Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.4, (2015), Issue 2, pp.106-115

THE PONCELET PENCIL'S HYPERBOLAS AS LOCUS GEOMETRIC AND THEIR EQUATIONS IN BARYCENTRIC COORDINATES

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ABSTRACT. In this paper we shown that any hyperbolas of the Poncelet pencil may be consider as a locus geometric.

1. INTRODUCTION

The isogonal conjugation determined by a triangle ABC transform a general line m to a conic K passing through A, B, C. Let $px + qy + rz = 0$ the equation of the line m. We use the barycentric coordinates with respect to the triangle ABC: $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$. In the plan of the triangle ABC we consider a varying point $M = (u : v : w)$, so that $uvw \neq 0$. The isogonal transform of M is the point $M' = (a^2vw : b^2wu : c^2uv)$ and of line m the circumconic $pa^2yz + qb^2zx + rc^2xy = 0$. Consequently, the point M is on the line m, if and only if the conic K passes through the M'. Indeed:

$$M \in m \Leftrightarrow pu + qv + rw = 0 \Leftrightarrow a^2b^2c^2uvw(pu + qv + rw) = 0 \Leftrightarrow$$
$$pa^2 \cdot b^2wu \cdot c^2uv + qb^2 \cdot c^2uv \cdot a^2vw + rc^2 \cdot a^2vw \cdot b^2wu = 0 \Leftrightarrow M' \in K.$$

By the isogonal conjugation the pencil of lines through the circumcenter O is transformed to a pencil of rectangular hyperbolas passing through the vertices A, B, C and the orthocenter H of triangle ABC (see [1], Theorem 1).

The straight line connecting the circumcenter O and the incenter I of a triangle ABC is transformed into the Feuerbach hyperbola. There are three other excentral Feuerbach hyperbolas which are obtained by applying the isogonal conjugation to the lines OI_a, OI_b, OI_c, where I_a, I_b and I_c denote the excenters, of the triangle ABC. In [2] are established another derivation, namely as locus geometric and some properties of these Feuerbach hyperbolas.

In this paper we shown that any hyperbolas of the Poncelet pencil may be consider as a locus geometric. Let $\mu = u + v + w \neq 0$, S the twice of the area of triangle ABC, $S_A = bc \cos A$, $S_B = ca \cos B$, $S_C = ab \cos C$, so that $S_AS_BS_C \neq 0$ and X, Y, Z the projections of M on the sidelines BC, CA, AB, respectively.

2010 *Mathematics Subject Classification.* 51M04.

Key words and phrases. Isogonal conjugate, Poncelet Pencil's Hyperbolas.

2. The length of the segments MX, MY, MZ

Since the equation of the line MX is

$$(vS_B - wS_C)x - (wa^2 + uS_B)y + (va^2 + uS_C)z = 0,$$

the barycentric coordinates of the point X are

$$X = (0 : va^2 + uS_C : wa^2 + uS_B).$$

The absolut barycentric coordinates of the point X are

$$X = \left(0, \frac{va^2 + uS_C}{\mu a^2}, \frac{wa^2 + uS_B}{\mu a^2}\right) = \left(0, \frac{v}{\mu} + \frac{uS_C}{\mu a^2}, \frac{w}{\mu} + \frac{uS_B}{\mu a^2}\right).$$

Now we calculate the length of the segment MX:

$$\begin{aligned} MX^2 &= \frac{u^2}{\mu^2}\left[S_A + \frac{S_B S_C}{a^4}(S_B + S_C)\right] = \frac{u^2}{\mu^2}\left(S_A + \frac{S_B S_C}{a^2}\right) \\ &= \frac{u^2}{a^2\mu^2}(a^2 S_A + S_B S_C) = \frac{u^2 S^2}{a^2 \mu^2} \end{aligned}$$

consequently $MX = \dfrac{uS}{a\mu}$. The equations of the line MY and MZ are:

$$(wb^2 + vS_A)x + (wS_C - uS_A)y - (ub^2 + vS_C)z = 0,$$
$$-(vc^2 + wS_A)x + (uc^2 + wS_B)y + (uS_A - vS_B)z = 0.$$

