is a cubic [5]. When P coincides with X_{1157}, we obtain X_{1141}, the only perspector lies on circumcircle other than A, B, C.

2. Generalization of Musselman's theorem and some properties around its configuration

2.1. Generalization theorem.

Theorem 4. *(Generalization of Musselman's theorem, [6]) Let P, Q be isogonal conjugate points with respect to $\triangle ABC$.*
PA, PB, PC intersects $(PBC), (PCA), (PAB)$ at $D, E, F \neq P$, respectively.
Then $(AQD), (BQE), (CQF)$ are coaxial.

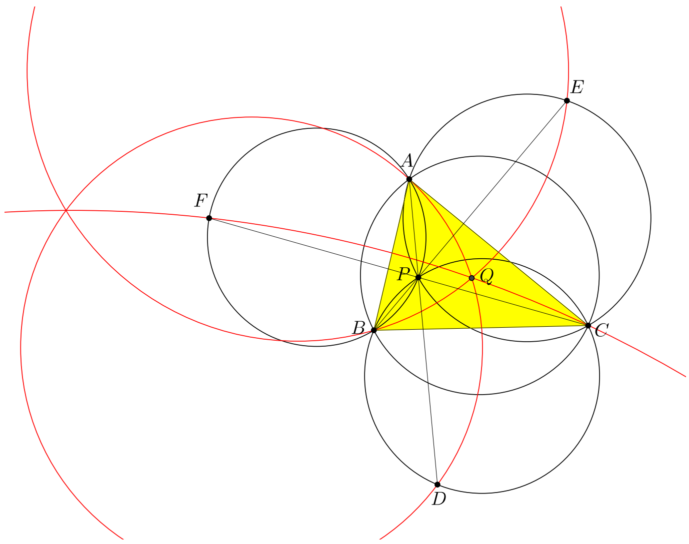

Figure 4. Generalization of Musselman's theorem

If P coincides with orthocenter of $\triangle ABC$, we have Musselman's theorem.

Proof. Let QA, QB, QC intersect $(QBC), (QCA), (QAB)$ at $X, Y, Z \neq Q$.
First, we need some lemmas.

Lemma 5. *PQ is parallel to DX, EY, FZ.*

Proof. Since P, Q are isogonal conjugate:

$$(AB, AP) = (AQ, AC) = (AX, AC)$$

$$(XA, XC) = (XQ, XC) = (BQ, BC) = (BA, BP)$$

Therefore $\triangle ABP$ and $\triangle AXC$ are directly similar (angle-angle).
Thus $AB.AC = AP.AX$. Similarly, $AB.AC = AQ.AD$.

$$\Rightarrow AP.AX = AQ.AD \Leftrightarrow \frac{AP}{AD} = \frac{AQ}{AX}$$

$\Rightarrow PQ \parallel DX$. Similarly, we can prove $PQ \parallel EY, FZ$. $\square$

Lemma 6. *D and X, E and Y, F and Z are isogonal conjugate points with respect to $\triangle ABC$.*
BF, CE pass through X; CD, AF pass through Y; AE, BD pass through Z.
BZ, CY pass through D; CX, AZ pass through E; AY, BX pass through F.

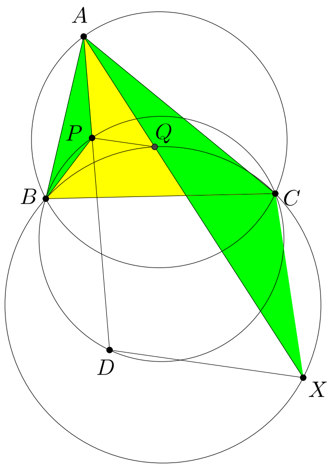

Figure 5

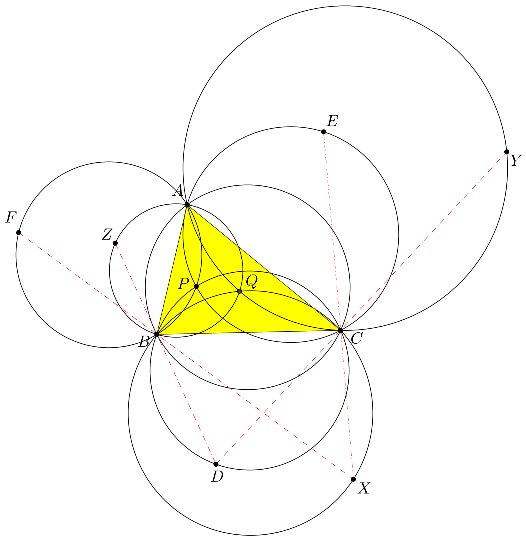

Figure 6

Proof. (See figure 6) Similar to the proof of lemma 5, we have $\triangle APC$ and $\triangle ABX$ are directly similar, $\triangle APB$ and $\triangle ACX$ are directly similar.

