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AN INSCRIBED SQUARE OF A RIGHT TRIANGLE ASSOCIATED WITH AN ARBELOS

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Abstract. The inscribed square of a right triangle with a side along the hypotenuse is constructed from an arbelos. The arbelos with the square yields several dozens of congruent circles, which are not Archimedean circles of the arbelos, but Archimedean circles of several generalized arbeloi called an arbelos with overhang, a collinear arbelos, an arbelos in n-aliquot parts, and a skewed arbelos associated with the arbelos and the square.

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1. Introduction

Let us consider an arbelos consisting of three semicircles α, β and γ with diameters AO, BO and AB, respectively for a point O on the segment AB in the plane. We denote the arbelos by (α, β, γ). There are two inscribed squares of a right triangle, one with two sides lying on sides of the triangle, and the other with a side lying on the hypotenuse. In [1], we have considered the inscribed square of an arbelos, which are also the inscribed square of a right triangle in the former case.

Let a and b be the radii of α and β, respectively. Circles of radius $r_{\mathrm{A}} = ab/(a+b)$ form a special class of circles and are called Archimedean circles of (α, β, γ). Archimedean circles are also defined for several generalized arbeloi, such as an arbelos with overhang [2], an arbelos in n-aliquot parts [6], a collinear arbelos [3, 4], and a skewed arbelos [3, 5, 7]. Let I be the point of intersection of the semicircle γ and the radical axis of the semicircles α and β. In this article we consider the inscribed square of the right triangle ABI in the latter case, i.e., its one side lies on the segment AB. We show that the arbelos with the square yields several dozens of congruent circles, which are not Archimedean circles of (α, β, γ), but Archimedean circles of the four generalized arbeloi just mentioned above associated with (α, β, γ) and the square.

2. Some properties of pencils of circles

In this section we consider some properties of the arbelos and pencil of circles, which will be needed in later sections. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$, respectively, where we assume that the semicircles α, β and γ are constructed in the region $y > 0$. For two points P and Q, (PQ) and $P(Q)$ denote the circle with diameter PQ and the circle with center P passing through Q, respectively. However if their centers lie on AB, we regard them as semicircles with their diameters lying along AB constructed in the region $y > 0$.

For two circles δ_1 and δ_2, one of which may be a line, we denote the pencil of circles determined by δ_1 and δ_2 by $\mathrm{P}(\delta_1, \delta_2)$. The concept of pencil of circles is also valid for semicircles if we consider them as circles. The other concepts on circles are also valid for semicircles in a similar way. In this section we assume that V and W are points on the semicircle γ having the x-coordinates v and w, and V' and W' are the feet of perpendiculars from V and W to the line AB, respectively. Let y_w be the y-coordinate of W. Then

$$y_w = \sqrt{(2a - w)(w + 2b)}, \tag{1}$$

and the line AW has the equation

$$y = (x - 2a)y_w/(w - 2a). \tag{2}$$

The perpendicular from a point X to AB is denoted by $\mathcal{P}_X$.

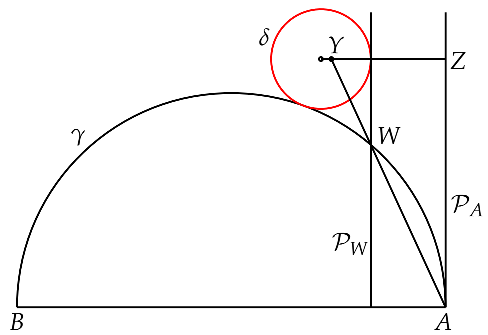

Figure 1.

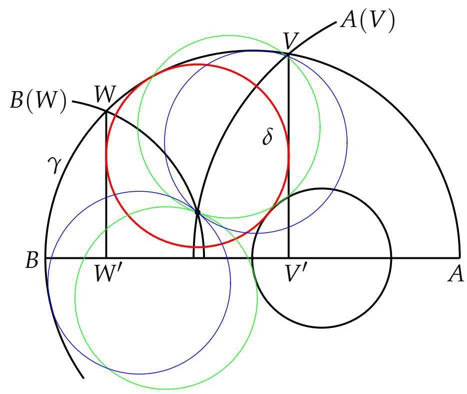

Figure 2.