The barycentric, respectively absolute barycentric coordinates of the points Y and Z are

$$\begin{aligned} Y = (ub^2 + vS_C : 0 : wb^2 + vS_A) &= \left(\frac{ub^2 + vS_C}{\mu b^2}, 0, \frac{wb^2 + vS_A}{\mu b^2}\right) \\ &= \left(\frac{u}{\mu} + \frac{vS_C}{ub^2}, 0, \frac{w}{\mu} + \frac{vS_A}{\mu b^2}\right), \end{aligned}$$

$$\begin{aligned} Z = (uc^2 + wS_B : vc^2 + wS_A : 0) &= \left(\frac{uc^2 + wS_B}{\mu c^2}, \frac{vc^2 + wS_A}{\mu c^2}, 0\right) \\ &= \left(\frac{u}{\mu} + \frac{wS_B}{\mu c^2}, \frac{v}{\mu} + \frac{wS_A}{\mu c^2}, 0\right). \end{aligned}$$

Similarly we obtain the length of the segment MY and MZ:

$$MY = \frac{vS}{b\mu},\ MZ = \frac{wS}{c\mu}.$$

Let $X(\alpha), Y(\beta)$ and $Z(\gamma)$ points on the half-line MX, MY, MZ respectively, and $MX(\alpha) = \alpha$, $MY(\beta) = \beta$, $MZ(\gamma) = \gamma$.

3. The barycentric coordinates of the points $X(\alpha)$, $Y(\beta)$ and $Z(\gamma)$

We determine the absolute barycentric coordinates of the point $X(\alpha)$. Since $\dfrac{MX(\alpha)}{X(\alpha)X} = \dfrac{\alpha}{MX-\alpha}$, for this

$$\begin{aligned} X(\alpha) &= \frac{(MX-\alpha)M + \alpha X}{MX} \\ &= \frac{MX-\alpha}{MX}\left(\frac{u}{\mu}, \frac{v}{\mu}, \frac{w}{\mu}\right) + \frac{\alpha}{MX}\left(0, \frac{v}{\mu} + \frac{uS_C}{\mu a^2}, \frac{w}{\mu} + \frac{uS_B}{\mu a^2}\right) \\ &= \left(\frac{u}{\mu}, \frac{v}{\mu}, \frac{w}{\mu}\right) + \frac{\alpha}{MX}\left(-\frac{u}{\mu}, \frac{uS_C}{\mu a^2}, \frac{uS_B}{\mu a^2}\right) \\ &= \left(\frac{u}{\mu} - \frac{a\alpha}{S}, \frac{v}{\mu} + \frac{\alpha S_C}{aS}, \frac{w}{\mu} + \frac{\alpha S_B}{aS}\right). \end{aligned}$$

So the barycentric coordinates of the point $X(\alpha)$ are

$$X(\alpha) = (auS - \mu\alpha a^2 : avS + \mu\alpha S_C : awS + \mu\alpha S_B).$$

Similarly we obtain

$$\begin{aligned} Y(\beta) &= (buS + \mu\beta S_C : bvS - \mu\beta b^2 : bwS + \mu\beta S_A), \\ Z(\gamma) &= (cuS + \mu\gamma S_B : cvS + \mu\gamma S_A : cwS - \mu\gamma c^2). \end{aligned}$$

In general, the lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$ are not concurrent. What is the condition of concurrence of these lines?

4. The condition of concurrence of the lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$

Proposition 4.1. *The lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$ are concurrent if and only if*

$$\begin{aligned} &S\left[bc(vS_B - wS_C)u\alpha + ca(wS_C - uS_A)v\beta + ab(uS_A - vS_B)w\gamma\right] \\ &= \mu\left[aS_A(vS_B - wS_C)\beta\gamma + bS_B(wS_C - uS_A)\gamma\alpha + cS_C(uS_A - vS_B)\alpha\beta\right]. \end{aligned} \tag{1}$$

Proof. The equations of the lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$ are

$$\begin{aligned} (awS + \mu\alpha S_B)y - (avS + \mu\alpha S_C)z &= 0, \\ (bwS + \mu\beta S_A)x - (buS + \mu\beta S_C)z &= 0, \\ (cvS + \mu\gamma S_A)x - (cuS + \mu\gamma S_B)y &= 0. \end{aligned}$$