$$(BC, BD) = (PC, PD) = (PC, PA) = (BX, BA)$$

$$(CB, CD) = (PB, PD) = (PB, PA) = (CX, CA)$$

So D, X are isogonal conjugate.

$$(BX, BF) = (BX, BA) + (BA, BF) = (PC, PA) + (PA, PF) = 0$$

Hence BF passes through X.

□

Lemma 7. (ABC), (APX), (AQD) *are coaxial.*

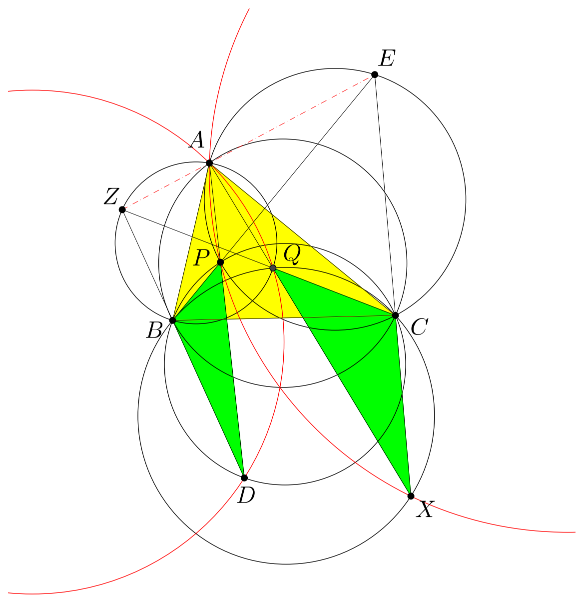

Figure 7

Proof. Considering the inversion $\mathbf{I}(A, AB.AC)$:

$$B, C, P, Q, D, X \mapsto B', C', P', Q', D', X'$$
$$(ABC), (APX), (AQD) \to B'C', P'X', Q'D'$$

Since $AP.AX = AQ.AD = AB.AC$, these pairs of points: (B, C'), (C, B'), (P, X'), (Q, D'), (D, Q'), (X, P') are symmetrically through bisector of $\angle BAC$.
Hence, instead of prove $B'C'$, $P'X'$, $Q'D'$ are concurrent, we prove BC, PX, QD are concurrent.
Considering $\triangle BPD$ and $\triangle CXQ$:
According to lemma 6:
BD intersects CQ at Z, BP intersects CX at E, PD intersects QX at A and Z, A, E are collinear.
Then by Desargues's theorem, BC, PX, QD are concurrent. □

Back to the main proof.
From lemma 7: $(ABC), (APX), (AQD)$ have two common points A, A'
$(ABC), (BPY), (BQE)$ have two common points B, B'
$(ABC), (CPZ), (CQF)$ have two common points C, C'
Let N be midpoint of PQ.

