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CHEN'S INEQUALITY FOR INVARIANT SUBMANIFOLDS IN A GENERALIZED (κ, μ)-SPACE FORMS

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ABSTRACT. In this paper, we obtain a basic Chen's inequality for an invariant submanifold in a generalized (κ, μ)-contact space forms involving intrinsic invariants. Inequalities between the squared mean curvature and Ricci curvature and between the squared mean curvature and Ricci curvature are also obtained.

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*Keywords:*generalized Sasakian-space-form, invariant submanifold, k-Ricci curvature, (κ, μ)-space forms, generalized (κ, μ)-contact space forms, Chen's inequality.

1. INTRODUCTION

One of the most fundamental problems in the theory of submanifolds is the immersibility of a Riemannian manifold in a Euclidean space (or, more generally, in a space form). According to the well-known theorem of J. Nash in 1956 [13], every Riemannian manifold can be isometrically embedded in some Euclidean space $\mathbb{E}^{n(n+1)(3n+11)/2}$, and this Nash's theorem enables us to consider any Riemannian manifold as a submanifold of Euclidean space; and this provides a natural motivation for the study of submanifolds of Riemannian manifolds. To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the basic interests in the submanifold theory.

In [8] B.-Y. Chen defined a Riemannian invariant $\delta_M = \tau - \inf K$ for any Riemannian manifold M, where τ is the scaler curvature of M and $(\inf K)(p) = \inf\{K(\pi)|$ plane sections $\pi \subset T_pM\}$. Also, in [8] Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature and squared mean curvature. This inequalities are also sharp, and many nice classes of submanifolds realize equality in inequalities. In [9], he gave a sharp relationship between the squared mean curvature and the Ricci curvature for the submanifolds in a real space form. This inequalities are also sharp, and many nice classes of submanifolds realize equality in inequalities. In [3], the authors studied a spacial class of contact manifolds as (κ, μ)-spaces for which the characteristic vector field belongs to the (κ, μ)-nullity distribution and in [7] A. Carriazo, V. Martín Molina and M. M. Tripathi introduce generalized (κ, μ)-space forms as an almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, \langle , \rangle)$ whose curvature tensor can be written as $R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6$, where $f_1, f_2, f_3, f_4, f_5, f_6$ are differentiable functions

on $\tilde{M}$, and $R_1, R_2, R_3, R_4, R_5, R_6$ are the tensors defined by (13). In [6] A. Carriazo and V. Martín-Molina defined generalized (κ, μ)-space forms with divided the tensor field R_5 into two parts $R_{5,1}(X,Y)Z = \langle hY, Z\rangle hX - \langle hX, Z\rangle hY$, $R_{5,2}(X,Y)Z = \langle \phi hY, Z\rangle \phi hX - \langle \phi hX, Z\rangle \phi hY$ for vector fields X, Y, Z on $\tilde{M}$.
In recent years, a lot of articles studied Chen invariants and inequalities, warped product spaces[11, 12] and C-totally real submanifolds in a (κ, μ)-contact space form[10, 15]. Other interesting types of submanifolds can be find in [4].
M. M. Tripathi [14] studied the relationship between the scaler curvature, the sectional curvature and the squared mean curvature for invariant submanifolds in a non-Sasakian (κ, μ)-space.
In this paper, we improved Theorems in [14] for invariant submanifold of generalized (κ, μ)-space forms with divided R_5. The paper is organized as follows. In section 2, we recall some necessary details background on Riemannian manifolds, contact metric manifold, invariant submanifolds, contact metric manifolds and generalized (κ, μ)-contact space forms. In section 3, we establish a basic Chen's inequality for invariant submanifolds in a generalized (κ, μ)-space form with divided R_5. Finely, in sections 4 contain an inequality between the squared mean curvature, k-Ricci curvature and Ricci curvature.

2. Preliminaries

Let M be an n-dimensional Riemannian manifold. We denote by $K(\pi)$ the sectional curvature of M for a plane section π in T_pM. For any orthonormal basis $\{e_1, \ldots, e_n\}$ for T_pM, The scalar curvature $\tau(p)$ of M at p is defined by $\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j)$, where $K(e_i \wedge e_j)$ is the sectional curvature of the plane section spanned by e_i and e_j at $p \in M$.
Let $\mathbf{P}_k$ be a k-plane section of T_pM and $\{e_1, \ldots, e_k\}$ any orthonormal basis of $\mathbf{P}_k$. The scalar curvature $\tau(\mathbf{P}_k)$ of $\mathbf{P}_k$ is given by

$$\tau(\mathbf{P}_k) = \sum_{1\le i<j\le k} K(e_i \wedge e_j). \tag{1}$$

The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau(p) = \tau(T_pM)$. Also, we denote by $(\inf K)(p) = \inf\{K(\pi) | \pi \subset T_pM, \dim \pi = 2\}$, and introduce the first Chen invariant $\delta_M(p) = \tau(p) - (\inf K)(p)$, which is certainly an intrinsic character of M.
Suppose $\mathbf{P}$ is a k-plane section of T_pM and U a unit vector in $\mathbf{P}$. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of $\mathbf{P}$ such that $e_1 = U$. The Ricci curvature $\mathrm{Ric}_{\mathbf{P}}$ of $\mathbf{P}$ at U is given by

$$\operatorname*{Ric}_{\mathbf{P}}(U) = K_{12} + \cdots + K_{1k}, \tag{2}$$

where K_{ij} is the sectional curvature of the plane section spanned by e_i and e_j. The $\mathrm{Ric}_{\mathbf{P}}(U)$ is called a k-Ricci curvature. For each integer K, $2 \le k \le n$, the Riemannian invariant θ_k on n-dimensional Riemannian Manifold M is defined by

$$\theta_k(p) = \left(\frac{1}{k-1}\right) \inf_{\mathbf{P},X} \operatorname*{Ric}_{\mathbf{P}}(X), p \in M, \tag{3}$$

where $\mathbf{P}$ is k-plane sections in T_pM and X is unit vector in $\mathbf{P}$ [9].
Let M be an n-dimensional submanifold in a manifold $\tilde{M}$equipped with a Riemannian metric $\langle , \rangle$. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y =$