Theorem 1. If δ is a circle touching the semicircle γ and the line $\mathcal{P}_W$ from the side opposite to A or B according as it touches γ externally or internally, and Y is the point of intersection of the line AW and the diameter of δ parallel to AB, then Y is one of the limiting points of the pencil $\mathrm{P}(\delta, \mathcal{P}_A)$.

Proof. We assume that δ touches γ externally (see Figure 1). If r and y_d are the radius and the y-coordinate of the center of δ, then $y_d^2 = (r + a + b)^2 - ((a - b) - (w - r))^2 = (2b + w)(2a + 2r - w)$. Therefore Y has the x-coordinates $2a - \sqrt{(2a - w)(2a + 2r - w)}$ by (2). Hence if Z is the foot of perpendicular from Y to the line $\mathcal{P}_A$, then $|YZ|^2 = (2a - w)(2a + 2r - w)$. On the other hand, the power of Z with respect to δ also equals $(2a - (w - r))^2 - r^2 = (2a - w)(2a + 2r - w)$. Therefore Y is one of the limiting points of $\mathrm{P}(\delta, \mathcal{P}_A)$. The case δ touching γ internally is proved similarly. □

A point D is said to generate circles of radius r with a circle δ, if two congruent circles of radius r touch externally at D and also touch δ at points different from D. The definition is also valid in the case δ being a semicircle, if we regard it as a circle. We use the next theorem [3, Theorem 7] (see Figure 2).

Theorem 2. Let δ be a circle of radius r touching the semicircle γ and the lines $\mathcal{P}_V$ and $\mathcal{P}_W$. The following statements are true.
(i) The perpendicular bisector of $V'W'$ is the radical axis of the semicircles $A(V)$ and $B(W)$.
(ii) If $A(V)$ and $B(W)$ intersect and the point of intersection and δ lies on the same side of γ, the point generates circles of radius r with any circle with center on AB touching δ.

We investigate further properties of the figure of Theorem 2. Let $e = a + b$.

Theorem 3. If we assume the hypothesis of (ii) of Theorem 2 in the situation of the theorem, the point of intersection of $A(V)$ and $B(W)$ is one of the limiting points of the pencil $\mathrm{P}(\delta, AB)$.

Proof. Since $|AV|^2 = |AV'|^2 + |VV'|^2 = (2a-v)^2 + (2a-v)(v+2b) = 2e(2a-v)$, the semicircle $A(V)$ has the equation

$$(x-2a)^2 + y^2 = 2e(2a-v). \tag{3}$$

The point of intersection of $A(V)$ and $B(W)$ has the x-coordinate $m = (v+w)/2$ by (i) of Theorem 2. Therefore the square of the distance between this point and AB equals

$$2e(2a-v) - (m-2a)^2. \tag{4}$$

If $v > w$, then $r = (v-w)/2$ and δ touches γ internally and has center with coordinates (m, l), where l satisfies $l^2 = (e-r)^2 - (m-(a-b))^2$. If D is the foot of perpendicular from the center of δ to AB, the power of D with respect to δ is $l^2 - r^2$, which also equals (4). Therefore the theorem is proved. The case $v < w$ is proved in a similar way. □

The center of a circle or a semicircle δ is denoted by O_δ.

Proposition 2.1. *If r and y_d are the radius and the y-coordinate of the center of the circle touching the semicircles α externally γ internally and the line $\mathcal{P}_W$ from the side opposite to B, then*

$$r = b(2a-w)/(2e) \ \ and \ \ y_d = y_w\sqrt{a/e}. \tag{5}$$

Proof. From the two right triangles formed by the segments joining two of the points O_α, O_γ and O_δ and the perpendicular from O_δ to AB, we get $(w+r-a)^2 + y_d^2 = (a+r)^2$ and $(w+r-(a-b))^2 + y_d^2 = (e-r)^2$. Solving the equation, we get (5). □

Theorem 4. If δ is the circle touching α externally γ internally and the line $\mathcal{P}_W$ from the side opposite to B, and X is a point of intersection of the semicircle α and the line $\mathcal{P}_V$, then the following statements are equivalent.
(i) The points A, X and W are collinear. (ii) $|O_\beta W| = |O_\beta X|$.
(iii) The line $\mathcal{P}_V$ is the tangent of δ different from $\mathcal{P}_W$.
(iv) The line $\mathcal{P}_W$ is the radical axis of the semicircles α and $A(V)$.
(v) The diameter of δ parallel to AB and the line AW and $A(V)$ meet in a point.

