THE PONCELET PENCIL’S HYPERBOLAS AS LOCUS GEOMETRIC AND
THEIR EQUATIONS IN BARYCENTRIC COORDINATES

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ABSTRACT. In this paper we shown that any hyperbolas of the Poncelet pencil may be consider as a locus geometric.

1. INTRODUCTION
The isogonal conjugation determined by a triangle ABC transform a general line m to a conic K passing through A, B, C. Let \( px + qy + rz = 0 \) the equation of the line \( m \). We use the barycentric coordinates with respect to the triangle ABC: \( A = (1 : 0 : 0),\ B = (0 : 1 : 0),\ C = (0 : 0 : 1) \). In the plan of the triangle ABC we consider a varying point \( M = (u : v : w) \), so that \( uvw \neq 0 \). The isogonal transform of \( M \) is the point \( M' = (a^2vw : b^2wu : c^2uv) \) and of line \( m \) the circumconic \( pa^2yz + qb^2zx + rc^2xy = 0 \). Consequently, the point \( M \) is on the line \( m \), if and only if the conic \( K \) passes through the \( M' \). Indeed:
\[
M \in m \iff pu + qv + rw = 0 \iff a^2b^2c^2uvw(pu + qv + rw) = 0 \iff
pa^2 \cdot b^2wu \cdot c^2uv + qb^2 \cdot c^2uv \cdot a^2vw + rc^2 \cdot a^2vw \cdot b^2wu = 0 \iff M' \in K.
\]
By the isogonal conjugation the pencil of lines through the circumcenter O is transformed to a pencil of rectangular hyperbolas passing through the vertices A, B, C and the orthocenter H of triangle ABC (see [1], Theorem 1).

The straight line connecting the circumcenter O and the incenter I of a triangle ABC is transformed into the Feuerbach hyperbola. There are three other excentral Feuerbach hyperbolas which are obtained by applying the isogonal conjugation to the lines \( OI_a, OI_b, OI_c \), where \( I_a, I_b, I_c \) denote the excenters, of the triangle ABC. In [2] are established another derivation, namely as locus geometric and some properties of these Feuerbach hyperbolas.
In this paper we shown that any hyperbolas of the Poncelet pencil may be consider as a locus geometric. Let \( \mu = u + v + w \neq 0 \), \( S \) the twice of the area of triangle ABC, \( S_A = bc \cos A, S_B = ca \cos B, S_C = ab \cos C \), so that \( S_A S_B S_C \neq 0 \) and \( X, Y, Z \) the projections of \( M \) on the sidelines \( BC, CA, AB \), respectively.

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2. THE LENGTH OF THE SEGMENTS $MX, MY, MZ$

Since the equation of the line $MX$ is

$$(vS_B - wS_C)x - (wa^2 + uS_B)y + (va^2 + uS_C)z = 0,$$

the barycentric coordinates of the point $X$ are

$$X = (0 : va^2 + uS_C : wa^2 + uS_B).$$

The absolut barycentric coordinates of the point $X$ are

$$X = (0, v \frac{a^2 + uS_C}{\mu a^2}, w \frac{a^2 + uS_B}{\mu a^2}).$$

Now we calculate the length of the segment $MX$:

$$MX^2 = \frac{u^2}{\mu^2} \left[ S_A + \frac{S_B S_C}{a^2} (S_B + S_C) \right] = \frac{u^2}{\mu^2} \left( S_A + \frac{S_B S_C}{a^2} \right)$$

consequently $MX = \frac{uS}{a\mu}$. The equations of the line $MY$ and $MZ$ are:

$$(wb^2 + vS_A)x + (wS_C - uS_A)y - (ub^2 + vS_C)z = 0,$$

$$-(vc^2 + wS_A)x + (uc^2 + wS_B)y + (uS_A - vS_B)z = 0.$$

The barycentric, respectively absolute barycentric coordinates of the points $Y$ and $Z$ are

$$Y = (ub^2 + vS_C : 0 : wb^2 + vS_A) = \left( \frac{ub^2 + vS_C}{\mu b^2}, 0, \frac{wb^2 + vS_A}{\mu b^2} \right)$$