$$(DA', DP) = (DA', DA) = (QA', QA) = (QA', QX)$$
$$(PA', PD) = (PA', PA) = (XA', XA) = (XA', XQ)$$

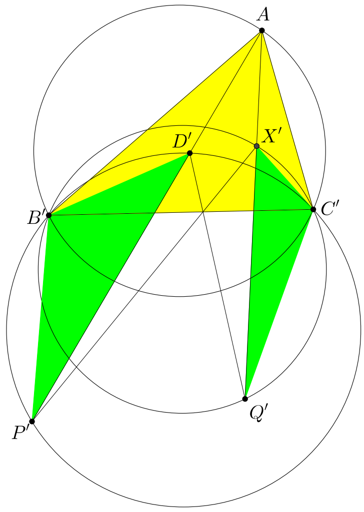

Figure 8. Inverse

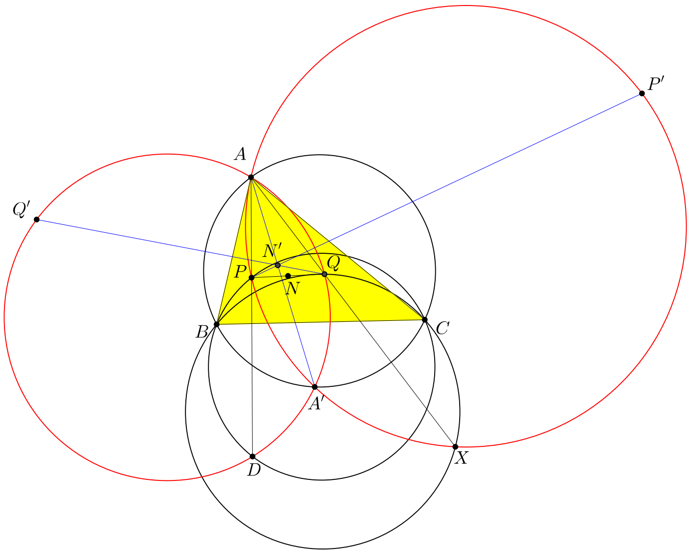

Figure 9

Hence, $\triangle A'DP$ and $\triangle A'QX$ are similar.

$$\Rightarrow \frac{AP}{AQ} = \frac{PD}{QX} = \frac{d(A', AP)}{d(A', AQ)}$$

(Note that $d(M, \ell)$ is distance from M to the line ℓ).
This means distances from A' to AP, AQ are proportional to AP, AQ.
So AA' is the symmedian of $\triangle APQ$ then AN, AA' are isogonal lines with respect to $\angle BAC$. Similarly, BB', BN are isogonal lines with respect to $\angle ABC$; CC', CN are isogonal

lines with respect to $\angle ACB$.
So AA', BB', CC' are concurrent at N' - isogonal conjugate of N with respect to $\triangle ABC$(when P, Q coincide with orthocenter and circumcenter, N' become Kosnita point). Let P', Q' be two points on PN', QN' such that:

$$\overline{N'P}.\overline{N'P'} = \overline{N'Q}.\overline{N'Q'} = \mathcal{P}_{N'/(ABC)}$$

Then (AQD), (BQE), (CQF) pass through Q' and (APX), (BPY), (CPZ) pass through P'.
$\Longrightarrow$ (AQD), (BQE), (CQF) are coaxial, (APX), (BPY), (CPZ) are coaxial. □

Theorem 8. *The circles (AEF), (BFD), (CDE) pass through Q'.*

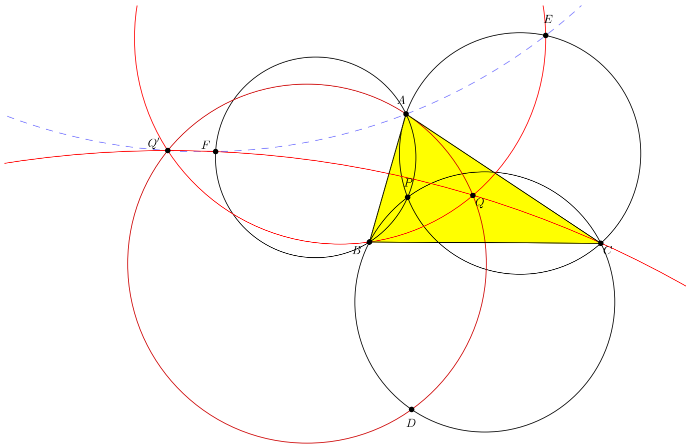

Figure 10

Proof.

$$\begin{aligned}(Q'E, Q'F) &= (Q'E, Q'Q) + (Q'Q, Q'F)\\
&= (BE, BQ) + (CQ, CF) \quad (B, Q, E, Q' \text{ are concyclic and } C, Q, F, Q' \text{ are concyclic})\\
&= (BE, BA) + (BA, BQ) + (CQ, CA) + (CA, CF)\\
&= (BP, BA) + (BP, BC) + (CB, CP) + (CA, CP) \quad (P, Q \text{ are isogonal conjugate})\\
&= (BP, BA) + (AB, AC) + (CA, CP) + (AC, AB) + (PB, PC)\\
&= (AC, AB) + 2(PB, PC)\\
&= (AC, AB) + (PB, PF) + (PE, PC)\\
&= (AC, AB) + (AB, AF) + (AE, AC)\\
&= (AE, AF)\end{aligned}$$

$\Rightarrow Q'$ lies on (AEF).
Similarly, Q' lies on (BFD), (CDE). □

Theorem 9. *(Generalization of Gibert point) Let the lines that pass through Q' and parallel to PA, PB, PC intersects (AQD), (BQE), (CQF) at A_Q, B_Q, $C_Q \neq Q$.*
AA_Q, BB_Q, CC_Q are concurrent at a point on (ABC).