Proof. The three points in (i) are collinear if and only if the triangles $AW'W$ and $AV'X$ are similar, which is equivalent to $(2a-v)/a = (2a-w)/e$ (see Figure 3). The last equation is equivalent to

$$v = a(2b+w)/e. \tag{6}$$

Since X has the y-coordinates $y_x = \sqrt{(2a-v)v}$, $|O_\beta X|^2 - |O_\beta W|^2 = (b+v)^2 + y_x^2 - ((b+w)^2 + y_w^2) = 2e(v - a(2b+w)/e)$ by (1). Therefore (i) and (ii) are equivalent. If r is the radius of δ, we get the left equation of (5). Hence (iii) holds if and only if $w + b(2a - w)/e = v$. The last equation is also equivalent to (6), i.e., (i) and (iii) are equivalent. Subtracting the equation $(x-2a)x + y^2 = 0$ of α from (3), we get $x = v + b(v/a - 2)$. The last equation expresses $\mathcal{P}_W$ if and only if $v + b(v/a-2) = w$, which is equivalent to (6). Therefore (iv) and (i) are equivalent. If the point of intersection of AW and the diameter of δ parallel to AB has the coordinates (x_d, y_d), then $x_d = 2a + (w-2a)\sqrt{a/e}$ and $y_d = y_w\sqrt{a/e}$ by (2) and the right equation of (5). Then $(x_d - 2a)^2 + y_d^2 - 2e(2a - v) = 2e(v - a(2b+w)/e)$, which shows the equivalence of (v) and (i) by (3). □

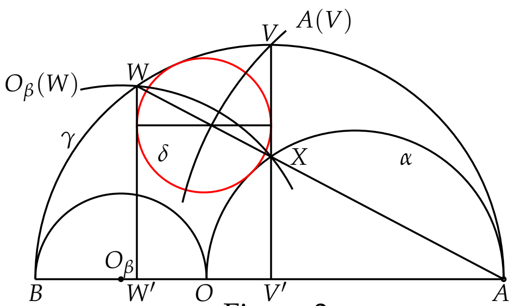

Figure 3.

3. The inscribed square of the triangle ABI

We call the radical axis of α and β the axis of (α, β, γ), which overlaps with the y-axis. Let Q and R be points on the segment AI and BI, and let P and S be the feet of perpendiculars from Q and R to AB, respectively. We assume that $PQRS$ is a square (see Figure 4). Let $f = \sqrt{ab}$, $g = e + f$. Since $|IO| = 2f$ and the triangles ABI and QRI are similar, $(|IO| - |PQ|) : |PQ| = |IO| : |AB|$ holds. This implies

$$|PQ| = 2ef/g. \tag{7}$$

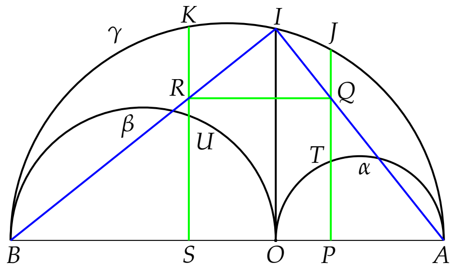

Figure 4.