$\nabla_X Y + \sigma(X,Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$ for all$X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and σ is the second fundamental form related to the shape operator A_N in the direction of N by $\langle \sigma(X,Y), N\rangle = \langle A_N X, Y\rangle$. Then, the Gauss equation is given by

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) &- \langle \sigma(X,W), \sigma(Y,Z)\rangle \\ &+ \langle \sigma(X,Z), \sigma(Y,W)\rangle\end{aligned} \tag{4}$$

for all $X, Y, Z, W \in TM$, where $\tilde{R}$ and R are the curvature tensors of $\tilde{M}$ and M respectively. The mean curvature vector H is expressed by $nH = \text{trace}(\sigma)$. The submanifold M is totally geodesic in $\tilde{M}$ if $\sigma = 0$, and minimal if $H = 0$. If $\sigma(X,Y) = \langle X, Y\rangle H$, for all $X, Y \in TM$, then M is totally umbilical.
A $(2m+1)$-dimensional differentiable manifold $\tilde{M}$ is called an almost contact metric manifold if there is an almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a compatible Riemannian metric $\langle, \rangle$ satisfying

$$\begin{aligned}&\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \phi\xi = 0, \eta o \phi = 0, \\ &\langle X, Y\rangle = \langle \phi X, \phi Y\rangle + \eta(X)\eta(Y), \\ &\langle X, \phi Y\rangle = -\langle \phi X, Y\rangle, \langle X, \xi\rangle = \eta(X),\end{aligned} \tag{5}$$

for all $X, Y \in \Gamma(T\tilde{M})$. An almost contact metric structure becomes a contact metric structure if $d\eta = \Phi$, where $\Phi(X,Y) = \langle X, \phi Y\rangle$ is the fundamental 2-form of $\tilde{M}$.
An almost contact metric structure of $\tilde{M}$ is said to be normal if the Nijenhuis torsion $[\phi, \phi]$ of ϕ equals $-2d\eta \otimes \xi$. A normal contact metric manifold is called a Sasakian manifold. It can be proved that an almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = \langle X, Y\rangle \xi - \eta(Y)X, \tag{6}$$

for any $X, Y \in \Gamma(T\tilde{M})$ or equivalently, a contact metric structure is a Sasakian structure if and only if $\tilde{R}$ satisfies

$$\tilde{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{7}$$

for $X, Y \in \Gamma(T\tilde{M})$. In a contact metric manifold $\tilde{M}$, the $(1,1)$-tensor field h is defined by $2h = \mathcal{L}_\xi \phi$, which is the Lie derivative of ϕ in the characteristic direction ϕ. It is symmetric and satisfies

$$h\xi = 0, \quad h\phi + \phi h = 0, \tag{8}$$

$$\tilde{\nabla}\xi = -\phi - \phi h, \quad \text{trace}(h) = \text{trace}(\phi h) = 0, \tag{9}$$

where $\tilde{\nabla}$ is Levi-Civita connection.
Given an almost contact metric manifold (ϕ, ξ, η, g), a ϕ-section of M at $p \in M$ is a section $\mathbf{P} \subset T_p\tilde{M}$ spanned by a unit vector X_p orthogonal to ξ_p, and ϕX_p. The ϕ-sectional curvature of $\mathbf{P}$ is defined by $\tilde{K}(X, \phi X) = \tilde{R}(X, \phi X, \phi X, X)$. A Sasakian manifold with constant ϕ-sectional curvature c is called a Sasakian space form and is denoted by $\tilde{M}(c)$.
A contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is said to be a (κ, μ)-contact manifold if its curvature tensor satisfies the condition

$$\tilde{R}(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{10}$$

where κ and μ are real constant numbers. If the (κ,μ)-contact metric manifold $\tilde{M}$ has constant ϕ-sectional curvature c, then it is said to be a (κ,μ)-contact space form.
A submanifold M an almost contact metric manifold $\tilde{M}$ with the structure $(\phi,\xi,\eta,\langle,\rangle)$ is called an invariant submanifold if $\phi T_pM \subset T_pM$. If $\tilde{M}$ is contact also, then $\xi \in \Gamma(TM)$, $\sigma(X,\xi)=0$ and M is minimal [2]. On the other hand, we have the following

Proposition 2.1 ([14]). *Every totally umbilical submanifold M of a contact metric manifold such that $\xi \in TM$ is minimal and consequently totally geodesic.*

Proposition 2.2 ([14]). *In an n-dimensional invariant submanifold of a contact metric manifold, we have*

$$\|P\|^2 = n-1, \quad \mathrm{trace}(h^T) = \mathrm{trace}((\phi h)^T) = 0, \quad \|(\phi h)^T\|^2 = \|h^T\|^2. \tag{11}$$

Definition 2.3. ([7]) We say that an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,\langle,\rangle)$ is a generalized (κ,μ)-space form if there exist functions f_1,f_2,f_3,f_4,f_5,f_6 defined on M such that

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{12}$$

where R_1,R_2,R_3,R_4,R_5,R_6 are the following tensors

$$\begin{aligned}
R_1(X,Y)Z &= \langle Y,Z\rangle X - \langle X,Z\rangle Y,\\
R_2(X,Y)Z &= \langle X,\phi Z\rangle \phi Y - \langle Y,\phi Z\rangle \phi X + 2\langle X,\phi Y\rangle \phi Z,\\
R_3(X,Y)Z &= \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + \langle X,Z\rangle \eta(Y)\xi - \langle Y,Z\rangle \eta(X)\xi,\\
R_4(X,Y)Z &= \langle Y,Z\rangle hX - \langle X,Z\rangle hY + \langle hY,Z\rangle X - \langle hX,Z\rangle Y,\\
R_5(X,Y)Z &= \langle hY,Z\rangle hX - \langle hX,Z\rangle hY + \langle \phi hX,Z\rangle \phi hY - \langle \phi hY,Z\rangle \phi hX,\\
R_6(X,Y)Z &= \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + \langle hX,Z\rangle \eta(Y)\xi - \langle hY,Z\rangle \eta(X)\xi,
\end{aligned} \tag{13}$$

for all vector fields X,Y,Z on $\tilde{M}$, where $2h = \mathcal{L}_\xi\phi$ and $\mathcal{L}$ is the usual Lie derivative. We will denote such a manifold by $\tilde{M}(f_1,\ldots,f_6)$. (κ,μ)-space forms are examples of generalized (κ,μ)-space forms, with constant functions

$$f_1 = \frac{c+3}{4}, f_2 = \frac{c-1}{4}, f_3 = \frac{c+3}{4} - \kappa, f_4 = 1, f_5 = \frac{1}{2}, f_6 = 1-\mu.$$

Generalized Sasakian space forms $\tilde{M}(f_1,f_2,f_3)$ introduced in [1] are generalized (κ,μ)-space forms, with $f_4 = f_5 = f_6 = 0$.