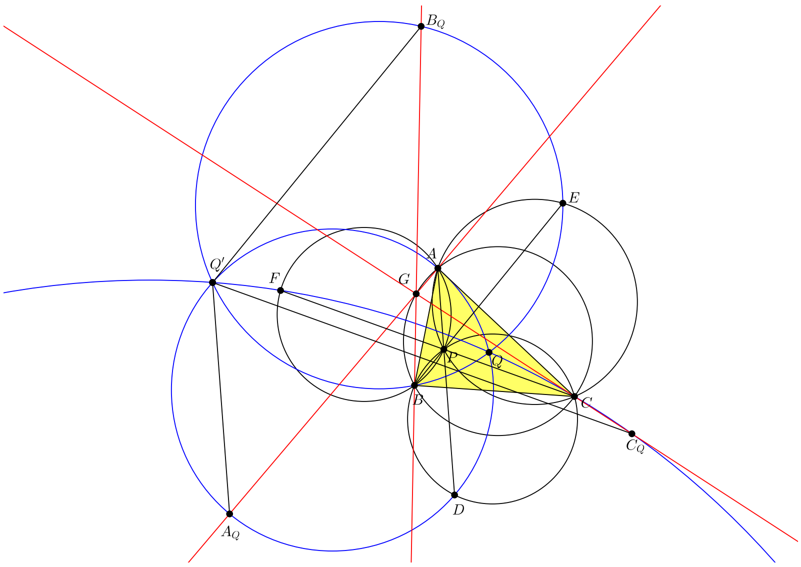

Figure 11

Proof. Let G be intersection of AA_Q and BB_Q. We show that G lies on (ABC).

$$\begin{aligned}
(GA, GB) &= (AA_Q, BB_Q) \\
&= (AA_Q, Q'A_Q) + (Q'A_Q, Q'B_Q) + (Q'B_Q, BB_Q) \\
&= (QA, QQ') + (PA, PB) + (QQ', QB) \\
&= (PA, PB) + (QA, QB) \\
&= (AP, AB) + (BA, BP) + (AQ, AB) + (BA, BQ) \\
&= (AP, AB) + (BA, BP) + (AC, AP) + (BP, BC) \quad (P, Q \text{ are isogonal conjugate}) \\
&= (CA, CB)
\end{aligned}$$

Similarly, the intersections of BB_Q, CC_Q lies on (ABC), therefore AA_Q, BB_Q, CC_Q are concurrent at a point on (ABC). □

2.2. Some properties.

Proposition 10. *The following sets of 4 points are concyclic:*
(B, C, F, Y), (B, C, E, Z).
(C, A, D, Z), (C, A, F, X).
(A, B, E, X), (A, B, D, Y).

Proof.