From the similar triangles AOI and APQ, and BOI and BSR, we have

$$|AP| = 2ae/g \text{ and } |BS| = 2be/g. \tag{8}$$

Therefore we have

$$|OP| = 2af/g \ \text{and} \ |OS| = 2bf/g. \tag{9}$$

Let the line PQ intersect α and γ at points T and J, and let the line RS intersect β and γ at points U and K, respectively. By (9), the y-coordinates of the points J and K are

$$2e\sqrt{a(b+f)/g} \ \text{ and } \ 2e\sqrt{b(a+f)/g}. \tag{10}$$

4. Circles of radius r_s

Let H be the point of intersection of the axis and the segment QR. We denote the radius of the circle (HI) by r_s. Since $|IO| = 2f$, $r_s = ab/g$ by (7). In this section we show that more than a dozen of circles of radius r_s are obtained from (α, β, γ) with the square $PQRS$. Let δ_α (resp. δ_β) be the circle touching α (resp. β) externally, γ internally, and the line JP (resp. KS) from the side opposite to B (resp. A) (see Figure 5). Notice that if we exchange the roles of A and B, then α and β, J and K, P and S, T and U are also interchanged, respectively by their definitions.

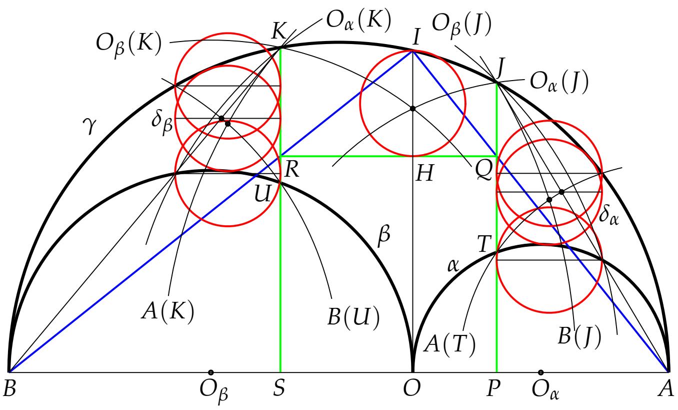

Figure 5.

Theorem 5. The following statements hold, and similar results are also true if we exchange the roles of A and B.
(i) The circles δ_α has radius r_s.
(ii) The semicircles α, $O_\beta(J)$ and the line AJ meet in a point, and its distance from the line JP equals $2r_s$. Also the distance between JP and the point of intersection of semicircles γ and $A(T)$ is $2r_s$. The diameter of δ_α parallel to AB, $A(T)$ and AJ meet in a point, which is one of the limiting points of the pencil $\mathrm{P}(\delta_\alpha, \mathcal{P}_A)$.
(iii) The point of intersection of the semicircles $A(T)$ and $B(J)$ is one of the limiting points of the pencil $\mathrm{P}(\delta_\alpha, AB)$. The point generates circles of radius r_s with each of γ and α.
(iv) The point of intersection of the semicircles $O_\alpha(J)$ and $O_\beta(K)$ is one of the limiting points of the pencil $\mathrm{P}((HI), AB)$.

Proof. If $w = 2af/g$, then $b(2a-w)/(2e) = r_s$. This proves (i) by the left equation of (5) and (9). The part (ii) is proved by (i) and Theorems 1 and 4. The first part of (iii) is proved by Theorem 3. The rest of (iii) is proved by (ii) of Theorem 2. We prove (iv). $|O_\alpha J|^2 = (a - |OP|)^2 + |JP|^2 = a^2 + 4er_s$ by (9) and (10). Hence $O_\alpha(J)$ has the equation $(x-a)^2 + y^2 = a^2 + 4er_s$. Therefore if M is the point of intersection of $O_\alpha(J)$ and the axis, then $|OM|^2 = 4er_s$. Since the right side of the equation is symmetric in a and b, the semicircle $O_\beta(K)$ also passes through M. While the power of O with respect to (HI) also equals $(|IO| - r_s)^2 - r_s^2 = 4er_s$. This proves (iv). □

5. Two generalized arbeloi with Archimedean circles of radius r_s

In this section and in the next two sections, we show that circles of radius r_s are Archimedean circles of some generalized arbeloi associated with (α, β, γ) with the square $PQRS$. In this section we consider an arbelos with overhang and an arbelos in n-aliquot parts. Let A_h and B_h be the points with coordinates $(2(a+h), 0)$ and $(-2(b+h), 0)$ for a real number $h > -\min(a, b)$, respectively, and let $\alpha_h = (A_h O)$ and $\beta_h = (B_h O)$. The configuration consisting of the three semicircles α_h, β_h and γ is called an arbelos with overhang h and is denoted by $(\alpha_h, \beta_h, \gamma)$. We use the next proposition [2, Propositions 1 and 6]. Recall that $r_{\rm A} = ab/e$.