$$Y = \left( \frac{u}{\mu} + \frac{vS_C}{ub^2}, 0, \frac{w}{\mu} + \frac{vS_A}{\mu b^2} \right),$$

$$Z = (uc^2 + wS_B : vc^2 + wS_A : 0) = \left( \frac{uc^2 + wS_B}{\mu c^2}, \frac{vc^2 + wS_A}{\mu c^2}, 0 \right)$$

$$Z = \left( \frac{u}{\mu} + \frac{wS_B}{\mu c^2}, \frac{v}{\mu} + \frac{wS_A}{\mu c^2}, 0 \right).$$

Similarly we obtain the length of the segment $MY$ and $MZ$:

$$MY = \frac{vS}{b\mu}, \quad MZ = \frac{wS}{c\mu}.$$

Let $X(\alpha), Y(\beta)$ and $Z(\gamma)$ points on the half-line $MX, MY, MZ$ respectively, and $MX(\alpha) = \alpha, MY(\beta) = \beta, MZ(\gamma) = \gamma$. 

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3. THE BARYCENTRIC COORDINATES OF THE POINTS X(α), Y(β) AND Z(γ)

We determine the absolute barycentric coordinates of the point X(α). Since \( \frac{MX(\alpha)}{X(\alpha)X} = \frac{\alpha}{MX - \alpha} \), for this

\[
X(\alpha) = \frac{(MX - \alpha)M + \alpha X}{MX} = \frac{MX - \alpha}{MX} \left( \frac{u}{\mu}, \frac{v}{\mu}, \frac{w}{\mu} \right) + \frac{\alpha}{MX} \left( 0, \frac{vS_C}{\mu a^2}, \frac{w}{\mu} + \frac{uS_B}{\mu a^2} \right)
\]

\[
= \left( \frac{u}{\mu} \right) - \frac{\alpha}{S} \left( \frac{v}{\mu} + \frac{\alpha S_C}{\mu a} \right) + \frac{\alpha}{S} \left( \frac{w}{\mu} + \frac{\alpha S_B}{\mu a} \right).
\]

So the barycentric coordinates of the point X(α) are

\[
X(\alpha) = (auS - \mu a^2 : avS + \mu aS_C : awS + \mu aS_B).
\]

Similarly we obtain

\[
Y(\beta) = (buS + \mu bS_C : bvS - \mu b^2 : bwS + \mu bS_A),
\]

\[
Z(\gamma) = (cuS + \mu cS_B : cvS + \mu cS_A : cwS - \mu c^2).
\]

In general, the lines AX(α), BY(β), CZ(γ) are not concurrent. What is the condition of concurrence of these lines?

4. THE CONDITION OF CONCURRENCE OF THE LINES AX(α), BY(β), CZ(γ)

**Proposition 4.1.** The lines AX(α), BY(β), CZ(γ) are concurrent if and only if

\[
S \left[ bc(vS_B - wS_C)u\alpha + ca(wS_C - uS_A)v\beta + ab(uS_A - vS_B)w\gamma \right] = \mu \left[ aS_A(vS_B - wS_C)\beta \gamma + bS_B(wS_C - uS_A)\gamma \alpha + cS_C(uS_A - vS_B)\alpha \beta \right].
\]

**Proof.** The equations of the lines AX(α), BY(β), CZ(γ) are

\[
(awS + \mu aS_C)\gamma - (awS + \mu aS_C)z = 0,
\]

\[
(bwS + \mu bS_A)z - (buS + \mu bS_C)z = 0,
\]

\[
(cwS + \mu cS_B)x - (cuS + \mu cS_A)\gamma = 0.
\]

The concurrence of the lines AX(α), BY(β), CZ(γ) is equivalent with the

\[
(awS + \mu aS_C)(bwS + \mu bS_A)(cuS + \mu cS_B)
= (awS + \mu aS_B)(bwS + \mu bS_C)(cwS + \mu cS_A),
\]

which is equivalent with the condition (1). □
We introduce the following notations:
\[ E(u, v, w, \alpha, \beta, \gamma) = bc(vS_B - wS_C)u\alpha + ca(wS_C - uS_A)v\beta + ab(uS_A - vS_B)w\gamma, \]
\[ F(u, v, w, \alpha, \beta, \gamma) = aS_A(vS_B - wS_C)\beta\gamma + bS_B(wS_C - uS_A)\gamma\alpha + cS_C(uS_A - vS_B)\alpha\beta. \]