$$(FB, FC) = (FB, FP) = (AB, AP)$$

Since P, Q are isogonal conjugate

$$(AB, AP) = (AQ, AC) = (YQ, YC) = (YB, YC)$$

Hence, B, C, F, Y are concyclic. □

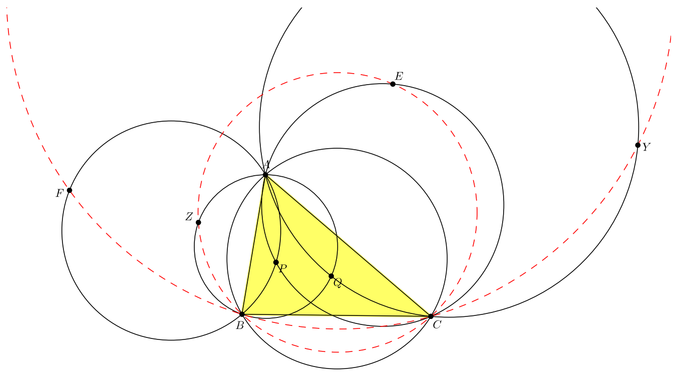

Figure 12

Proposition 11. *EF, YZ, BC are concurrent.*

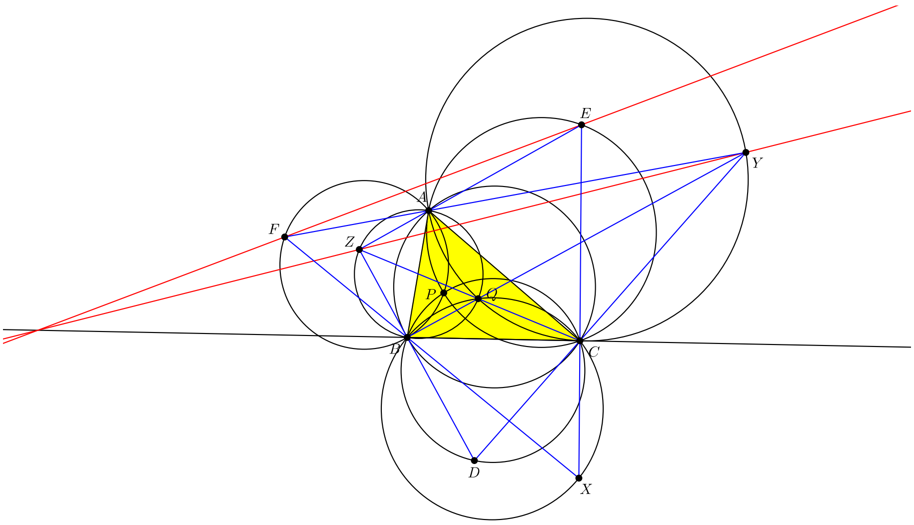

Figure 13

Proof. From lemma 6, FY intersect EZ at A, BF intersects CE at X. BY intersects CZ at Q. Since A, Q, X are collinear then by Desargues's theorem, EF, YZ, BC are concurrent. □

Proposition 12.

$$(DYZ), (EZX), (FXY), (PDX), (PEY), (PFZ) \text{ have a common point.}$$

$$(XEF), (YFD), (ZDE), (QDX), (QEY), (QFZ) \text{ have a common point.}$$

Proof. From lemma 6, D, Y, C are collinear and D, Z, B are collinear, then:
$(DY, DZ) = (DC, DB) = (PC, PB)$
Similarly:
$(EZ, EX) = (EA, EC) = (PA, PC)$
$(FX, FY) = (FB, FA) = (PB, PA)$
$\Rightarrow (DY, DZ) + (EZ, EX) + (FX, FY) = 0$. Hence $(DYZ), (EZX), (FXY)$ have a common

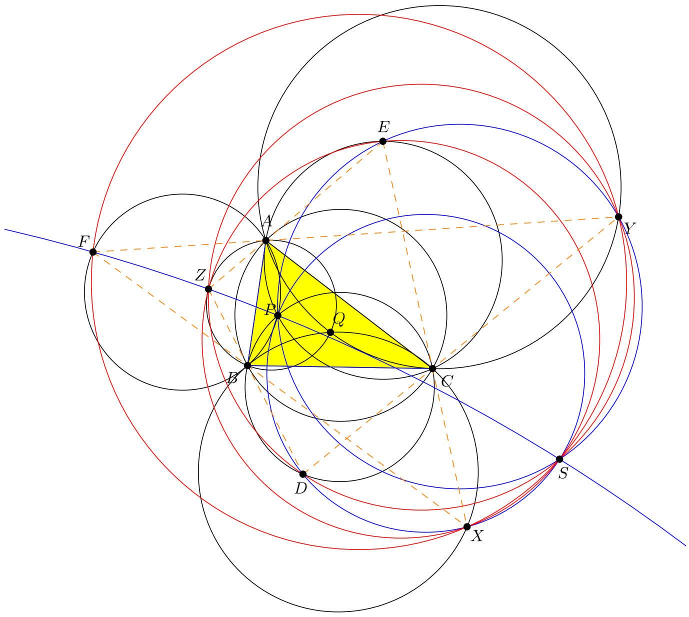

Figure 14

point S. Now from symmetry we only need to prove that S lies on (PDX).