Definition 2.4. ([6]) We say that an almost contact metric manifold $(\tilde{M},\phi,\xi,\eta,\langle,\rangle)$ is a generalized (κ,μ)-space form with divided R_5 if there exist function $f_1,f_2,f_3,f_4,f_{5,1}$, $f_{5,2},f_6$ defined on M such that

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_{5,1}R_{5,1} + f_{5,2}R_{5,2} + f_6R_6, \tag{14}$$

where $R_{5,1},R_{5,2}$ are the following tensors

$$\begin{aligned}
R_{5,1}(X,Y)Z &= \langle hY,Z\rangle hX - \langle hX,Z\rangle hY,\\
R_{5,2}(X,Y)Z &= \langle \phi hY,Z\rangle \phi hX - \langle \phi hX,Z\rangle \phi hY,
\end{aligned}$$

for all vector fields X, Y, Z on $\tilde{M}$. We will denote such a manifold by $\tilde{M}(f_1, f_2, f_3, f_4, f_{5,1}, f_{5,2}, f_6)$. It follows that $R_5 = R_{5,1} - R_{5,2}$. It is obvious that, if $\tilde{M}(f_1, \ldots, f_6)$ is a generalized (κ, μ)-space form then $\tilde{M}$ is a generalized (κ, μ)-space form with divided R_5 with $f_{5,1} = f_5$ and $f_{5,2} = -f_5$. A non-Sasakian (κ, μ)-space form is the generalized (κ, μ)-space form with divided R_5 with

$$f_1 = \frac{2-\mu}{2}, f_2 = -\frac{\mu}{2}, f_3 = \frac{2-\mu-2\kappa}{2}, f_4 = 1, f_{5,1} = \frac{2-\mu}{2(1-\kappa)}, f_{5,2} = \frac{2\kappa-\mu}{2(1-\kappa)}$$

and $f_6 = 1 - \mu$ but not the generalized (κ, μ)-space form.
We recall the following Some Theorems and Lemmas for later use.

Theorem 2.5. *([7]) If $M(f_1, \ldots, f_6)$ is a contact metric generalized (κ, μ)-space form, then it is a generalized (κ, μ)-space, with $\kappa = f_1 - f_3$ and $\mu = f_4 - f_6$.*

Lemma 2.6. *([8]) If $a_1, \ldots, a_n, a_{n+1}$ are $n + 1 (n > 1)$ real numbers such that*

$$\frac{1}{n-1}(\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + a_{n+1},$$

then $2a_1a_2 \geq a_{n+1}$, with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$.

Lemma 2.7. *([14]) If $a_1, \ldots, a_n$ are $n(n > 1)$ real numbers, then*

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \sum_{i=1}^{n} a_i^2,$$

with equality holding if and only if $a_1 = \ldots = a_n$.

3. Certain basic inequalities

For a vector field X on a submanifold M of an almost contact metric manifold $\tilde{M}$, let PX be the tangential part of ϕX. Thus, P is an endomorphism of the tangent bundle of M and satisfies $\langle X, PY \rangle = -\langle PX, Y \rangle$ for $X, Y \in \Gamma(TM)$. Let $\pi \subset T_pM$ be a plane section spanned by an orthonormal basis $\{e_1, e_2\}$. We define $\alpha(\pi)$ and $\beta(\pi)$ given by $\alpha(\pi) = \langle e_1, Pe_2 \rangle^2$ and $\beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2$, are a real number in the closed unit interval $[0, 1]$, which are independent of the choice of the orthonormal basis $\{e_1, e_2\}$. Let $\xi \in \Gamma(TM)$ and put

$$\gamma(\pi) = \eta(e_1)^2 \langle h^T e_2, e_2 \rangle + \eta(e_2)^2 \langle h^T e_1, e_1 \rangle - 2\eta(e_1)\eta(e_2) \langle h^T e_1, e_2 \rangle.$$

Then, $\gamma(\pi)$ is also real numbers and do not depend on the choice of the orthonormal basis $\{e_1, e_2\}$ [16].
In view of (14) and (4) we state the following Lemma.

Lemma 3.1. *In an n-dimensional submanifold M in a $(2m + 1)$-dimensional generalized (κ, μ)-space form with divided R_5 $\tilde{M}(f_1, \ldots, f_6)$ such that $\xi \in \Gamma(TM)$, the scalar curvature and the squared mean curvature satisfy*

$$\begin{aligned} 2\tau = {} & n(n-1)f_1 + 3f_2\|P\|^2 - 2(n-1)f_3 \\ & + f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} - f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \\ & + 2((n-1)f_4 - f_6)\,\mathrm{trace}(h^T) + n^2\|H\|^2 - \|\sigma\|^2, \end{aligned} \tag{15}$$

where

$$\|Q\|^2 = \sum_{i,j=1}^{n} \langle e_i, Qe_j\rangle^2, \quad Q \in \{P, (\phi h)^T, h^T\}, \quad \|\sigma\|^2 = \sum_{i,j=1}^{n} \langle \sigma(e_i,e_j), \sigma(e_i,e_j)\rangle,$$

and $(\phi h)^T X$ *and* $h^T X$ *are the tangential parts of* ϕhX *and* hX *respectively for* $X \in \Gamma(TM)$.

