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BIHARMONIC REEB CURVES IN SASAKIAN MANIFOLDS

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ABSTRACT. Sasakian manifolds provide explicit formulae of some Jacobi operators which describe the biharmonic equation of curves in Riemannian manifolds. In this paper we characterize non-geodesic biharmonic curves in Sasakian manifolds which are either tangent or normal to the Reeb vector field.
In the three-dimensional case, we prove that such curves are some helixes whose geodesic curvature and geodesic torsion satisfy a given relation.

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1. Introduction

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (cf. [6]).
For a map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds the energy functional E_1 is defined by

$$E_1(\phi) \;=\; \frac{1}{2}\int_M |d\phi|^2 v_g.$$

Critical points of E_1 are called harmonic maps and are then solutions of the corresponding Euler-Lagrange equation

$$\tau_1(\phi) = trace\nabla^\phi d\phi.$$

Here ∇^ϕ denotes the induced connection on the pull-back bundle $\phi^{-1}(TN)$ and $\tau_1(\phi)$ is called the tension field of ϕ.
Biharmonic maps are the critical points of the functional bienergy

$$E_2(\phi) \;=\; \frac{1}{2}\int_M |\tau_1(\phi)|^2 v_g,$$

whose Euler-Lagrange equation is given by the vanishing of the bitension field (cf. [10]) defined by

$$\tau_2(\phi) \quad = \quad -\Delta^\phi \tau_1(\phi) - traceR^N(d\phi, \tau_1(\phi))d\phi, \tag{1}$$

where $\Delta^\phi = -trace_g(\nabla^\phi\nabla^\phi - \nabla^\phi_\nabla)$ is the Laplacian on the sections of $\phi^{-1}(TN)$, and R^N is the Riemannian curvature operator of (N, h). Note that

$$\tau_2(\phi) = J_\phi(\tau_1(\phi)) \tag{2}$$

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where J_ϕ is the Jacobi operator along ϕ defined by

$$J_\phi(X) = -\Delta^\phi X - traceR^N(d\phi, X)d\phi, \quad \forall\, X \in \phi^{-1}(TN). \tag{3}$$

Harmonic maps are obviously biharmonic and are absolute minimum of the bienergy.
Nonminimal biharmonic submanifolds of the pseudo-euclidean space and of the spheres have been studied in [2] and [3].
Biharmonic curves have been investgated on many special Riemannian manifolds like Heisenberg groups [4], [7], invariant surfaces [9], Damek-Ricci spaces [5], Sasakian manifolds [8], etc.
As in the general theory of metric contact manifolds, an important role is played by the Reeb vector field whose dynamics can be used to study the structure of the contact manifold or even the underlying manifold using techniques of Floer homology such as symplectic field theory and embedded contact homology.
The main purpose of this work is to study non-geodesic biharmonic curves in a $(2n+1)$-dimensional Sasakian manifold, which are either tangent or normal to the Reeb vector field.

2. SASAKIAN MANIFOLDS

A contact manifold is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying almost contact structure (η, φ, ξ) where ξ is a global vector field (called the characteristic vector field) and φ a global tensor of type $(1,1)$ such that

$$\eta(\xi) = 1,\ \varphi\xi = 0,\ \eta\varphi = 0,\ \varphi^2 = -I + \eta \otimes \xi. \tag{4}$$

A Riemannian metric g can be found such that

$$\eta = g(\xi, .),\ d\eta = g(., \varphi.),\ g(., \varphi.) = -g(\varphi., .). \tag{5}$$

(M, η, g) or $(M, \eta, g, \xi, \varphi)$ is called a contact metric manifold. If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt}), \tag{6}$$

is integrable, $(M, \eta, g, \xi, \varphi)$ is said to be Sasakian.
The following relations play an important role in the present work:

Lemma 2.1. *[1] On a Sasakian manifold $(M, \eta, g, \xi, \varphi)$ we have*

$$R(X, \xi)X = -\xi \tag{7}$$

and

$$R(\xi, X)\xi = -X \tag{8}$$

for any unit vector field X orthogonal to the Reeb vector field ξ, where R denotes the Riemannian curvature of (M, g).