With these notations the condition (1) can write in the below form:
\[ S \cdot E(u, v, w, \alpha, \beta, \gamma) - \mu \cdot F(u, v, w, \alpha, \beta, \gamma) = 0. \] (2)

**Proposition 4.2.** The lines \( AX(\alpha), BY(\beta), CZ(\gamma) \) are concurrent, \( \forall t \in \mathbb{R} \), if
\[ \alpha = \frac{at}{u}, \ \beta = \frac{bt}{v}, \ \gamma = \frac{ct}{w}. \] (3)

**Proof.** If \( \alpha = \frac{at}{u}, \ \beta = \frac{bt}{v}, \ \gamma = \frac{ct}{w} \), we will demonstrate that the condition of concurrence (2) is come true. Indeed:
\[ E \left( u, v, w, \frac{at}{u}, \frac{bt}{v}, \frac{ct}{w} \right) = abct(vS_B - wS_C + wS_C - uS_A + uS_A - vS_B) = 0, \]
\[ F \left( u, v, w, \frac{at}{u}, \frac{bt}{v}, \frac{ct}{w} \right) = \frac{abct^2}{uvw} [uS_A(vS_B - wS_C) + vS_B(wS_C - uS_A) + wS_C(uS_A - vS_B)] = 0. \]

\[ \Box \]

**Lemoine’s theorem** [3]. Let \( ABC \) be a triangle, \( M \) a point in its plane, and \( X, Y, Z \) the projections of \( M \) on \( BC, CA, AB \), respectively. If \( A’, B’, C’ \) are points on the half-lines \( MX, MY, MZ \), respectively, such that
\[ MX \cdot MA’ = MY \cdot MB’ = MZ \cdot MC’ \] (4)
then \( AA’, BB’, CC’ \) are concurrent.

Let \( A’ = X \left( \frac{at}{u} \right), B’ = Y \left( \frac{bt}{v} \right), C’ = Z \left( \frac{ct}{w} \right) \). The conditions (3) and (4) are equivalent. Indeed:
\[ \alpha = \frac{at}{u}, \beta = \frac{bt}{v}, \gamma = \frac{ct}{w} \Leftrightarrow \frac{uat}{a} = \frac{vbt}{b} = \frac{wct}{c} = t \Leftrightarrow \]
\[ \Leftrightarrow \frac{uS}{a\mu} = \frac{vS}{b\nu} = \frac{wS}{c\mu} \Leftrightarrow MX \cdot MA’ = MY \cdot MB’ = MZ \cdot MC’. \]

Consequently the Proposition 4.2 is equivalent with the Lemoine’s theorem. The condition (3) is only sufficient for the concurrence of the lines \( AX(\alpha), BY(\beta), CZ(\gamma) \), but it is not also necessary.

In this case we introduce the following notation:
\[ K(M, t) = K \left( u, v, w, \frac{at}{u}, \frac{bt}{v}, \frac{ct}{w} \right) = AX \left( \frac{at}{u} \right) \cap BY \left( \frac{bt}{v} \right) \cap CZ \left( \frac{ct}{w} \right). \]
Remarks. 1) \( K(M, 0) = M \).
2) If the point \( M \) coincide with the incenter \( I = (a : b : c) \) and \( t = r \), then the point \( K(I, r) = K(a, b, c, r, r, r) = AX(r) \cap BY(r) \cap CZ(r) \) is the Gergonne point of the triangle \( ABC \).
3) If \( M \neq H \), then \( H = K(M, \infty) = AX(\infty) \cap BY(\infty) \cap CZ(\infty) \).

5. The locus of the points \( K(M, t) \)

Proposition 5.1. If \( M \neq H \), then the point \( K(M, t) \) describe the circumconic \( K_M \) with equation
\[
u(vS_B - wS_C)yz + v(wS_C - uS_A)zx + w(uS_A - vS_B)xy = 0,
\]
which is a hyperbola.