$$\begin{aligned}
(SD, SX) &= (SD, SY) + (SY, SX) \\
&= (ZD, ZY) + (FY, FX) \quad (S, D, Y, Z \text{ are concyclic, } S, X, Y, F \text{ are concyclic}) \\
&= (ZB, ZY) + (FA, FB) \quad (Z, D, B \text{ are collinear}) \\
&= (ZB, ZY) + (PA, PB) \quad (F, A, B, P \text{ are concyclic}) \\
(PD, PX) &= (PA, PX) \\
&= (P'A, P'X) \quad (A, P, X, P' \text{ are concyclic}) \\
&= (P'A, P'Z) + (P'Z, P'X) \\
&= (AY, YZ) + (BZ, BX) \quad (A, P', Y, Z \text{ are concyclic, } B, Z, X, P' \text{ are concyclic})
\end{aligned}$$

$$\begin{aligned}
(SD, SX) - (PD, PX) &= (PA, PB) + (BZ, AY) + (BX, BZ) \\
&= (PA, PB) + (BZ, BA) + (AB, AY) + (BX, BC) + (BC, BZ) \\
&= (PA, PB) + (BC, BF) + (AE, AC) + (BA, BD) + (BF, BA) \\
&= (PA, PB) + (BC, BD) + (AE, AC) \\
&= (PA, PB) + (PC, PD) + (PE, PC) \\
&= (PA, PB) + (PC, PA) + (PB, PC) \\
&= 0
\end{aligned}$$

Therefore, S lies on (PDX). □

Proposition 13. *(ADX), (AEY), (AFZ), (APQ) are tangent at A.*
(BDX), (BEY), (BFZ), (BPQ) are tangent at B.
(CDX), (CEY), (CFZ), (CPQ) are tangent at C.

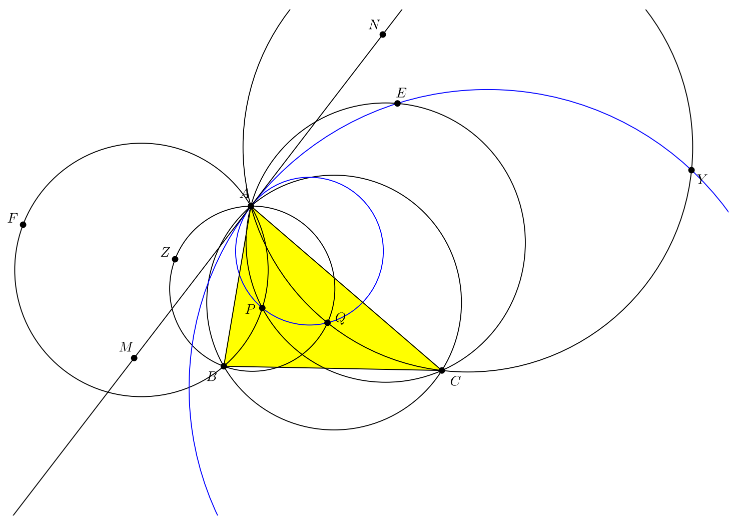

Figure 15

Proof. Since $EY \parallel FZ$ and EZ, FY pass through A, (AEY) and (AFZ) are tangent at A.
$DX \parallel PQ$, PD, QX pass through A so (APQ), (ADX) are tangent at A.
Let AM, AN be tangent lines of (APQ), (AEY) at A.

$$\begin{aligned}(AM, AN) &= (AM, AP) + (AP, AE) + (AE, AN)\\ &= (QA, QP) + (AP, AE) + (YE, YA) \quad (AN \text{ is tangent line of } (AEY))\end{aligned}$$

Since $PQ \parallel EY$:

$$\begin{aligned}(AM, AN) &= (AQ, AY) + (AP, AE)\\ &= (AQ, AC) + (AC, AY) + (AP, AB) + (AB, AE)\end{aligned}$$

Because P, Q and E, Y are isogonal conjugate with respect to $\triangle ABC$:

$$(AQ, AC) + (AP, AB) = 0 \qquad (AC, AY) + (AB, AE) = 0$$

$\Rightarrow (AM, AN) = 0$, then A, M, N are collinear.
Hence, (ADX), (AEY), (AFZ), (APQ) are tangent at A. □

Proposition 14. *Suppose that:*
ℓ_a is radical axis of (ADX), (AEY), (AFZ), (APQ)
ℓ_b is radical axis of (BDX), (BEY), (BFZ), (BPQ)
ℓ_c is radical axis of (CDX), (CEY), (CFZ), (CPQ)
Then ℓ_a, ℓ_b, ℓ_c are concurrent at a point on (ABC).