3. Biharmonic curves in Sasakian manifolds

Let $\gamma : I \longrightarrow (M^{2n+1}, \eta, g)$ be a regular curve parametrized by its arc lenght in a (2n+1)-dimensional Sasakian manifold and $\{T, N_1, ..., N_{2n}\}$ be the Frenet frame in M^{2n+1}, defined along γ, where $T = \gamma'$ is the unit tangent vector field of γ.
It holds:

Lemma 3.1. *[8] The Frenet equations of γ are given by*

$$\begin{cases} \nabla_T T &= \chi_1 N_1 \\ \nabla_T N_1 &= -\chi_1 T + \chi_2 N_2 \\ \vdots & \vdots \quad \vdots \\ \nabla_T N_k &= -\chi_k N_{k-1} + \chi_{k+1} N_{k+1}, \quad k = 2, ..., 2n-1, \\ \vdots & \vdots \quad \vdots \\ \nabla_T N_{2n} &= -\chi_{2n} N_{2n-1}, \end{cases}$$

where $\chi_1 = ||\nabla_T T||$, $\chi_2 = \chi_2(s), ..., \chi_{2n} = \chi_{2n}(s)$ are real valued functions, where s is the arc length of γ.

Definition 3.2. If the functions χ_k, $k = 1, ..., 2n$ are all constant, then γ is said to be a helix..

The tension field $\tau_1(\gamma)$ and the bitension field $\tau_2(\gamma)$ of the curve γ are given in the Frenet frame $(T, N_1, ..., N_{2n})$ by:

Proposition 3.3. [8]

$$\tau_1(\gamma) = \chi_1 N_1, \tag{9}$$

and

$$\begin{aligned} \tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2) N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2') N_2 + \chi_1\chi_2\chi_3 N_3 - \chi_1 R(T, N_1)T. \end{aligned} \tag{10}$$

From Proposition 3.3, we get:

Proposition 3.4. If γ is either tangent or normal to the Reeb vector field, then

$$\begin{aligned} \tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1) N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2') N_2 + \chi_1\chi_2\chi_3 N_3 \end{aligned} \tag{11}$$

Proof :
Let γ be non-geodesic biharmonic curve in a Sasakian manifold (M, η, g).
Assume that γ is tangent to the Reeb vector field ξ; that is $T = \xi$.
The relation (10) in Proposition 3.3 becomes

$$\begin{aligned} \tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2) N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2') N_2 + \chi_1\chi_2\chi_3 N_3 - \chi_1 R(\xi, N_1)\xi \\ &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2) N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2') N_2 + \chi_1\chi_2\chi_3 N_3 + \chi_1 N_1, \text{ according to lemma 2.1} \end{aligned}$$

It follows that

$$\begin{aligned}\tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1)N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2')N_2 + \chi_1\chi_2\chi_3 N_3\end{aligned} \tag{12}$$

We assume now that γ is normal to the Reeb vector field; that is $N_1 = \xi$.
The relation (10) in Proposition 3.3 becomes then

$$\begin{aligned}\tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2)N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2')N_2 + \chi_1\chi_2\chi_3 N_3 - \chi_1 R(T,\xi)T \\ &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2)N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2')N_2 + \chi_1\chi_2\chi_3 N_3 + \chi_1\xi \text{ according to lemma 2.1}\end{aligned}$$

We obtain then again

$$\begin{aligned}\tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1)N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2')N_2 + \chi_1\chi_2\chi_3 N_3\end{aligned} \tag{13}$$

Thus we get the relation (11) in both cases.

□

From Proposition 3.4, we get the following result.

Theorem 3.5. *Non-geodesic biharmonic curves in Sasakian manifolds which are either tangent or normal to the Reeb vector field are characterized by :*

$$\begin{cases} \chi_1 = constant \in [-1,0[\cup]0,1], \\ \chi_1^2 + \chi_2^2 = 1, \\ \chi_2\chi_3 = 0. \end{cases} \tag{14}$$

where χ_1, χ_2 and χ_3 are functions defined in lemma 3.1.

Proof :
$\tau_2(\gamma) = 0$ with $\chi_1 \neq 0$ implies $\chi_1' = 0$ according to the first component in (11). So χ_1 is constant. Then the second component gives $\chi_1^2 + \chi_2^2 = 1$. Thus (14) is satisfied.

□

Remark 3.6. (1) In general, non-geodesic biharmonic curves in Sasakian manifolds which are either tangent or normal to the Reeb vector field are not helixes, since if $\chi_1 = \pm 1$, χ_3 is not necessarily constant.
(2) Physically we can say that if a particle describes a biharmonic curve γ which is tangent or normal to the Reeb vector field in Sasakian manifold and parametrized by the arc length then the covariant derivative of his speed along γ has constant norm. In addition if the dimension of the Sasakian manifold is three and $\chi_1 = \pm 1$, the binormal vector field N_2 is parallel to γ.