Proposition 5.1. *The following statements are true.*
(i) *The circle touching α_h (resp. β_h) externally γ internally and the axis of (α, β, γ) from the side opposite to B (resp. A) has radius $ab/(e+h)$.*
(ii) *If $h > 0$, the circle touching α_h (resp. β_h) and γ internally and α (resp. β) externally has radius $hr_{\rm A}/(h + r_{\rm A})$.*

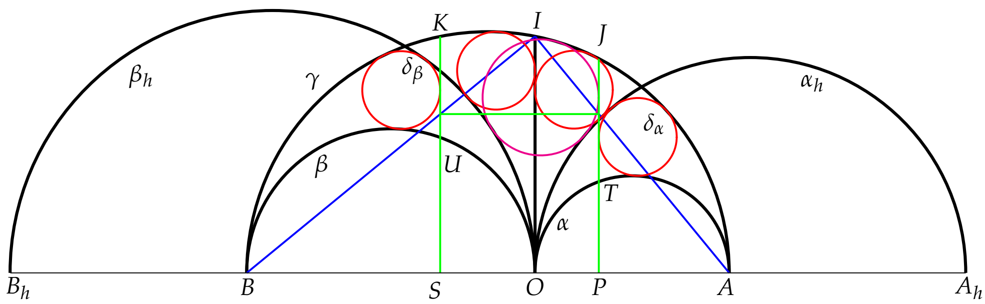

Figure 6.

Notice that $|AA_h| = |BB_h| = |IO|$ in the case $h = f$.

Theorem 6. If $h = f$, the following statements hold.
(i) The circle touching α_h (resp. β_h) externally γ internally and the axis from the side opposite to B (resp. A) has radius r_s.
(ii) The circle touching α_h (resp. β_h) and γ internally, α (resp. β) externally coincides with the circle δ_α (resp. δ_β).
(iii) The circle touching α_h and β_h externally and γ internally is an Archimedean circle of the arbelos (α, β, γ).

Proof. The parts (i) and (ii) follow from (i) and (ii) of Proposition 5.1. The circle in (iii) is an Archimedean circle of (α, β, γ) if and only if both the pairs γ and α_h and γ and β_h have the same external center of similitude [8, Theorem 3]. While they have the same external center of similitude with x-coordinate $2f(\sqrt{a} + \sqrt{b})/(\sqrt{b} - \sqrt{a})$. This proves (iii). □

We call the two circles of radius $ab/(e+h)$ in (i) of Proposition 5.1 the twin circles of Archimedes of $(\alpha_h, \beta_h, \gamma)$, and circles of radius $ab/(e+h)$ are called Archimedean circles of $(\alpha_h, \beta_h, \gamma)$. Hence if $h = f$, the two circles of radius r_s in (i) of Theorem 6 are the twin circles of Archimedes of $(\alpha_h, \beta_h, \gamma)$ and circles of radius r_s are Archimedean circles of $(\alpha_h, \beta_h, \gamma)$ (see Figure 6). Notice that the semicircles (A_hB) and (B_hA) have radius g and (7) shows that $|PQ|$ equals the diameter of Archimedean circles of the arbeloi $((AB), (AA_h), (A_hB))$ and $((AB), (BB_h), (B_hA))$. By (ii) of Theorem 2, the point of intersection of $A(T)$ and $B(J)$ also generates circles of radius r_s with α_h. Similar fact also holds for $B(U)$, $A(K)$ and β_h.