The concurrence of the lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$ is equivalent with the

$$\begin{aligned} &(avS + \mu\alpha S_C)(bwS + \mu\beta S_A)(cuS + \mu\gamma S_B) \\ &= (awS + \mu\alpha S_B)(buS + \mu\beta S_C)(cvS + \mu\gamma S_A), \end{aligned}$$

which is equivalent with the condition (1). □

We introduce the following notations:

$$\begin{aligned}E(u,v,w,\alpha,\beta,\gamma) &= bc(vS_B - wS_C)u\alpha + ca(wS_C - uS_A)v\beta \\ &\quad + ab(uS_A - vS_B)w\gamma,\\ F(u,v,w,\alpha,\beta,\gamma) &= aS_A(vS_B - wS_C)\beta\gamma + bS_B(wS_C - uS_A)\gamma\alpha \\ &\quad + cS_C(uS_A - vS_B)\alpha\beta.\end{aligned}$$

With these notations the condition (1) can write in the below form:

$$S \cdot E(u,v,w,\alpha,\beta,\gamma) - \mu \cdot F(u,v,w,\alpha,\beta,\gamma) = 0. \tag{2}$$

Proposition 4.2. *The lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$ are concurrent, $\forall\, t \in \mathbb{R}$, if*

$$\alpha = \frac{at}{u}, \quad \beta = \frac{bt}{v}, \quad \gamma = \frac{ct}{w}. \tag{3}$$

Proof. If $\alpha = \frac{at}{u}$, $\beta = \frac{bt}{v}$, $\gamma = \frac{ct}{w}$, we will demonstrate that the condition of concurrence (2) is come true. Indeed:

$$E\left(u,v,w,\frac{at}{u},\frac{bt}{v},\frac{ct}{w}\right) = abct(vS_B - wS_C + wS_C - uS_A + uS_A - vS_B) = 0,$$

$$F\left(u,v,w,\frac{at}{u},\frac{bt}{v},\frac{ct}{w}\right)$$

$$= \frac{abct^2}{uvw}\left[uS_A(vS_B - wS_C) + vS_B(wS_C - uS_A) + wS_C(uS_A - vS_B)\right] = 0.$$

□

Lemoine's theorem [3]. *Let ABC be a triangle, M a point in its plane, and X, Y, Z the projections of M on BC, CA, AB, respectively. If A', B', C' are points on the half-lines MX, MY, MZ, respectively, such that*

$$MX \cdot MA' = MY \cdot MB' = MZ \cdot MC' \tag{4}$$

then AA', BB', CC' are concurrent.

Let $A' = X\left(\frac{at}{u}\right)$, $B' = Y\left(\frac{bt}{v}\right)$, $C' = Z\left(\frac{ct}{w}\right)$. The conditions (3) and (4) are equivalent. Indeed:

$$\begin{aligned}&\alpha = \frac{at}{u}, \beta = \frac{bt}{v}, \gamma = \frac{ct}{w} \Leftrightarrow \frac{u\alpha}{a} = \frac{v\beta}{b} = \frac{w\gamma}{c} = t \Leftrightarrow \\ &\Leftrightarrow \frac{uS}{a\mu}\alpha = \frac{vS}{b\mu}\beta = \frac{wS}{c\mu}\gamma \Leftrightarrow MX \cdot MA' = MY \cdot MB' = MZ \cdot MC'.\end{aligned}$$

Consequently the Proposition 4.2 is equivalent with the Lemoine's theorem. The condition (3) is only sufficient for the concurrence of the lines $AX(\alpha)$, $BY(\beta)$, $CZ(\gamma)$, but it is not also necessary.

In this case we introduce the following notation:

$$K(M,t) = K\left(u,v,w,\frac{at}{u},\frac{bt}{v},\frac{ct}{w}\right) = AX\left(\frac{at}{u}\right) \cap BY\left(\frac{bt}{v}\right) \cap CZ\left(\frac{ct}{w}\right).$$

Remarks. 1) $K(M,0) = M$.
2) If the point M coincide with the incenter $I = (a : b : c)$ and $t = r$, then the point $K(I,r) = K(a,b,c,r,r,r) = AX(r) \cap BY(r) \cap CZ(r)$ is the Gergonne point of the triangle ABC.
3) If $M \neq H$, then $H = K(M,\infty) = AX(\infty) \cap BY(\infty) \cap CZ(\infty)$.

5. The locus of the points $K(M,t)$

Proposition 5.1. *If $M \neq H$, then the point $K(M,t)$ describe the circumconic K_M with equation*

$$u(vS_B - wS_C)yz + v(wS_C - uS_A)zx + w(uS_A - vS_B)xy = 0, \tag{5}$$

which is a hyperbola.

