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A PROOF OF DAO'S THEOREM

GIANG NGOC NGUYEN

ABSTRACT. We present a proof of Dao's generalization of Goormaghtigh theorem and Zaslavsky theorem and Carnot theorem.

1. INTRODUCTION

The Goormaghtigh's theorem and Zaslavsky's theorem are two nice theorems of Euclidean geometry, these theorems as follows:

Theorem 1.1 (Goormaghtigh [1]). *Let ABC be a triangle and point P distinct from A, B, C. Let a line Δ passes through P. A_1, B_1, C_1 belong to BC, CA, AB respectively such that PA_1, PB_1, PC_1 are the images of PA, PB, PC respetively by reflection R_Δ. Then, A_1, B_1, C_1 are collinear.*

Notation R_Δ refers to reflection against Δ.

Theorem 1.2 (Zaslavsky [2]). *Let $A'B'C'$ be the reflection of a triangle ABC through a given point P, and let three parallel lines through A', B', C' intersect BC, CA, AB at X, Y, Z respectively. Then the points X, Y, Z are collinear.*

A proof of the Zaslavsky due to Darij Grinberg, see [3].
In 2014, O.T.Dao expanded the Goormaghtigh theorem as follows:

Theorem 1.3 (Dao [4]). *Let ABC be a triangle and point P distinct from A, B, C. Lines L and L_0 cut at P. Points A_1, B_1, C_1 belong to BC, CA, AB respectively such that $(PA, PA_1, L, L_0) = (PB, PB_1, L, L_0) = (PC, PC_1, L, L_0) = -1$. Then three points A_1, B_1, C_1 are collinear.*

A proof of the Dao theorem due to Tran Hoang Son, see [5]. Continuing O.T.Dao expanded the theorem 1.2 and 1.3 as follows:

Theorem 1.4 (Dao [6]). *Let a conic (S) and a point P on the plane. Construct three lines d_a, d_b, d_c through P such that they meet the conic at A, A' ; B, B' ; C, C' respectively. Let D be a point on the polar of point P with respect to (S) or P lies on the conic (S). Let $DA' \cap BC = A_0$; $DB' \cap AC = B_0$; $DC' \cap AB = C_0$. Then A_0, B_0, C_0 are collinear.*

- When point P at infinity the theorem 1.4 is the theorem 1.3
- When the conic is an ellipse, and the polar line of P is the major axis (or the minor axis) of the ellipse, the theorem 1.4 is the Goormaghtigh theorem.
- When point P is the center of the conic theorem 1.4 is the Zaslavsky theorem.
- When D be a point on the conic, and conic is a circle and P be a point at infinity the theorem 1.4 is the Carnot theorem, you can see the Carnot theorem in [7]. Note that the Carnot theorem is a generalization of the famous Simson line theorem.