If we divide the area surrounded by α, β and γ by $n-1$ semicircles constructed in the region $y > 0$ belonging to $\mathrm{P}(\alpha, \beta)$ (one of which may be the axis) into n areas, and all their incircles are congruent, the configuration of $n+2$ semicircles is called an arbelos (α, β, γ) in n-aliquot parts. Circles congruent to those incircles are called Archimedean circles in n-aliquot parts of (α, β, γ) [6]. Therefore if $h = f$, the configuration consisting of α, α_h, the axis, β_h, β and γ form an arbelos (α, β, γ) in 4-aliquot parts, and circles of radius r_s are also Archimedean circles in 4-aliquot parts of (α, β, γ) (see Figure 6).

6. A collinear arbelos with Archimedean circles of radius r_s

Let X (resp. Y) be a point on the half line with initial point A (resp. B) passing through B (resp. A), and let $s = |AY|/2$ and $t = |BX|/2$. If X and Y lie inside γ, the circle touching (AX) (resp. (BY)) externally γ internally and the radical axis of (AX) and (BY) from the side opposite to B (resp. A) has radius $st/(s+t)$. Also if X and Y lie outside γ, the circle touching (AX) (resp. (BY)) internally γ externally and the radical axis of (AX) and (BY) from the side opposite to A (resp. B) has radius $st/(s+t)$ (see Figure 7). In both the cases the configuration consisting of the semicircles (AX), (BY) and γ is called a collinear arbelos and is denoted by $((AX), (BY), \gamma)$ [4]. The two congruent circles are called the twin circles of Archimedes of $((AX), (BY), \gamma)$, and circles of radius $st/(s+t)$ are called Archimedean circles of it. Note that X and Y are interchanged, if we exchange the roles of A and B.

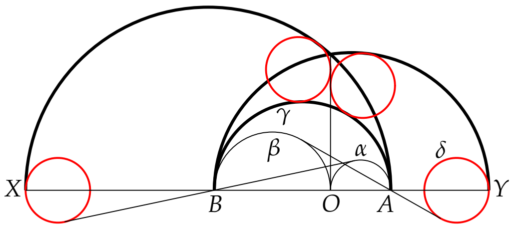

Figure 7.

If $X = S$ and $Y = P$, the radical axis of the two semicircles (AS) and (BP) coincides with the axis of (α, β, γ) by (9), and $s = |AP|/2$ and $t = |BS|/2$. This implies $st/(s+t) = r_s$ by (8). Therefore the two circles of radius r_s in (i) of Theorem 6 coincide with the twin circles of Archimedes of the collinear arbelos $((AS), (BP), \gamma)$, and circles of radius r_s are Archimedean circles of it (see Figure 8).

We have been giving many theorems that prove circles are Archimedean circles of some generalized arbeloi [2, 3, 4, 5]. Thereby if circles of radius r_s are Archimedean circles of such a generalized arbelos, we can get circles of radius r_s from those theorems. We consider a brief example for the collinear arbelos. The next theorem gives new Archimedean circles of the collinear arbelos.

Theorem 7. The following statements hold for a collinear arbelos $((AX),(BY),\gamma)$, and similar facts also hold if we exchange the roles of A and B.
(i) If the perpendicular bisector of the segment AY intersects the semicircle α, the point of intersection generates circles of radius $st/(s+t)$ with γ.
(ii) If a circle δ touches the tangent of β from A and also touches the semicircle (BY) externally or internally at Y according as Y lies inside or outside γ, then δ is an Archimedean circle of $((AX),(BY),\gamma)$.

Proof. For points V and W on the line AB, if the semicircle (AV) and the line $\mathcal{P}_W$ intersect, the point of intersection generates circles of radius $|AW||BV|/(2e)$ with γ [3, Lemma 3]. If W is the midpoint of AY and $V=O$, then using the fact $ta=sb$ [4, (1)], we get

$$\frac{|AW||BV|}{2e}=\frac{(|AY|/2)|BO|}{2e}=\frac{s\cdot 2b}{2e}=\frac{sb}{e}=\frac{sb}{sb/t+b}=\frac{st}{s+t}.$$

This proves (i). If we regard β as a circle, A is one the center of similitude of β and δ (see Figure 7). Hence if r is the radius of δ, we get $b/|AB|=r/|AY|$, i.e., $r=st/(s+t)$. □

Therefore the point of intersection of α and the perpendicular bisector of AP generates circles of radius r_s with γ by (i) of this theorem. (see Figure 8). Also the circle touching the tangent of β from A and (BP) externally at P has radius r_s by (ii). Similar results hold if we exchange the roles of A and B.