Proof. The equations of the lines \( AX(\frac{at}{u}) \), \( BY(\frac{bt}{v}) \), \( CZ(\frac{ct}{w}) \) are
\[
(uvw + \mu tS_B)y - (uvw + \mu tS_C)z = 0,
(uvw + \mu tS_A)x - (uvw + \mu tS_C)z = 0,
(uvw + \mu tS_A)x - (uvw + \mu tS_B)y = 0,
\]
From this equations \( t = \frac{uv(z - wy)}{(yS_B - zS_C)\mu} = \frac{v(wx - uz)S}{(zS_C - xS_A)\mu} = \frac{w(uy - vx)S}{(xS_A - yS_B)\mu} \), from where arise the equation (5). The circumconic \( pyz + qzx + rxy = 0 \) is hyperbola if and only if \( pS_A + qS_B + rS_C = 0 \). In our case the condition is true. Indeed:
\[
u(vS_B - wS_C)S_A + v(wS_C - uS_A)S_B + w(uS_A - vS_B)S_C = 0.
\]

\[\square\]

Remark. If \( M = H \), then the locus of the points \( K(M, t) \) is only the point \( H \).

Proposition 5.2. The circumconic \( K_M \) passes through the orthocenter \( H \) of the triangle \( ABC \), thus is a rectangular hyperbola. The point \( M \) is on the hyperbola \( K_M \), too.

Proof. The barycentric coordinates of the points \( H = \left( \frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C} \right) \) and \( M = (u : v : w) \) satisfy the equation of \( K_M \):
\[
H \in K_M \iff \frac{1}{S_AS_BS_C} \left[ uS_A(vS_B - wS_C) + vS_B(wS_C - uS_A) + wS_C(uS_A - vS_B) \right] = 0,
M \in K_M \iff uvw(vS_B - wS_C + wS_C - uS_A + uS_A - vS_B) = 0.
\]

\[\square\]

Proposition 5.3. The hyperbola \( K_M \) is the isogonal conjugate of the line \( OM' \), where \( O \) is the circumcenter of the triangle \( ABC \) and \( M' \) is the isogonal conjugate of the point \( M \).
Proof. The coordinates of the points $O$ and $M'$ are:

$$O = (a^2S_A : b^2S_B : c^2S_C), \quad M' = (a^2vw : b^2wu : c^2uv).$$

The equation of the line $OM'$ is

$$b^2c^2u(vS_B - wS_C)x + c^2a^2v(wS_C - uS_A)y + a^2b^2w(uS_A - vS_B)z = 0.$$  

The isogonal transform of this equation is

$$b^2c^2u(vS_B - wS_C)a^2yz + c^2a^2v(wS_C - uS_A)b^2zx + a^2b^2w(uS_A - vS_B)c^2xy = 0,$$

which is equivalent with the equation of the hyperbola $K_M$. \[\square\]

**Proposition 5.4.** The center $C_M$ of hyperbola $K_M$ has barycentric coordinates

$$x = u(vS_B - wS_C)[w(u + v)b^2 - v(u + w)c^2],$$

$$y = v(wS_C - uS_A)[u(v + w)c^2 - w(v + u)a^2],$$

$$z = w(uS_A - vS_B)[v(w + u)a^2 - u(w + v)b^2],$$

and it is on the nine-point circle of the triangle $ABC$.

Proof. The coordinates of the center $C_M$ are

$$x = p(-p + q + r), \quad y = q(p - q + r), \quad z = r(p + q - r),$$

where $p = u(vS_B - wS_C), \quad q = v(wS_C - uS_A), \quad r = w(uS_A - vS_B)$ and

$$-p + q + r = -u(vS_B - wS_C) + v(wS_C - uS_A) + w(uS_A - vS_B)$$

$$= w(u + v)b^2 - v(u + w)c^2.$$  

The equation of the nine-point circle is

$$S_Ax^2 + S_By^2 + S_Cz^2 - a^2yz - b^2zx - c^2xy = 0.$$  

We transform the left member of this equation:

$$S_Ax^2 + S_By^2 + S_Cz^2 - a^2yz - b^2zx - c^2xy$$

$$= S_Ax^2 + S_By^2 + S_Cz^2 - (S_B + S_C)yz - (S_C + S_A)zx - (S_A + S_B)xy$$

$$= -x(-x + y + z)S_A - y(x - y + z)S_B - z(x + y - z)S_C$$

$$= -\frac{xyz}{pqr} (pS_A + qS_B + rS_C) = 0,$$

since $x(-x + y + z) = \frac{xyz}{qr}, \quad y(x - y + z) = \frac{xyz}{rp}, \quad z(x + y - z) = \frac{xyz}{pq}$ and

$$pS_A + qS_B + rS_C = 0.$$  

\[\square\]
6. THE CONSTRUCTION OF THE POINTS $K(M, t)$

The question is how we can determine geometrical the points $X\left(\frac{at}{u}\right)$, $Y\left(\frac{bt}{v}\right)$, $Z\left(\frac{ct}{w}\right)$ which satisfy the conditions of concurrence (1)?

**Lemma 6.1.** The antiparallel $d_A$ of the side $BC$ is perpendicular to the line $OA$, when $O$ is the circumcenter of the triangle $ABC$.

**Proof.** Let $L$, $P$ and $N$ the points of intersection of the antiparallel $d_A$ with the lines $AB$, $OA$ and $BC$ (see Figure 1). Let furthermore $D$ the diametrically opposite of the vertices $A$.

![Figure 1](image)

Since the triangles $APL$ and $ABD$ are right-angled triangles, we have:

\[
\widehat{ALN} \equiv 90^\circ - \widehat{BAD} \equiv ADB \equiv ACB.
\]

Since $MX, MY, MZ$ are perpendicular to the sides $BC, CA, AB$, respectively, the circumcenters of the triangles $MYZ$, $MZX$, $MXY$ are the midpoints of the segments $MA, MB, MC$. The construction of the points $K(M, t)$ is following: we put on the half-line $MX$ a point arbitrary $A'$ and we pull a perpendicular to the lines $MC$ which intersect the line $MY$ in the point $B'$; answerably to the Lemma 6.1, the line $A'B'$ is the antiparallel with the line $XY$, consequently $MX \cdot MA' = MY \cdot MB'$; from $B'$ we pull a perpendicular to the lines $MA$ which intersect the line $MZ$ in the point $C'$; the line $B'C'$ is the antiparallel with the line $YZ$, consequently $MY \cdot MB' = MZ \cdot MC'$; the line $C'A'$ will be the antiparallel of the lines $ZX$; from the theorem of Lemoine result then the lines $AA', BB', CC'$ are concurrent (see Figure 2).
7. Special cases

7.1. To the points $A, B, C$ no correspond never a one hyperbole (5).

7.2. If $u = 0$ and $vw \neq 0$, then $M \in BC$, $M \neq B$, $M \neq C$ and the locus of the points $K(M, t)$ is the line $AH$, the altitude, joined with the point $M$.

7.3. The isogonal conjugate of the line $OA$ has equation $S_Czx - S_Bxy = 0$, which is a degenerate hyperbola consisting of the side $BC$ and the altitude $AH$. So, the Poncelet pencil has three degenerate rectangular hyperbolas.

8. Equations of some notable hyperbolas

8.1. If the point $M$ coincide with the centroid $G = (1 : 1 : 1)$ of the triangle $ABC$, then $u = v = w = 1$. In this case $\alpha = at$, $\beta = bt$, $\gamma = ct$ and the point

$$K(G, t) = K(1, 1, 1, at, bt, ct) = AX(at) \cap BY(bt) \cap CZ(ct)$$

describe the Kiepert hyperbola with equation

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0. \quad (6)$$

The Kiepert hyperbola is the isogonal transform of the line $GL$, where $L$ is the symmedian point (Lemoine point) of the triangle $ABC$.