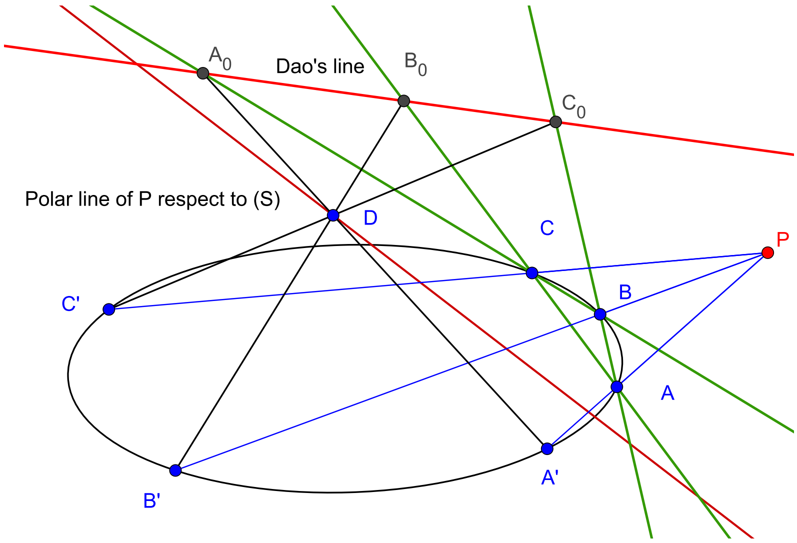

FIGURE 1. A_0, B_0, C_0 lies on the Dao's line

2. PROOF OF THEOREM 1.4

If P lies on the conic, you can see a proof by O.T.Dao in [8]. This paper the author gives a proof of case P lies on polar line of P respect to the conic (S).
Consider the projective target $\{A, B, C; P\}$.
We have $A = (1,\ 0,\ 0)\ ;\ B = (0,\ 1,\ 0)\ ;\ C = (0,\ 0,\ 1)$.
Since $A,\ B,\ C\ \in\ (S)$, the equation of the conic (S) is of the form :

$$ax_2x_3\ +\ bx_3x_1\ +\ cx_1x_2\ =\ 0.$$

The coordinates of the equation of the line PA are of the form:

$$\left[\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\right]$$

Thus, $(PA)\ :\ x_2\ =\ x_3$.
Since $A'\ =\ PA\ \cap\ (S)$, the coordinates of the point A' satisfy the system of equations:

$$\begin{cases} ax_2x_3\ +\ bx_3x_1\ +\ cx_1x_2\ =\ 0 \\ x_2 = x_3 \end{cases}$$

Thus, $A'\ =\ (-a\ ;\ b+c\ ;\ b+c)$.
Similarly, $B' = (a+c,\ -b,\ a+c)\ ;\ C' = (a+b,\ a+b,\ -c)$.
The coordinates of the equation of the line BC are of the form:

$$\left[\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}\right]\ =\ [1,\ 0,\ 0].$$

Thus, $BC\ :\ x_1\ =\ 0$.
Similarly, $CA\ :\ x_2\ =\ 0$. and $AB\ :\ x_3\ =\ 0$.
The equation of the polar d of the point P to the conic (S) is:

$$[1,\ 1,\ 1]\left[\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Leftrightarrow (b+c)x_1 + (c+a)x_2 + (a+b)x_3 = 0$$

Since D is on the line d, so the coordinates of the point $D = (m,\ n,\ p)$ satisfy the equation:

$$(b + c)m + (c+a)n + (a+b)p = 0 \Rightarrow p = -\frac{(b+c)m + (a+c)n}{a+b}.$$

Thus, the coordinates of the point D are of the form:

$$D = (m,\ n,\ -\frac{(b+c)m + (a+c)n}{a+b}) = ((a+b)m,\ (a+b)n,\ -(b+c)m - (c+a)n)$$

If the coordinates of the equation of the line DA' are $[x_1,\ x_2,\ x_3]$ then

$$x_1 = \begin{vmatrix} n(a+b) & -(b+c)m - (c+a)n \\ b+c & b+c \end{vmatrix} = (b+c)(n(a+b)+m(b+c)+(c+a)n)\ ;$$

$$x_2 = \begin{vmatrix} -(b+c)m - (c+a)n & m(a+b) \\ b+c & -a \end{vmatrix} = a\left((b+c)m + (c+a)n\right) - (b+c)m(a+b)$$

$$x_3 = \begin{vmatrix} m(a+b) & n(a+b) \\ -a & b+c \end{vmatrix} = (b+c)m(a+b) + a(a+b)n.$$

Since $A_0 = DA' \cap BC$, the coordinates of the point A_0 satisfy the system of equations:

$$\begin{cases} x_1 = 0 \\ ((b+c)(n(a+b)+m(b+c)+(c+a)n))\, x_1 + \left(a\left((b+c)m+(c+a)n\right) - (b+c)m(a+b)\right) x_2 \\ +\ ((b+c)m(a+b) + a(a+b)n)\, x_3 = 0 \end{cases}$$

Thus, $A_0 = (0,\ (b+c)m(a+b) + a(a+b)n,\ (b+c)m(a+b) - a\left((b+c)m + (c+a)n\right))$