Proof. The equations of the lines $AX\left(\frac{at}{u}\right)$, $BY\left(\frac{bt}{v}\right)$, $CZ\left(\frac{ct}{w}\right)$ are

$$(uwS + \mu tS_B)y - (uvS + \mu tS_C)z = 0,$$
$$(vwS + \mu tS_A)x - (vuS + \mu tS_C)z = 0,$$
$$(wvS + \mu tS_A)x - (wuS + \mu tS_B)y = 0,$$

From this equations $t = \dfrac{u(vz - wy)S}{(yS_B - zS_C)\mu} = \dfrac{v(wx - uz)S}{(zS_C - xS_A)\mu} = \dfrac{w(uy - vx)S}{(xS_A - yS_B)\mu}$, from where arise the equation (5). The circumconic $pyz + qzx + rxy = 0$ is hyperbola if and only if $pS_A + qS_B + rS_C = 0$. In our case the condition is true. Indeed:

$$u(vS_B - wS_C)S_A + v(wS_C - uS_A)S_B + w(uS_A - vS_B)S_C = 0.$$

□

Remark. If $M = H$, then the locus of the points $K(M,t)$ is only the point H.

Proposition 5.2. *The circumconic K_M passes through the orthocenter H of the triangle ABC, thus is a rectangular hyperbola. The point M is on the hyperbola K_M, too.*

Proof. The barycentric coordinates of the points $H = \left(\frac{1}{S_A} : \frac{1}{S_B} : \frac{1}{S_C}\right)$ and $M = (u : v : w)$ satisfy the equation of K_M:

$$H \in K_M \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{S_A S_B S_C}\left[uS_A(vS_B - wS_C) + vS_B(wS_C - uS_A) + wS_C(uS_A - vS_B)\right] = 0,$$

$$M \in K_M \Leftrightarrow uvw(vS_B - wS_C + wS_C - uS_A + uS_A - vS_B) = 0.$$

□

Proposition 5.3. *The hyperbola K_M is the isogonal conjugate of the line OM', where O is the circumcenter of the triangle ABC and M' is the isogonal conjugate of the point M.*

Proof. The coordinates of the points O and M' are:

$$O = (a^2S_A : b^2S_B : c^2S_C), \quad M' = (a^2vw : b^2wu : c^2uv).$$

The equation of the line OM' is

$$b^2c^2u(vS_B - wS_C)x + c^2a^2v(wS_C - uS_A)y + a^2b^2w(uS_A - vS_B)z = 0.$$

The isogonal transform of this equation is

$$b^2c^2u(vS_B - wS_C)a^2yz + c^2a^2v(wS_C - uS_A)b^2zx + a^2b^2w(uS_A - vS_B)c^2xy = 0,$$

which is equivalent with the equation of the hyperbola K_M. □

Proposition 5.4. *The center C_M of hyperbola K_M has barycentric coordinates*

$$x = u(vS_B - wS_C)[w(u+v)b^2 - v(u+w)c^2],$$
$$y = v(wS_C - uS_A)[u(v+w)c^2 - w(v+u)a^2],$$
$$z = w(uS_A - vS_B)[v(w+u)a^2 - u(w+v)b^2],$$

and it is on the nine-point circle of the triangle ABC.

Proof. The coordinates of the center C_M are

$$x = p(-p+q+r), \quad y = q(p-q+r), \quad z = r(p+q-r),$$

where $p = u(vS_B - wS_C)$, $q = v(wS_C - uS_A)$, $r = w(uS_A - vS_B)$ and

$$\begin{aligned} -p+q+r &= -u(vS_B - wS_C) + v(wS_C - uS_A) + w(uS_A - vS_B) \\ &= w(u+v)b^2 - v(u+w)c^2. \end{aligned}$$

The equation of the nine-point circle is

$$S_Ax^2 + S_By^2 + S_Cz^2 - a^2yz - b^2zx - c^2xy = 0.$$

We transform the left member of this equation:

$$\begin{aligned} & S_Ax^2 + S_By^2 + S_Cz^2 - a^2yz - b^2zx - c^2xy \\ &= S_Ax^2 + S_By^2 + S_Cz^2 - (S_B+S_C)yz - (S_C+S_A)zx - (S_A+S_B)xy \\ &= -x(-x+y+z)S_A - y(x-y+z)S_B - z(x+y-z)S_C \\ &= -\frac{xyz}{pqr}(pS_A + qS_B + rS_C) = 0, \end{aligned}$$

since $x(-x+y+z) = \dfrac{xyz}{qr}$, $y(x-y+z) = \dfrac{xyz}{rp}$, $z(x+y-z) = \dfrac{xyz}{pq}$ and

$$pS_A + qS_B + rS_C = 0.$$

□

6. THE CONSTRUCTION OF THE POINTS $K(M,t)$

The question is how we can determine geometrical the points $X\left(\frac{at}{u}\right)$, $Y\left(\frac{bt}{v}\right)$, $Z\left(\frac{ct}{w}\right)$ which satisfy the conditions of concurrence (1)?