8.2. If the point $M$ coincide with the circumcenter $O = (a^2S_A : b^2S_B : c^2S_C)$ of the triangle $ABC$, then $u = a^2S_A$, $v = b^2S_B$, $w = c^2S_C$. In this case $\alpha = \frac{t}{aS_A}$, $\beta = \frac{t}{bS_B}$.
\[ \gamma = \frac{t}{cS_C} \]
and the point
\[ K(O, t) = K \left( a^2 S_A \cdot b^2 S_B \cdot c^2 S_C \cdot \frac{t}{aS_A} \cdot \frac{t}{bS_B} \cdot \frac{t}{cS_C} \right) \]
\[ = AX \left( \frac{t}{aS_A} \right) \cap BY \left( \frac{t}{bS_B} \right) \cap CZ \left( \frac{t}{cS_C} \right) \]
describe the \textit{Jerabek hyperbola} with equation
\[ a^2 S_A (b^2 - c^2) yz + b^2 S_B (c^2 - a^2) zx + c^2 S_C (a^2 - b^2) xy = 0. \] (7)

The Jerabek hyperbola is the isogonal transform of the Euler line OH.

8.3. If the point \( M \) coincide with the incenter \( I = (a : b : c) \) of the triangle \( ABC \), then \( u = a, v = b, w = c \). In this case \( \alpha = \beta = \gamma = t \) and the point
\[ K(I, t) = K(a, b, c, t, t, t) = AX(t) \cap BY(t) \cap CZ(t) \]
describe the \textit{Feuerbach hyperbola} with equation
\[ a(bS_B - cS_C)yz + b(cS_C - aS_A)zx + c(aS_A - bS_B)xy = 0. \] (8)

8.4. If the point \( M \) coincide with the A-excenter \( I_a = (-a : b : c) \) of the triangle \( ABC \), then \( u = -a, v = b, w = c \). In this case \( \alpha = \beta = \gamma = t \) and the point
\[ K(I_a, t) = K(-a, b, c, -t, t, t) = AX(-t) \cap BY(t) \cap CZ(t) \]
describe the \textit{A-ex-Feuerbach hyperbola} with equation
\[ a(bS_B - cS_C)yz - b(cS_C + aS_A)zx + c(aS_A + bS_B)xy = 0. \] (9)

8.5. If the point \( M \) coincide with the B-excenter \( I_b = (a : -b : c) \) of the triangle \( ABC \), then \( u = a, v = -b, w = c \). In this case \( \alpha = t, \beta = t, \gamma = t \) and the point
\[ K(I_b, t) = K(a, -b, c, t, -t, t) = AX(t) \cap BY(-t) \cap CZ(t) \]
describe the \textit{B-ex-Feuerbach hyperbola} with equation
\[ a(bS_B + cS_C)yz + b(cS_C - aS_A)zx - c(aS_A + bS_B)xy = 0. \] (10)

8.6. If the point \( M \) coincide with the C-excenter \( I_c = (a : b : -c) \) of the triangle \( ABC \), then \( u = a, v = b, w = -c \). In this case \( \alpha = t, \beta = t, \gamma = -t \) and the point
\[ K(I_c, t) = K(a, b, -c, t, t, -t) = AX(t) \cap BY(t) \cap CZ(-t) \]
describe the \textit{C-ex-Feuerbach hyperbola} with equation
\[ -a(bS_B + cS_C)yz + b(cS_C + aS_A)zx + c(aS_A - bS_B)xy = 0. \] (11)

8.7. The isogonal transform of the tangent line to Jerabek hyperbola at \( O \) is the \textit{Huygens' hyperbola}. The equation of the tangent is
\[ b^2 c^2 S_B S_C (b^2 - c^2) x + c^2 a^2 S_C S_A (c^2 - a^2) y + a^2 b^2 S_A S_B (a^2 - b^2) z = 0. \]

So the equation of the Huygens' hyperbola is
\[ S_B S_C (b^2 - c^2) yz + S_C S_A (c^2 - a^2) zx + S_A S_B (a^2 - b^2) xy = 0. \] (12)
9. FURTHER RESEARCH

Further research may tackle the determination of the axes, foci, vertex and asymptotes of the hyperbolas $K_M$.

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