If the coordinates of the equation of the line DB' are $[x_1,\ x_2,\ x_3]$ then

$$x_1 = \begin{vmatrix} n(a+b) & -(c+b)m - (c+a)n \\ -b & a+c \end{vmatrix} = (a+c)n(a+b) - b\left((c+b)m + (c+a)n\right)$$

$$x_2 = \begin{vmatrix} -(b+c)m - (c+a)n & m(a+b) \\ c+a & c+a \end{vmatrix} = -(c+a)\left((b+c)m + (c+a)n + m(a+b)\right)$$

$$x_3 = \begin{vmatrix} m(a+b) & n(a+b) \\ c+a & -b \end{vmatrix} = -mb(a+b) - (c+a)n(a+b).$$

Since $B_0 = DB' \cap CA$, the coordinates of the point B_0 satisfy the system of equations:

$$\begin{cases} x_2 = 0 \\ ((a+c)n(a+b) - b\left((c+b)m + (c+a)n\right))\, x_1 + \left(-(c+a)\left((b+c)m + (c+a)n + m(a+b)\right)\right) x_2 \\ +\ (-mb(a+b) - (c+a)n(a+b))\, x_3 = 0 \end{cases}$$

Thus, $B_0 = (mb(a+b) + (c+a)n(a+b), 0, (a+c)n(a+b) - b\left((c+b)m + (c+a)n\right))$

If the coordinates of the equation of the line DC' are $[x_1,\ x_2,\ x_3]$ then

$$x_1 = \begin{vmatrix} n(a+b) & -(b+c)m - (c+a)n \\ a+b & -c \end{vmatrix} = -cn(a+b) + (a+b)\left((b+c)m + (c+a)n\right)\ ;$$

$x_2 = \begin{vmatrix} -(b+c)m - (c+a)n & m(a+b) \\ -c & a+b \end{vmatrix} = -(a+b)\left((b+c)m + (c+a)n\right) + cm(a+b)$

$x_3 = \begin{vmatrix} m(a+b) & n(a+b) \\ a+b & a+b \end{vmatrix} = (a+b)^2(m - n)$.

Since $C_0 = DC' \cap AB$, the coordinates of the point C_0 satisfy the system of equations:

$$\begin{cases} x_3 = 0 \\ (-cn(a+b) + (a+b)((b+c)m + (c+a)n))\,x_1 + (-(a+b)((b+c)m + (c+a)n) + cm(a+b))\,x_2 \\ + \left((a+b)^2 (m - n)\right) x_3 = 0 \end{cases}$$

Thus, $C_0 = ((a+b)((b+c)m + (c+a)n) - cm(a+b),\ -cn(a+b) + (a+b)((b+c)m + (c+a)n),\ 0)$

Consider the determinant

$$\Delta = \begin{vmatrix} 0 & (b+c)m(a+b) + a(a+b)n & (b+c)m(a+b) - a((b+c)m + (c+a)n) \\ mb(a+b) + (c+a)n(a+b) & 0 & (a+c)n(a+b) - b((c+b)m + (c+a)n) \\ (a+b)((b+c)m + (c+a)n) - cm(a+b) & -cn(a+b) + (a+b)((b+c)m + (c+a)n) & 0 \end{vmatrix}$$

We need to prove: $\Delta = 0$

$\Delta = -[(mb + cn + an).(a+b)].(a+b).$
$[cn - (bm + cm + cn + an)].\left[(bm+cm)(a+b) - (amb + amc + anc + a^2n)\right]$
$+ (a+b)[(b+c)m + (c+a)n - cm].[a+b].[bm+cm+an].\left[(an+cn)(a+b) - \left(bmc + b^2m + bnc + bam\right)\right]$

We need to prove

$(mb + cn + an).(bm + cm + an).\left(bma + b^2m + cma + cmb - amb - amc - anc - a^2n\right) +$
$[bm + cm + cn + an - cm].[bm + cm + an].\left[a^2n + abn + can + cbn - bmc - b^2m - bnc - ban\right] = 0$

It is equivalent to

$$b^2m + bmc - anc - a^2n + a^2n + can - bmc - b^2m = 0$$

This is obviously. Thus $\Delta = 0$, therefore $A_0,\ B_0,\ C_0$ are collinear.

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DOCTORÂS DEGREE STUDENT, OF THE VIETNAM INSTITUTE OF EDUCATIONAL SCIENCES, 101 TRAN HUNG DAO, HANOI, VIET NAM.
E-mail address: nguyenngocgiang.net@gmail.com