Proof. We choose a local orthonormal frame $\{e_1,\dots,e_n\}$ such that $e_1,\dots,e_n$ are tangent to M, e_{n+1} is parallel to the mean curvature vector H. Then from equation of Gauss we have

$$K(e_i \wedge e_j) = \tilde{K}(e_i \wedge e_j) + \sum_{r=n+1}^{2m+1} (\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2). \tag{16}$$

From equation (16) we get

$$2\tau = 2\tilde{\tau}(T_pM) + n^2\|H\|^2 - \|\sigma\|^2. \tag{17}$$

By using (1) and (14), we obtain

$$\begin{aligned}
2\tilde{\tau}(T_pM) = & \sum_{1\leqslant i\neq j\leqslant n} \tilde{R}(e_i,e_j,e_j,e_i) \\
=& n(n-1)f_1 + 3f_2 \sum_{1\leqslant i\neq j\leqslant n} \langle e_i, Pe_j\rangle^2 \\
& - 2(n-1)f_3 + 2(n-1)f_4 \operatorname{trace}(h^T) \\
& + f_{5,1}\Big\{(\operatorname{trace}(h^T))^2 - \sum_{1\leqslant i\neq j\leqslant n} \langle e_i, h^Te_j\rangle^2\Big\} \\
& - f_{5,2}\Big\{\sum_{1\leqslant i\neq j\leqslant n} \langle e_i, (\phi h)^T e_j\rangle^2 - (\operatorname{trace}(\phi h)^T)^2\Big\} - 2f_6 \operatorname{trace}(h^T) \\
& = n(n-1)f_1 + 3f_2\|P\|^2 - 2(n-1)f_3 + 2(n-1)f_4 \operatorname{trace}(h^T) \\
& + f_{5,1}\left\{(\operatorname{trace}(h^T))^2 - \|h^T\|^2\right\} - f_{5,2}\left\{\|(\phi h)^T\|^2 - (\operatorname{trace}(\phi h)^T)^2\right\} \\
& - 2f_6 \operatorname{trace}(h^T).
\end{aligned} \tag{18}$$

Now if we put (18) in (17), obtain (15). □

From Lemma 3.1 we directly obtain:

Proposition 3.2. *In an n-dimensional submanifold M in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-space form with divided* R_5 $\tilde{M}(f_1,\dots,f_6)$ *such that* $\xi \in \Gamma(TM)$*, the scalar curvature and the squared mean curvature satisfy*

$$\begin{aligned}
2\tau \leq\ & n^2\|H\|^2 + n(n-1)f_1 + 3f_2\|P\|^2 - 2(n-1)f_3 \\
& + f_{5,1}\left\{(\operatorname{trace}(h^T))^2 - \|h^T\|^2\right\} - f_{5,2}\left\{\|(\phi h)^T\|^2 - (\operatorname{trace}(\phi h)^T)^2\right\} \\
& + 2((n-1)f_4 - f_6)\operatorname{trace}(h^T).
\end{aligned} \tag{19}$$

The equality in (19) *holds if and only if M is totally geodesic.*

Proposition 3.3. *In an n-dimensional invariant submanifold M in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-space form* $\tilde{M}(f_1,\dots,f_6)$ *such that* $\xi\in\Gamma(TM)$*, the scalar curvature and the squared mean curvature satisfy*

$$2\tau=(n-1)(nf_1+3f_2-2f_3)+n^2\|H\|^2-\|\sigma\|^2, \tag{20}$$

Corollary 3.4. *In an n-dimensional invariant submanifold M in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-space form with divided* R_5 $\tilde{M}(f_1,\dots,f_6)$ *such that* $\xi\in\Gamma(TM)$*, the scalar curvature and the squared mean curvature satisfy*

$$2\tau\le(n-1)(nf_1+3f_2-2f_3)-(f_{5,1}+f_{5,2})\|h^T\|^2. \tag{21}$$

The equality in (21) *holds if and only if M is totally geodesic.*

Corollary 3.5. *In an n-dimensional invariant submanifold M in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-space form* $\tilde{M}(f_1,\dots,f_6)$ *such that* $\xi\in\Gamma(TM)$*, the scalar curvature and the squared mean curvature satisfy*

$$2\tau\le(n-1)(nf_1+3f_2-2f_3). \tag{22}$$

The equality in (22) *holds if and only if M is totally geodesic.*

Proposition 3.6. *Let M be an n-dimensional* $(n\ge3)$ *submanifold isometrically immersed in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-contact space form with divided* R_5 $\tilde{M}(f_1,\dots,f_6)$ *such that* $\xi\in\Gamma(TM)$*. Then, for each point* $p\in M$*, we have*

$$\begin{aligned}2\tau\le{}&n(n-1)\|H\|^2+n(n-1)f_1+3f_2\|P\|^2-2(n-1)f_3\\&+f_{5,1}\left\{(\mathrm{trace}(h^T))^2-\|h^T\|^2\right\}-f_{5,2}\left\{\|(\phi h)^T\|^2-(\mathrm{trace}(\phi h)^T)^2\right\}\\&+2((n-1)f_4-f_6)\,\mathrm{trace}(h^T),\end{aligned} \tag{23}$$

The equality in (23) *holds if and only if the p is a totally umbilical point.*

Proof. We choose an orthonormal basis $\{e_1,\dots,e_n,e_{n+1},\dots,e_{2m+1}\}$ at p for $T_p\tilde{M}$ such that $\{e_1,\dots,e_n\}$ is basis for T_pM, $A_{n+1}(e_i)=a_ie_i$ for $i=1,\dots,n$, where A_{n+1} is the shape operator in the direction of e_{n+1}. Therefore, we have

$$A_{n+1}=\begin{pmatrix}a_1&\cdots&0\\0&a_2&0\\\vdots&\ddots&\vdots\\0&0&a_n\end{pmatrix},\mathrm{trace}(A_r)=0,r\in\{n+2,\dots,2m+1\}, \tag{24}$$

where A_r is the shape operator in the direction of e_r for $r=n+2,\dots2m+1$.
From (15) we have

$$\begin{aligned}2\tau={}&n(n-1)f_1+3f_2\|P\|^2-2(n-1)f_3\\&+f_{5,1}\left\{(\mathrm{trace}(h^T))^2-\|h^T\|^2\right\}-f_{5,2}\left\{\|(\phi h)^T\|^2-(\mathrm{trace}(\phi h)^T)^2\right\}\\&+2((n-1)f_4-f_6)\,\mathrm{trace}(h^T)+n^2\|H\|^2-\sum_{i=1}^n a_i^2-\sum_{r=n+2}^{2m+1}\sum_{i,j=1}^n(\sigma_{ij}^r)^2,\end{aligned} \tag{25}$$

By using Lemma 2.7, we have

$$n\|H\|^2 \leq \sum_{i=1}^{n} a_i^2. \tag{26}$$

In view of (25) and (26), we obtain

$$\begin{aligned}2\tau \leq\ & n(n-1)\|H\|^2 + n(n-1)f_1 + 3f_2\|P\|^2 - 2(n-1)f_3 \\ & + f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} - f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \\ & + 2((n-1)f_4 - f_6)\,\mathrm{trace}(h^T).\end{aligned} \tag{27}$$

The equality in (23) holds if and only if $a_1 = \cdots = a_n$, that is, $A_{n+1} = a_1 I$. Therefore, The equality holds if and only if the p is a totally umbilical point.