Proof. ℓ_a, ℓ_b, ℓ_c are tangent lines at A, B, C of $(APQ), (BPQ), (CPQ)$.
Tangent line at A of (APQ) is isogonal line of the line that passes through A and parallel to PQ with respect to $\angle BAC$. Therefore, ℓ_a passes through isogonal conjugate of infinity point on PQ, which lies on (ABC). Hence ℓ_a, ℓ_b, ℓ_c are concurrent at a point on (ABC). □

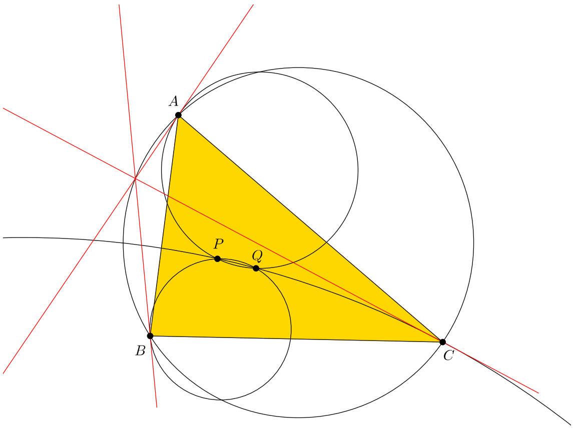

Figure 16

Proposition 15. *The following sets of 4 points are concyclic:*

$$(Q', D, X, P'), (Q', E, Y, P'), (Q', F, Z, P'), (Q', P, Q, P')$$

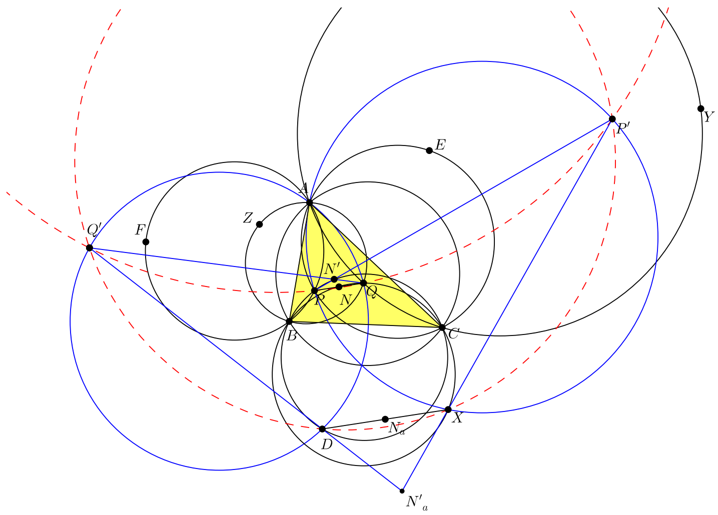

Figure 17

Proof. Let N_a, N_b, N_c be midpoints of DX, EY, FZ and N'_a, N'_b, N'_c be isogonal conjugate of N_a, N_b, N_c with respect to $\triangle ABC$. In the proof of theorem 4, we had:

$$\overline{N'P}.\overline{N'P'} = \overline{N'Q}.\overline{N'Q'} = \mathcal{P}_{N'/(ABC)}$$

So P, Q, P', Q' are concyclic.
Since D, X are isogonal conjugate with respect to $\triangle ABC$ and DA, DB, DC intersect (DBC), (DCA), (DAB) at P, Z, Y. Then by theorem 4, (AXP), (BXZ), (CXY) are coaxial and from theorem 5, (AXP), (BXZ), (CXY) pass through X and P'. Similarly, (ADQ), (BDF), (CDE) pass through D and Q', so DN'_a, XN'_a pass through Q', P', respectively, and:

$$\overline{N'_aD}.\overline{N'_aQ'} = \overline{N'_aX}.\overline{N'_aP'} = \mathcal{P}_{N'_a/(ABC)}$$

Hence, D, X, P', Q' are concyclic. □

Proposition 16. *The following sets of lines are concurrent:*
$(NN', N_aN'_a, BC)$, $(NN', N_bN'_b, CA)$, $(NN', N_cN'_c, AB)$.
$(N_bN'_b, N_cN'_c, BC)$, $(N_cN'_c, N_aN'_a, CA)$, $(N_aN'_a, N_bN'_b, AB)$.