Lemma 6.1. *The antiparallel d_A of the side BC is perpendicular to the line OA, when O is the circumcenter of the triangle ABC.*

Proof. Let L, P and N the points of intersection of the antiparallel d_A with the lines AB, OA and BC (see Figure 1). Let furthermore D the diametrically opposite of the vertices A.

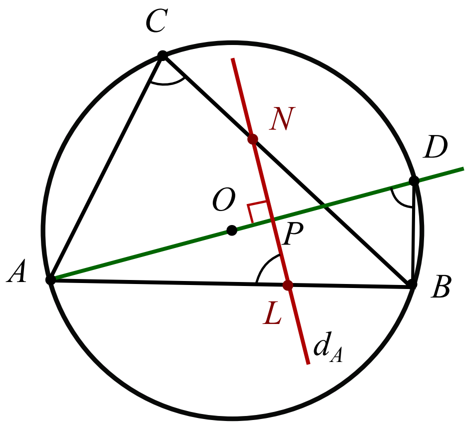

Figure 1

Since the triangles APL and ABD are right-angled triangles, we have:

$$\widehat{ALN} \equiv 90^\circ - \widehat{BAD} \equiv \widehat{ADB} \equiv \widehat{ACB}.$$

□

Since MX, MY, MZ are perpendicular to the sides BC, CA, AB, respectively, the circumcenters of the triangles MYZ, MZX, MXY are the midpoints of the segments MA, MB, MC. **The construction of the points** $K(M,t)$ **is following**: we put on the half-line MX a point arbitrary A' and we pull a perpendicular to the lines MC which intersect the line MY in the point B'; answerably to the Lemma 6.1, the line $A'B'$ is the antiparallel with the line XY, consequently $MX \cdot MA' = MY \cdot MB'$; from B' we pull a perpendicular to the lines MA which intersect the line MZ in the point C'; the line $B'C'$ is the antiparallel with the line YZ, consequently $MY \cdot MB' = MZ \cdot MC'$; the line $C'A'$ will be the antiparallel of the lines ZX; from the theorem of Lemoine result then the lines AA', BB', CC' are concurrent (see Figure 2).

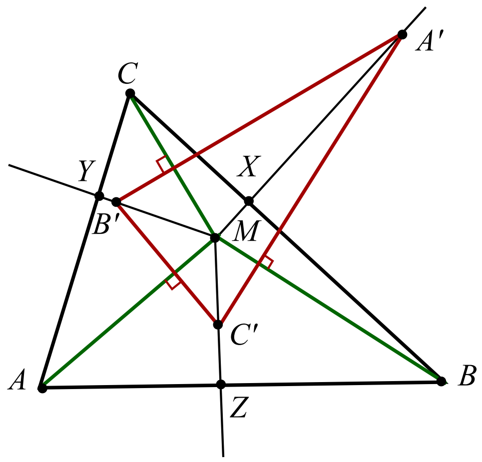

Figure 2

□

7. Special cases

7.1. To the points A, B, C no correspond never a one hyperbole (5).

7.2. If $u = 0$ and $vw \neq 0$, then $M \in BC$, $M \neq B$, $M \neq C$ and the locus of the points $K(M,t)$ is the line AH, the altitude, joined with the point M.

7.3. The isogonal conjugate of the line OA has equation $S_C zx - S_B xy = 0$, which is a degenerate hyperbola consisting of the side BC and the altitude AH. So, the Poncelet pencil has three degenerate rectangular hyperbolas.