□

Theorem 3.7. *Let M be an n-dimensional $(n \geq 3)$ submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form with divided R_5 $\tilde{M}(f_1,\ldots,f_6)$ such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \leqslant\ & \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{(n+1)(n-2)}{2}f_1 + \frac{3}{2}f_2\{\|P\|^2 - 2\alpha(\pi)\} \\ & + (1-n+\beta(\pi))f_3 + f_4\{(n-1)\,\mathrm{trace}(h^T) - \mathrm{trace}(h|_\pi)\} \\ & + \frac{1}{2}f_{5,1}\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2 - 2\det(h|_\pi)\} \\ & - \frac{1}{2}f_{5,2}\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2 + 2\det((\phi h)|_\pi)\} \\ & + f_6\{\gamma(\pi) - \mathrm{trace}(h^T)\}.\end{aligned} \tag{28}$$

The equality in (28) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}\}$ of $T_p^\perp M$ such that (a) $\pi = Span\{e_1,e_2\}$ and (b) the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1,\ldots,2m+1$, become

$$A_{n+1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)I_{n-2} \end{bmatrix}, \tag{29}$$

$$A_r = \begin{bmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{bmatrix}, \quad r = n+2,\ldots,2m+1. \tag{30}$$

Proof. Let $\pi \subset T_pM$ be a plane section. Choose an orthonormal basis $\{e_1,e_2,\ldots,e_n\}$ for T_pM and $\{e_{n+1},\ldots,e_{2m+1}\}$ for the normal space $T_p^\perp M$ at p such that $\pi = Span\{e_1,e_2\}$ and the mean curvature vector H is in the direction of the normal vector to e_{n+1}. We rewrite (15) as

$$\frac{1}{n-1}\Big(\sum_{i=1}^{n}\sigma_{ii}^{n+1}\Big)^2 = \sum_{i=1}^{n}(\sigma_{ii}^{n+1})^2 + \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^{r})^2 + \delta, \tag{31}$$

where

$$\begin{aligned}\delta = 2\tau - \frac{n^2(n-2)}{n-1}\|H\|^2 - n(n-1)f_1 - 3f_2\|P\|^2 + 2(n-1)f_3 \\ - 2(n-1)f_4 \operatorname{trace}(h^T) - f_{5,1}\{(\operatorname{trace}(h^T))^2 - \|h^T\|^2\} \\ + f_{5,2}\{\|(\phi h)^T\|^2 - (\operatorname{trace}(\phi h)^T)^2\} + 2f_6 \operatorname{trace}(h^T)\end{aligned} \tag{32}$$

and $\sigma_{ij}^r = \langle \sigma(e_i, e_j), e_r \rangle, \quad i,j \in \{1,\ldots,n\}; \quad r \in \{n+1,\ldots,2m+1\}$. Now, applying Lemma 2.6 to (31), we obtain

$$2\sigma_{11}^{n+1}\sigma_{22}^{n+1} \geqslant \sum_{i\neq j}(\sigma_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(\sigma_{ij}^r)^2 + \delta. \tag{33}$$

From equation (14) and (4) it also follows that

$$\begin{aligned}K(\pi) = f_1 + 3f_2\alpha(\pi) - f_3\beta(\pi) + f_4 \operatorname{trace}(h|_\pi) + f_{5,1}\det(h|_\pi) \\ + f_{5,2}\det((\phi h)|_\pi) - f_6\gamma(\pi) + \sigma_{11}^{n+1}\sigma_{22}^{n+1} - (\sigma_{12}^{n+1})^2 \\ + \sum_{r=n+2}^{2m+1}(\sigma_{11}^r\sigma_{22}^r - (\sigma_{12}^r)^2).\end{aligned} \tag{34}$$

Thus, from (33) and (34) we have

$$\begin{aligned}K(\pi) \geq f_1 + 3f_2\alpha(\pi) - f_3\beta(\pi) + f_4 \operatorname{trace}(h|_\pi) + f_{5,1}\det(h|_\pi) \\ + f_{5,2}\det((\phi h)|_\pi) - f_6\gamma(\pi) + \frac{\delta}{2} + \frac{1}{2}\sum_{2<i\neq j\leqslant n}(\sigma_{ij}^{n+1})^2 \\ + \frac{1}{2}\sum_{r=n+2}^{2m+1}(\sigma_{11}^r + \sigma_{22}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{i,j>2}(\sigma_{ij}^r)^2.\end{aligned} \tag{35}$$

In view of (32) and (35), we get (28). If the equality in (28) holds, then the inequalities given by (33) and (35) become equalities. In this case, we have

$$\begin{aligned}\sigma_{1j}^{n+1} = 0, \quad \sigma_{2j}^{n+1} = 0, \quad \sigma_{ij}^{n+1} = 0, \quad i \neq j > 2, \\ \sigma_{1j}^r = \sigma_{2j}^r = \sigma_{ij}^r = 0, \quad r = n+2,\ldots,2m+1; \quad i,j = 3,\ldots,n, \\ \sigma_{11}^{n+2} + \sigma_{22}^{n+2} = \cdots = \sigma_{11}^{2m+1} + \sigma_{22}^{2m+1} = 0.\end{aligned} \tag{36}$$

Now, we choose e_1 and e_2 so that $\sigma_{12}^{n+1} = 0$. Applying Lemma 2.6 we also have