In three-dimensional Sasakian manifolds the Frenet equations are given by

$$\begin{cases} \nabla_T T &= \chi_1 N_1 \\ \nabla_T N_1 &= -\chi_1 T + \chi_2 N_2 \\ \nabla_T N_2 &= -\chi_2 N_1, \end{cases} \tag{15}$$

where $\chi_1 = ||\nabla_T T||$, $\chi_2 = \chi_2(s)$ are real valued functions, and s is the arc length of γ. The equation characterizing the non-geodesic biharmonic curves which are either tangent or normal to Reeb vector field is then reduced to

$$\begin{aligned} \tau_2(\gamma) &= -3\chi_1\chi_1' T + (\chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1)N_1 \\ &\quad -(2\chi_1'\chi_2 + \chi_1\chi_2')N_2. \end{aligned} \tag{16}$$

So we have the following result.

Theorem 3.7. *Non-geodesic biharmonic curves which are either tangent or normal to the Reeb vector field in three-dimensional Sasakian manifolds are helixes whose geodesic curvature χ_1 and geodesic torsion χ_2 are related by:*

$$\chi_1^2 + \chi_2^2 = 1, \text{ with } \chi_1 \neq 0. \tag{17}$$

From Theorem 3.7 we get the following characterization of non-geodesic biharmonic curves which are normal to the Reeb vector field in three-dimensional Sasakian manifold.

Corollary 3.8. *In a three-dimensional Sasakian manifold $(M, \eta, g, \xi, \varphi)$, a non-geodesic curve γ is biharmonic and normal to the Reeb vector field ξ if and only if it is a solution of the following geodesic equation with a second member*

$$\left\{ \begin{aligned} \nabla_T T &= \chi_1 \xi \\ \nabla_T \xi &= -\chi_1 T + \chi_2 N \\ \nabla_T N &= -\chi_2 \xi, \end{aligned} \right. \tag{18}$$

under the constraint

$$g(\nabla_T \xi,\ \nabla_T \xi) = 1,$$

where $(T,\ \xi,\ N)$ is the Frenet frame associated to γ, and χ_1, χ_2 are constant reals such that $\chi_1 \neq 0$.

Proof :
We obtain (18) according to (15), since $\xi = N_1$. Now γ being a non-geodesic biharmonic curve normal to ξ, then χ_1 and χ_2 are constant reals such that $\chi_1 \neq 0$ and $\chi_1^2 + \chi_2^2 = 1$. Indeed $g(\nabla_T\xi,\ \nabla_T\xi) = g(-\chi_1 T + \chi_2 N,\ -\chi_1 T + \chi_2 N) = \chi_1^2 + \chi_2^2 = 1$.

□

We deduce again from Theorem 3.7 the following necessary condition for the existence of non-geodesic biharmonic curves which are tangent to the Reeb vector field in a three-dimensional Sasakian manifold.

Corollary 3.9. *If a three-dimensional Sasakian manifold $(M, \eta, g, \xi, \varphi)$ admits a non-geodesic biharmonic curve $\gamma : I \to M$ which is tangent to the Reeb vector field ξ, we have on $\gamma(I)$,*

$$g(\nabla_\xi \xi,\ \nabla_\xi \xi) = g(\nabla_\xi \nabla_\xi \xi,\ \nabla_\xi \nabla_\xi \xi) = constant \in]0, 1]. \tag{19}$$

Proof :
Using the Frenet frame (15) and since γ is tangent to the Reeb vector field ξ, we have $T = \xi$. Then

$$\nabla_\xi \xi = \chi_1 N_1, \tag{20}$$

so $g(\nabla_\xi \xi,\ \nabla_\xi \xi) = \chi_1^2$. Now γ is non-geodesic biharmonic curve, so χ_1^2 is in $]0,\ 1]$ and is contant on I.

In addition (20) implies
$$\nabla_\xi \nabla_\xi \xi = \chi_1 \nabla_\xi N_1 = \chi_1(-\chi_1 \xi + \chi_2 N_2). \tag{21}$$
So $g(\nabla_\xi \nabla_\xi \xi,\ \nabla_\xi \nabla_\xi \xi) = \chi_1^2(\chi_1^2 + \chi_2^2) = \chi_1^2$, since γ is non-geodesic biharmonic.

□

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