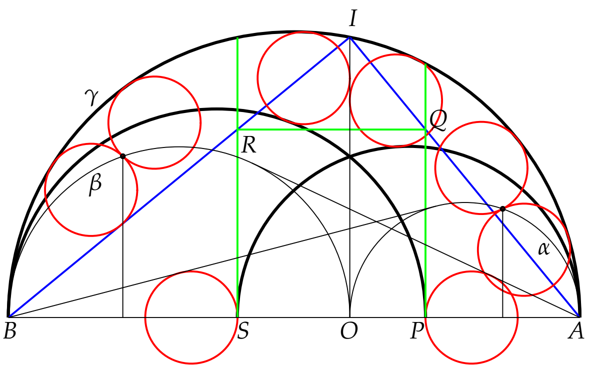

Figure 8.

7. A SKEWED ARBELOS WITH ARCHIMEDEAN CIRCLES OF RADIUS r_s

In this section we consider α and β as circles. Let us consider the circle γ_z expressed by the equation

$$\left(x-\frac{b-a}{z^2-1}\right)^2+\left(y-\frac{2z\sqrt{ab}}{z^2-1}\right)^2=\left(\frac{e}{z^2-1}\right)^2 \tag{11}$$

for a real number $z\neq\pm1$. It touches α and β internally if $|z|<1$ and externally if $|z|>1$. The configuration consisting of α, β and γ_z is also denoted by (α,β,γ_z) and is called a skewed arbelos [7].
Let A_z (resp. B_z) be the point of tangency of the circles γ_z and α (resp. β). Let δ_z^α be the circle different from β and touching α and the tangents of β from A_z. The circle δ_z^β is defined similarly. The circles δ_z^α and δ_z^β have common radius $|1-z^2|r_A$ [5], and circles of the same radius are called Archimedean circles of (α,β,γ_z) [3]. While the product of

the radius of γ_z and $|1-z^2|r_A$ equals ab by (11). Therefore (α,β,γ_z) has Archimedean circles of radius r_s if and only if γ_z has radius g, which is also equivalent to $z^2 = f/g$ or $z^2 = 2 - f/g$ (see Figure 9). Notice that γ_z has the same radius as the semicircles (A_hB) and (B_hA) with $h = f$ in this event.

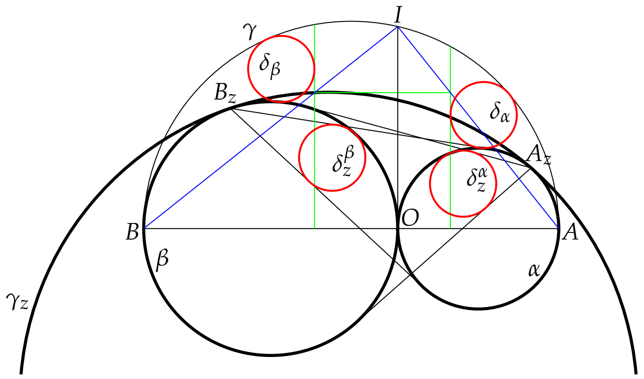

Figure 9: $z = \sqrt{f/g}$

8. Some more circles of radius r_s

In this section we show that some more circles of radius r_s are obtained from the arbelos (α,β,γ) with the square $PQRS$.

Theorem 8. The semicircles α and $O_\beta(K)$ and the line AK meet in a point L. If M is the point of intersection of the segment KP and the axis, then ML is parallel to AB and $|ML| = 2r_s$. Similar results are also true if we exchange the roles of A and B.