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1}. \tag{37}$$

Thus, choosing a suitable orthonormal basis $\{e_1,\ldots,e_{2m+1}\}$, the shape operator of M becomes of the form given by (29) and (30). The converse is easy to follow. □

Corollary 3.8. *Let M be an n-dimensional ($n \geq 3$) invariant submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form with divided R_5 $\tilde{M}(f_1,\ldots,f_6)$*

such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have

$$\begin{aligned} \tau - K(\pi) \leqslant & \frac{(n+1)(n-2)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - 2\alpha(\pi)\} \\ & + (1 - n + \beta(\pi)) f_3 - f_4 \,\mathrm{trace}(h|_\pi) \\ & - \frac{1}{2}(f_{5,1} + f_{5,2}) \|h^T\|^2 \\ & - f_{5,1} \det(h|_\pi) - f_{5,2} \det((\phi h)|_\pi) + f_6 \gamma(\pi). \end{aligned} \tag{38}$$

The equality in (38) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that (a) $\pi = Span\{e_1, e_2\}$ and (b) the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, 2m+1$, become of the forms given by (29) *and* (30).

For a submanifold M of an almost contact metric manifold tangential to the structure vector field ξ, we write the orthogonal direct decomposition $TM = \mathcal{D} \oplus \langle \xi \rangle$. Moreover, if the ambient manifold is contact also, then

$$\nabla_\xi \xi = 0 \quad \text{and} \quad \sigma(\xi, \xi) = 0. \tag{39}$$

Next, we prove the following theorem.

Theorem 3.9. *Let M be an n-dimensional ($n \geq 3$) submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5 $\tilde{M}(f_1, \ldots, f_6)$ such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset \mathcal{D}_p$, we have*

$$\begin{aligned} \tau - K(\pi) \leqslant & \frac{n^2(n-2)}{2(n-1)} \|H\|^2 + \frac{(n+1)(n-2)}{2} f_1 + \frac{3}{2} f_2 \{\|P\|^2 - 2\alpha(\pi)\} \\ & + (1-n) f_3 + f_4 \{(n-1)\,\mathrm{trace}(h^T) - \mathrm{trace}(h|_\pi)\} \\ & + \frac{1}{2} f_{5,1} \{(\mathrm{trace}(h^T))^2 - \|h^T\|^2 - 2\det(h|_\pi)\} \\ & - \frac{1}{2} f_{5,2} \{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2 + 2\det((\phi h)|_\pi)\} \\ & - f_6 \,\mathrm{trace}(h^T). \end{aligned} \tag{40}$$

The equality in (40) holds at $p \in M$ if and only if there exist an orthonormal basis$\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^\perp M$ such that (a) $e_n = \xi$, (b) $\pi = Span\{e_1, e_2\}$ and (c) the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, 2m+1$, become (30) and

$$A_{n+1} = \begin{bmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0_{n-2} \end{bmatrix}. \tag{41}$$

Proof. Let $\pi \subset \mathcal{D}_p$ be a plane section at $p \in M$. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_n = \xi\}$ for T_pM and $\{e_{n+1}, \ldots, e_{2m+1}\}$ for the normal space $T_p^\perp M$ at p such that $\pi = Span\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+1}. Using $\eta(e_1) = 0 = \eta(e_2)$, we get $\beta(\pi) = 0 = \gamma(\pi)$. Thus, proof of (40) is similar to that of (28). In equality case, using (39), (37) becomes

$$\sigma_{11}^{n+1} + \sigma_{22}^{n+1} = \sigma_{33}^{n+1} = \cdots = \sigma_{nn}^{n+1} = 0, \tag{42}$$

and thus (29) is modified to (41). ☐

Proposition 3.10. *Let M be an n-dimensional $(n \geq 3)$ invariant submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form with divided R_5 $\tilde{M}(f_1,\ldots,f_6)$ such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset \mathcal{D}_p$, we have*

$$\begin{aligned}\tau - K(\pi) \leqslant \frac{(n+1)(n-2)}{2} f_1 + \frac{3}{2} f_2(n-3) + (1-n) f_3 \\ - (f_{5,1} + f_{5,1}) \left(\frac{1}{2} \|h^T\|^2 + \det(h|_\pi) \right).\end{aligned} \tag{43}$$

The equality in (43) holds at $p \in M$ if and only if there exist an orthonormal basis$\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}\}$ of $T_p^\perp M$ such that (a) *$e_n = \xi$,* (b) *$\pi = Span\{e_1,e_2\}$ and* (c) *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1,\ldots,2m+1$, become (30) and* (41).

Proof. Let $\pi \subset \mathcal{D}_p$ be a plane section at $p \in M$. We can choose unit vectors U and PU such that $\pi = Span\{U, PU\}$. Thus, we get

$$\alpha(\pi) = 1, \mathrm{trace}(h|_\pi) = 0, \det(h|_\pi) = \det((\phi h)|_\pi). \tag{44}$$

From (11), (44) and (40), we have (43). ☐

From Proposition 3.10 we obtain:

Corollary 3.11. *Let M be an n-dimensional $(n \geq 3)$ invariant submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form $\tilde{M}(f_1,\ldots,f_6)$ such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset \mathcal{D}_p$, we have*

$$\tau - K(\pi) \leqslant \frac{(n+1)(n-2)}{2} f_1 + \frac{3}{2} f_2(n-3) + (1-n) f_3. \tag{45}$$

The equality in (45) holds at $p \in M$ if and only if there exist an orthonormal basis$\{e_1,\ldots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\ldots,e_{2m+1}\}$ of $T_p^\perp M$ such that (a) *$e_n = \xi$,* (b) *$\pi = Span\{e_1,e_2\}$ and* (c) *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1,\ldots,2m+1$, become (30) and* (41).

Corollary 3.12. *Let M be an n-dimensional $(n \geq 3)$ invariant submanifold isometrically immersed in a $(2m+1)$-dimensional generalized (κ,μ)-contact space form $\tilde{M}(f_1,\ldots,f_6)$ such that $\xi \in \Gamma(TM)$. Then, for each point $p \in M$ and each plane section $\pi \subset \mathcal{D}_p$, we have*

$$\delta_M^{\mathcal{D}} \leqslant \frac{(n+1)(n-2)}{2} f_1 + \frac{3}{2} f_2(n-3) + (1-n) f_3, \tag{46}$$

where $\delta_M^{\mathcal{D}}(p) = \tau(p) - \inf\{K(\pi) : \text{plane sections } \pi \subset \mathcal{D}_p\}$ defined in [5].