8. Equations of some notable hyperbolas

8.1. If the point M coincide with the centroid $G = (1 : 1 : 1)$ of the triangle ABC, then $u = v = w = 1$. In this case $\alpha = at$, $\beta = bt$, $\gamma = ct$ and the point

$$K(G,t) = K(1,1,1,at,bt,ct) = AX(at) \cap BY(bt) \cap CZ(ct)$$

describe the *Kiepert hyperbola* with equation

$$(b^2 - c^2)yz + (c^2 - a^2)zx + (a^2 - b^2)xy = 0. \tag{6}$$

The Kiepert hyperbola is the isogonal transform of the line GL, where L is the symmedian point (Lemoine point) of the triangle ABC.

8.2. If the point M coincide with the circumcenter $O = (a^2 S_A : b^2 S_B : c^2 S_C)$ of the triangle ABC, then $u = a^2 S_A$, $v = b^2 S_B$, $w = c^2 S_C$. In this case $\alpha = \dfrac{t}{aS_A}$, $\beta = \dfrac{t}{bS_B}$

$\gamma = \dfrac{t}{cS_C}$ and the point

$$K(O,t) = K\left(a^2S_A \cdot b^2S_B \cdot c^2S_C \cdot \frac{t}{aS_A} \cdot \frac{t}{bS_B} \cdot \frac{t}{cS_C}\right)$$
$$= AX\left(\frac{t}{aS_A}\right) \cap BY\left(\frac{t}{bS_B}\right) \cap CZ\left(\frac{t}{cS_C}\right)$$

describe the *Jerabek hyperbola* wit equation

$$a^2S_A(b^2-c^2)yz + b^2S_B(c^2-a^2)zx + c^2S_C(a^2-b^2)xy = 0. \tag{7}$$

The Jerabek hyperbola is the isogonal transform of the Euler line OH.

8.3. If the point M coincide with the incenter $I = (a : b : c)$ of the triangle ABC, then $u = a, v = b, w = c$. In this case $\alpha = \beta = \gamma = t$ and the point

$$K(I,t) = K(a,b,c,t,t,t) = AX(t) \cap BY(t) \cap CZ(t)$$

describe the *Feuerbach hyperbola* with equation

$$a(bS_B - cS_C)yz + b(cS_C - aS_A)zx + c(aS_A - bS_B)xy = 0. \tag{8}$$

8.4. If the point M coincide with the A-excenter $I_a = (-a : b : c)$ of the triangle ABC, then $u = -a, v = b, w = c$. In this case $\alpha = -t, \beta = \gamma = t$ and the point

$$K(I_a,t) = K(-a,b,c,-t,t,t) = AX(-t) \cap BY(t) \cap CZ(t)$$

describe the *A−ex–Feuerbach hyperbola* with equation

$$a(bS_B - cS_C)yz - b(cS_C + aS_A)zx + c(aS_A + bS_B)xy = 0. \tag{9}$$

8.5. If the point M coincide with the B-excenter $I_b = (a : -b : c)$ of the triangle ABC, then $u = a, v = -b, w = c$. In this case $\alpha = t, \beta = -t, \gamma = t$ and the point

$$K(I_b,t) = K(a,-b,c,t,-t,t) = AX(t) \cap BY(-t) \cap CZ(t)$$

describe the *B-ex-Feuerbach hyperbola* with equation

$$a(bS_B + cS_C)yz + b(cS_C - aS_A)zx - c(aS_A + bS_B)xy = 0. \tag{10}$$

8.6. If the point M coincide with the C-excenter $I_C = (a : b : -c)$ of the triangle ABC, then $u = a, v = b, w = -c$. In this case $\alpha = t, \beta = t, \gamma = -t$ and the point

$$K(I_c,t) = K(a,b,-c,t,t,-t) = AX(t) \cap BY(t) \cap CZ(-t)$$

describe the *C-ex-Feuerbach hyperbola* with equation

$$-a(bS_B + cS_C)yz + b(cS_C + aS_A)zx + c(aS_A - bS_B)xy = 0. \tag{11}$$

8.7. The isogonal transform of the tangent line to Jerabek hyperbola at O is the *Huygens'hyperbola*. The equation of the tangent is

$$b^2c^2S_BS_C(b^2-c^2)x + c^2a^2S_CS_A(c^2-a^2)y + a^2b^2S_AS_B(a^2-b^2)z = 0.$$

So the equation of the Huygens'hyperbola is

$$S_BS_C(b^2-c^2)yz + S_CS_A(c^2-a^2)zx + S_AS_B(a^2-b^2)xy = 0. \tag{12}$$

9. Further research

Further research may tackle the determination of the axes, foci, vertex and asymptotes of the hyperbolas K_M.

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