Proof. The first part of (i) follows from Theorem 4. The y-coordinate of L equals $|SK|a/e$ by the similar triangles ABK and AOL (see Figure 10). While M has y-coordinate $|SK||OP|/|PQ| = |SK|a/e$ by the similar triangles PSK and POM. Therefore ML is parallel to AB, and $|ML| = |BS|a/e = 2r_s$ by (8). □

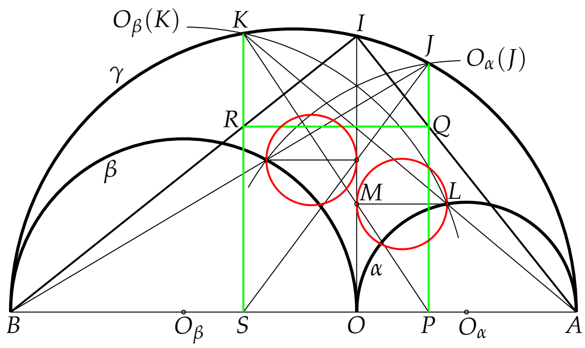

Figure 10.

Let C be the center of the square $PQRS$, and let E, F and G be the images of the point I by the rotations about C through $-90°$, $-180°$ and $-270°$, respectively. Hence the three points lie on the circle $C(I)$. Also let E' and G' be the reflections of the points E and G in the line $\mathcal{P}_C$, respectively (see Figure 11).

Theorem 9. The following statement hold.
(i) The distance from the segment QR (resp. SP) to each of the points of intersection of the axis and the circle (QR) (resp. (SP)) equals $2r_s$, and one of the points lies on the circle $C(I)$. Similar facts also hold for the lines EE', $\mathcal{P}_F$, and GG' and the corresponding sides of $PQRS$.
(ii) The circles $C(I)$ and (PQ) and the semicircle (AP) and the line AI, meet in the point E, and its distance from PQ is $2r_s$. The circle (RS) and the semicircle $P(I)$ meet in the point E', and its distance from RS is $2r_s$. Similar facts hold for the points G and G'.

Proof. The part (i) is obvious, because I lies on the circle (QR). Let E'' be the the point of intersection of (AP) and AI. The distance between PQ and E'' is $2b|AP|/|AB| = 2r_s$ by the similar triangles ABI and APE''. While rotating the line BI and the segment QR about C through $-90°$, we get the line AI and the segment PQ. Therefore E lies on AI, and whose distance from PQ equals $2r_s$, i.e., E and E'' coincide. Let D be the point of intersection of EE' and PQ. Since $|OP| = |DP|$, the triangles IOP and $E'DP$ are congruent. Hence $|E'P| = |IP|$, i.e., E' lies on $P(I)$. The rest of (ii) is obvious. □

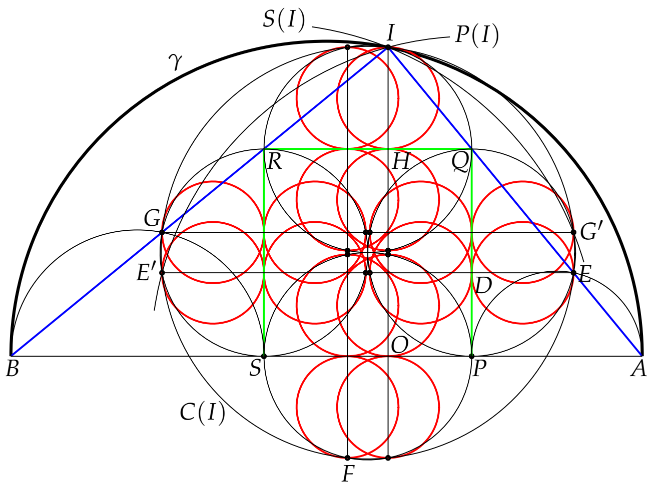

Figure 11.

9. Conclusion

We get many circles of radius r_s from the arbelos (α, β, γ) with the square $PQRS$. The circles are not Archimedean circles of (α, β, γ), but Archimedean circles of several generalized arbeloi associated with (α, β, γ). Our result suggests that if we find several circles of certain radius associated with (α, β, γ), and they are not Archimedean circles of (α, β, γ), then they might be Archimedean circles of some generalized arbelos associated with (α, β, γ).

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