4. Ricci and k-Ricci curvature

In [14] Tripathi establish a sharp relationship between Ricci curvature and the squared mean curvature for a submanifold in a non-Sasakian (κ,μ)-space as follows.

Theorem 4.1 ([14]). *Let M be an n-dimensional $(n \geq 3)$ submanifold in a $(2m+1)$-dimensional non-Sasakian (κ,μ)-space $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then for each point $p \in M$*

(i) *For all unit vector $U \in T_pM$, we have*

$$\begin{aligned} Ric(U) \leq & \frac{1}{4}n^2\|H\|^2 + \kappa + \frac{1}{2}(n-2)(2-\mu) + \mathrm{trace}(h^T) \\ & - \frac{3\mu}{2}\|PU\|^2 f_2 - \mathbf{t}_2(U) + (\mu + n - 3)\langle h^T U, U\rangle \\ & \{(n-2)(\kappa - 1 + \frac{\mu}{2}) + (\mu - 1)\,\mathrm{trace}(h^T)\}\eta(U)^2, \end{aligned} \tag{47}$$

where
$\mathbf{t}_2(U) = \frac{1-\mu/2}{1-\kappa}\left\{\|h^T U\|^2 - \mathrm{trace}(h^T)\langle h^T U, U\rangle\right\} + \frac{\kappa-\mu/2}{1-\kappa}\left\{\|(\phi h)^T U\|^2 - \mathrm{trace}((\phi h)^T)\langle(\phi h)^T U, U\rangle\right\}$.

(ii) *For $H(p) = 0$, a unit tangent vector $U \in T_pM$ satisfies the equality case of* (47) *if and only if U belongs to the relative null space $\mathcal{N}_p$.*

(iii) *the equality in* (47) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

In this section, we prove Theorem 4.1 for sub manifolds tangent to the structure vector field in a generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$.

Theorem 4.2. *Let M be an n-dimensional $(n \geq 3)$ submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$ such that $\xi \in \Gamma(TM)$. Then for each point $p \in M$*

(i) *For all unit vector $U \in T_pM$, we have*

$$\begin{aligned} Ric(U) \leq & \frac{1}{4}n^2\|H\|^2 + (n-1)f_1 + 3\|PU\|^2 f_2 \\ & + \big(2n - 3 - (n-2)\eta(U)^2\big) f_3 \\ & + f_4\Big(\mathrm{trace}(h^T) + (n-2)\langle h^T U, U\rangle\Big) \\ & + f_{5,1}\big(\mathrm{trace}(h^T)\langle h^T U, U\rangle - \|h^T U\|^2\big) \\ & + f_{5,2}\big(\mathrm{trace}((\phi h)^T)\langle(\phi h)^T U, U\rangle - \|(\phi h)^T U\|^2\big) \\ & - \Big(\langle h^T U, U\rangle + \big(\mathrm{trace}(h^T) - \langle h^T U, U\rangle\big)\eta(U)^2\Big) f_6. \end{aligned} \tag{48}$$

(ii) *For $H(p) = 0$, a unit tangent vector $U \in T_pM$ satisfies the equality case of* (48) *if and only if U belongs to the relative null space $\mathcal{N}_p$.*

(iii) *the equality in* (48) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Proof. Let $U \in T_pM$ be a unit tangent vector. We choose an orthonormal basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, ..., e_n$ are tangential to M at p with $e_1 = U$. Then, the squared second fundamental form and the squared mean curvature satisfy the following relation

$$\begin{aligned} \|\sigma\|^2 = & \frac{1}{2}n^2\|H\|^2 + \frac{1}{2}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r \cdots - \sigma_{nn}^r)^2 \\ & + 2\sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n}(\sigma_{ii}^r\sigma_{jj}^r - (\sigma_{ij}^r)^2). \end{aligned} \tag{49}$$

From (15) and (49) we get

$$
\begin{aligned}
\frac{1}{4}n^2\|H\|^2 =& \tau - \frac{n(n-1)}{2}f_1 - \frac{3}{2}\|P\|^2 f_2 - (n-1)f_3 \\
& - \big((n-1)f_4 - f_6\big)\,\mathrm{trace}(h^T) \\
& - \frac{1}{2}f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} \\
& + \frac{1}{2}f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \\
& + \frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma^r_{11} - \sigma^r_{22}\cdots - \sigma^r_{nn})^2 \\
& + \sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma^r_{1j})^2 - \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2).
\end{aligned}
\tag{50}
$$

From (4) and (14), we also have

$$
\begin{aligned}
\sum_{2\le i<j\le n} K_{ij} =& \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}(\sigma^r_{ii}\sigma^r_{jj} - (\sigma^r_{ij})^2) + \frac{(n-1)(n-2)}{2}f_1 \\
& + \frac{3}{2}(\|P\|^2 - 2\|Pe_1\|^2)f_2 - (n-2)\big(1-\eta(e_1)^2\big)f_3 \\
& + f_4(n-2)(\mathrm{trace}(h^T) - \langle h^T e_1, e_1\rangle) \\
& + \frac{1}{2}f_{5,1}\Big\{(\mathrm{trace}(h^T))^2 - 2\,\mathrm{trace}(h^T)\langle h^T e_1, e_1\rangle \\
& \qquad - \|h^T\|^2 + 2\|h^T e_1\|^2\Big\} \\
& - \frac{1}{2}f_{5,2}\Big\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2 - 2\|(\phi h)^T e_1\|^2 \\
& \qquad + 2\,\mathrm{trace}((\phi h)^T)\langle(\phi h)^T e_1, e_1\rangle\Big\} \\
& - \big(\mathrm{trace}(h^T) - \langle he_1, e_1\rangle\big)\big(1-\eta(e_1)^2\big)f_6.
\end{aligned}
\tag{51}
$$