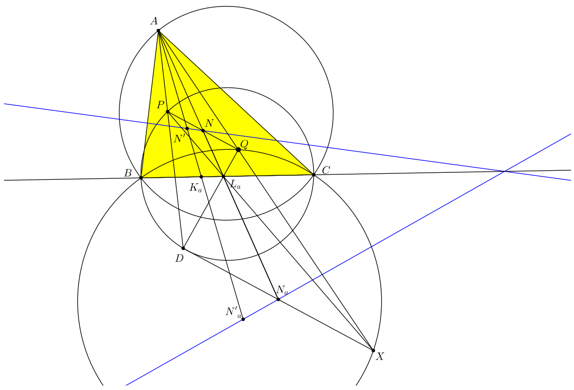

Figure 18

Proof. From lemma 5 and lemma 7, $PQXD$ is a trapezoid, the intersection L_a of PX, QD lies on BC.
Then A, N, L_a, N_a are collinear and $(AL_aNN_a) = -1$ so $B(AL_aNN_a) = -1$.
Since BA, BC, BN, BN_a are reflections of BC, BA, BN', BN'_a in bisector of $\angle ABC$

$$\Rightarrow B(CAN'N'_a) = B(AL_aNN_a) = -1$$

AN', AN'_a are isogonal lines of AN, AN_a with respect to $\angle BAC$ so A, N', N'_a are collinear.
Let AN' intersects BC at K_a.

$$\Rightarrow B(CAN'N'_a) = (K_aAN'N'_a) = (AK_aN'N'_a) = -1 = (AL_aNN_a)$$

So BC, NN', $N_aN'_a$ are concurrent.

□

Proposition 17. *Suppose that P is inside $\triangle ABC$. Let $\mathcal{R}$, $\mathcal{R}_a$, $\mathcal{R}_b$, $\mathcal{R}_c$ be radii of pedal circles of P, D, E, F with respect to $\triangle ABC$. Then:*

$$\frac{1}{\mathcal{R}} = \frac{1}{\mathcal{R}_a} + \frac{1}{\mathcal{R}_b} + \frac{1}{\mathcal{R}_c}$$

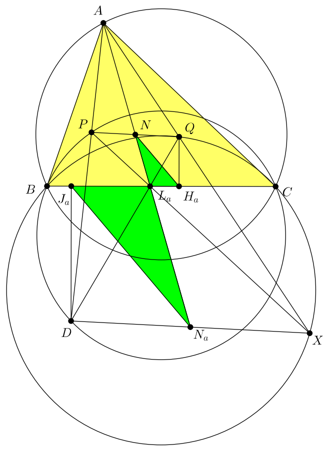

Figure 19

Proof. H_a, J_a are orthogonal projections of Q, D on BC. It is well-known that N is center of pedal circle of P with respect to $\triangle ABC$ and H_a lies on it. So $\mathcal{R} = NH_a$. Similarly, $\mathcal{R}_a = N_aJ_a$. By Thales's theorem:

$$\frac{L_aH_a}{L_aJ_a} = \frac{L_aQ}{L_aD} = \frac{L_aN}{L_aN_a}$$

Hence,

$$NH_a \parallel N_aJ_a \text{ and } \frac{NH_a}{N_aJ_a} = \frac{L_aN}{L_aN_a} = \frac{AN}{AN_a} = \frac{AP}{AD}$$

From the proof of lemma 5:

$$\frac{AP}{AD} = \frac{AP.AQ}{AQ.AD} = \frac{AP.AQ}{AB.AC}$$

Therefore,

$$\frac{\mathcal{R}}{\mathcal{R}_a} = \frac{AP.AQ}{AB.AC}$$

According to IMO Shortlist 1998, geometric problem 4(see [7]):

$$\frac{AP.AQ}{AB.AC} + \frac{BP.BQ}{BC.BA} + \frac{CP.CQ}{CA.CB} = 1$$

$$\Rightarrow \frac{\mathcal{R}}{\mathcal{R}_a} + \frac{\mathcal{R}}{\mathcal{R}_b} + \frac{\mathcal{R}}{\mathcal{R}_c} = 1 \Longrightarrow \frac{1}{\mathcal{R}_a} + \frac{1}{\mathcal{R}_b} + \frac{1}{\mathcal{R}_c} = \frac{1}{\mathcal{R}}$$

□

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