From (50) and (51), we get

$$\begin{aligned} Ric(U) =&\frac{1}{4}n^2\|H\|^2 + (n-1)f_1 + 3\|PU\|^2 f_2 \\ &+ \big((2n-3) - (n-2)\eta(U)^2\big)f_3 \\ &+ f_4\bigg(\operatorname{trace}(h^T) + (n-2)\langle h^T U, U\rangle\bigg) \\ &+ f_{5,1}\big(\operatorname{trace}(h^T)\langle h^T U, U\rangle - \|h^T U\|^2\big) \\ &+ f_{5,2}\big(\operatorname{trace}((\phi h)^T)\langle (\phi h)^T U, U\rangle - \|(\phi h)^T U\|^2\big) \\ &- \bigg(\langle h^T U, U\rangle + \big(\operatorname{trace}(h^T) - \langle h^T U, U\rangle\big)\eta(U)^2\bigg)f_6 \\ &- \frac{1}{4}\sum_{r=n+1}^{2m+1}(\sigma_{11}^r - \sigma_{22}^r \cdots - \sigma_{nn}^r)^2 \\ &- \sum_{r=n+1}^{2m+1}\sum_{j=2}^{n}(\sigma_{1j}^r)^2, \end{aligned} \tag{52}$$

which implies (48).
From (52), the equality case of (48) is valid if and only if

$$\begin{aligned} &\sigma_{11}^r = \sigma_{22}^r + \cdots + \sigma_{nn}^r, \\ &\sigma_{12}^r = \cdots = \sigma_{1n}^r = 0, r = n+1, \ldots, 2m+1. \end{aligned} \tag{53}$$

If $H(p) = 0$, (53) implies that $U = e_1$ lies in the relative null space $\mathcal{N}_p$. Conversely, if $U = e_1$ lies in the relative null space $\mathcal{N}_p$, then (53) holds, since $H(p) = 0$ is assumed. Thus (ii) is proved.
Now we prove (iii). The equality case of (48) for all unit tangent vectors to M at p happens if and only if

$$2\sigma_{ii}^r = \sigma_{11}^r + \cdots + \sigma_{nn}^r, i = 1, \ldots, n, r = n+1, \ldots, 2m+1, \tag{54}$$
$$\sigma_{ij}^r = 0, i \neq j, r = n+1, \ldots, 2m+1.$$

Thus, we have two cases, namely either $n = 2$ or $n \neq 2$. In the first case p is a totally umbilical point, while in the second case p is a totally geodesic point. The proof of converse part is straightforward. □

Corollary 4.3. *Let M be an n-dimensional ($n \geq 3$) invariant submanifold in a $(2m+1)$-dimensional generalized (κ, μ)-contact space form with divided R_5, $\tilde{M}(f_1, \ldots, f_6)$ such that $\xi \in \Gamma(TM)$. Then for each point $p \in M$*

(1) *For all unit vector $U \in T_pM$, we have*

$$\begin{aligned} Ric(U) \leq&(n-1)f_1 + 3\|\phi U\|^2 f_2 \\ &+ \big((2n-3) - (n-2)\eta(U)^2\big)f_3 + f_4(n-2)\langle h^T U, U\rangle \\ &- (f_{5,1}\|h^T U\|^2 + f_{5,2}\|(\phi h)^T U\|^2) \\ &- \langle h^T U, U\rangle\bigg(1 - \eta(U)^2\bigg)f_6, \end{aligned} \tag{55}$$

with equality case of (55) *if and only if U belongs to the relative null space* $\mathcal{N}_p$.

(2) *the equality in* (55) *holds identically for all unit tangent vectors at p if and only if either p is a totally geodesic point.*

Theorem 4.4. *Let M be an n-dimensional* $(n \geq 3)$ *submanifold in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-contact space form with divided* R_5, $\tilde{M}(f_1,\ldots,f_6)$ *with* $\xi \in \Gamma(TM)$. *Then we have*

$$\begin{aligned} n(n-1)\|H\|^2 \geq\, & n(n-1)\theta_k(p) - n(n-1)f_1 - 3\|P\|^2 f_2 - 2(n-1)f_3 \\ & - f_{5,1}\left\{(\mathrm{trace}(h^T))^2 - \|h^T\|^2\right\} \\ & + f_{5,2}\left\{\|(\phi h)^T\|^2 - (\mathrm{trace}(\phi h)^T)^2\right\} \\ & - 2\big((n-1)f_4 - f_6\big)\,\mathrm{trace}(h^T). \end{aligned} \tag{56}$$

Proof. Let $\{e_1,...,e_n\}$ be an orthonormal basis of T_pM. We denote by $L_{i_1\ldots i_k}$ the k-plane section spanned by $e_{i_1},...,e_{i_k}$. From (1) and (2), it follows that

$$\tau(L_{i_1\ldots i_k}) = \frac{1}{2}\sum_{i\in\{e_{i_1},\ldots,e_{i_k}\}} \mathrm{Ric}_{L_{i_1\ldots i_k}}(e_i), \tag{57}$$

and

$$\tau(p) = \frac{1}{C_{k-2}^{n-2}}\sum_{1\leq i_1<\cdots<i_k\leq n}\tau(L_{i_1\ldots i_k}). \tag{58}$$

Combining (3), (57) and (58), we obtain

$$\tau(p) \geq \frac{n(n-1)}{2}\theta_k(p), \tag{59}$$

which in view of (23) implies (56). □

Corollary 4.5. *Let M be an n-dimensional* $(n \geq 3)$ *invariant submanifold in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-contact space form with divided* R_5, $\tilde{M}(f_1,\ldots,f_6)$. *Then we have*

$$n\theta_k(p) \leq n f_1 + 3f_2 + 2f_3 - \frac{1}{n-1}(f_{5,1}+f_{5,2})\|h^T\|^2. \tag{60}$$

Corollary 4.6. *Let M be an n-dimensional* $(n \geq 3)$ *invariant submanifold in a* $(2m+1)$*-dimensional generalized* (κ,μ)*-contact space form* $\tilde{M}(f_1,\ldots,f_6)$. *Then we have*

$$n\theta_k(p) \leq n f_1 + 3f_2 + 2f_3. \tag{61}$$

Similar inequalities are proved in [10] for C-totally real submanifolds of generalized (κ,μ)-space forms with divided R_5.

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