Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.4, (2015), Issue 2, pp.79-85

ON A CHANGE OF m-TH ROOT FINSLER METRICS

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ABSTRACT. In this paper, we define a new β-change of Finsler metrics $F \to c_1F + c_2\beta + c_3\beta^2/F$, where $\beta = b_i(x)y^i$ is a 1-form on a manifold M and c_1, c_2 and c_3 are real constant. Then, we consider this new change on the class of m-th root Finsler metrics. We find necessary and sufficient condition under which a the new class of m-th root metrics obtained by this change be locally dually flat.

Keywords: Locally dually flat metric, m-th root metric.[1]

1. INTRODUCTION

Let (M, F) be an n-dimensional Finsler manifold and $\beta(x, y) = b_i(x)y^i$ be a non-zero 1-form on M. Then we have a transormation of Finsler defined by following

$$F(x, y) \to \bar{F}(x, y) = f(F, \beta)$$

where $f(F, \beta)$ is a positively homogeneous function of Finsler function F. This transformation of Finsler metrics is called a β-change of metric. By a simple calculation, it follows that if

$$B := \sup_{F(x,y)=1} |b_i(x)y^i| < 1,$$

then $\bar{F}$ is again a Finsler metric. In [5], Hashiguchi-Ichijyō showed that if the 1-form β is closed, then the new metric $\bar{F}$ is pointwise projective to F. There are many well-known β-change of Finsler metrics, namely, Randers change $\bar{F} = F + \beta$. The notion and geometric meaning of Randers change has been introduced by Matsumoto in [6], named by Hashiguchi-Ichijyō in [5] and studied in detail by Shibata in [9]. If F reduces to a Riemannian metric then $\bar{F}$ reduces to a Randers metric. Due to this reason the mentioned transformation has been called the Randers change. For other β-changes see [9][18][19].
In this paper, we consider the following special change

$$\bar{F} = c_1F + c_2\beta + c_3\frac{\beta^2}{F}, \tag{1}$$

where c_1, c_2 and c_3 are real constants. If $c_1 = c_2 = 1$ and $c_3 = 0$, then (1) reduces to the Randers change of Finsler metrics, i.e., $F \to F + \beta$. If $F = \alpha$ is a Riemannian metric and $c_1 = c_2 = 1$ and $c_3 = 0$ then we get the Randers metric $F = \alpha + \beta$. For other β-changes, see [19].
In [2], Amari-Nagaoka study the information geometry on Riemannian manifolds and introduced the notion of dually flat Riemannian metrics. Information geometry has

emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [8]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure. A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}$$

where $H = H(x,y)$ is a C^∞ homogeneous scalar function on TM_0. Such a coordinate system is called an adapted coordinate system [11][16][17]. Indeed, a Finsler metric F on an open subset $U \subset R^n$ is called dually flat if it satisfies

$$\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k = 2\frac{\partial F^2}{\partial x^l}.$$

Let (M,F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let F be the following function on M, by

$$F = \sqrt[m]{A},$$

where A is given by $A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ with $a_{i_1 \ldots i_m}$ symmetric in all its indices (see [3][4][7][10][11][12][13][14][15]). Then F is called an m-th root Finsler metric. Suppose that A_{ij} define a positive definite tensor and A^{ij} denotes its inverse. For an m-th root metric F, put

$$A_i = \frac{\partial A}{\partial y^i}, \ A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \ A_{x^i} = \frac{\partial A}{\partial x^i}, \ A_0 = A_{x^i} y^i.$$

Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. Let $F = \sqrt[m]{A}$ be a Finsler metric on M, where A is given by $A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ with $a_{i_1 \ldots i_m}$ symmetric in all its indices ...... Then F is called an m-th root Finsler metric. Suppose that A_{ij} define a positive definite tensor and A^{ij} denotes its inverse. For an m-th root metric F, put

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In this paper, we consider special change (1) of an m-th root Finsler metric and find necessary and sufficient condition under which a special change (1) of an m-th root metric be locally dually flat. More precisely, we prove the following.

Definition 1.1. *A polynomial $A := a_{i_1 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}$ of degree m are said to be reducible if $A = BC$, where B and C are polynomial such that $degree(B) < m$, $degree(C) < m$ and $degree(A) + degree(B) = m$. A polynomial A is called irreducible if it is not reducible.*

Theorem 1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset R^n$, where A is irreducible. Suppose that $\bar{F} = c_1 F + c_2 \beta + c_3 \beta^2 / F$ be special change of F where $\beta = b_i(x) y^i$ is a non-zero 1-form on M and $c_2^2 + 2c_1 c_3 \neq 0$. Then $\bar{F}$ is locally dually flat if and only if there* *exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following hold*

$$A_{x^l} = \frac{1}{3m}[mA\theta_l + 2\theta A_l], \tag{2}$$

$$\beta_{x^l} = \frac{-1}{3m2(c_2^2 + 2c_1c_3)}\Big[(\theta\beta)_l + \theta\beta_l\Big], \tag{3}$$

and

$$3\beta(2\beta_0\beta_l + \beta\beta_{0l} - 2\beta\beta_{x^l})A^2 - \beta^2(3\beta_0A_l + 3\beta_lA_0 + \beta A_{0l} + \beta A_{x^l})A + \Big(\frac{1}{m} + 1\Big)\beta^3A_0A_l = 0, \tag{4}$$

$$2mA^2(\beta\beta_{0l} + 3\beta_0\beta_l - 2\beta\beta_{x^l}) - \beta A(4\beta_0A_l + \beta A_{0l} + 4\beta_lA_0 - 2\beta A_{x^l}) + \Big(\frac{2}{m} + 1\Big)\beta^2A_0A_l = 0, \tag{5}$$

where $\beta_{0l} = \beta_{x^ky^l}y^k$, $\beta_{x^l} = (b_i)_{x^l}y^i$, $\beta_0 = \beta_{x^l}y^i$ and $\beta_{0l} = (b_l)_0$.

Acknowledgements. The authors would like to thank the Professor Akbar Tayebi for his helpful suggestions. Also, the authors would like to thank the referee for his careful reading of the manuscript and very exact comments.

2. Preliminary

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where

$$G^i := \frac{1}{4}g^{il}\left[\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\right], \quad y \in T_xM.$$

$\mathbf{G}$ is called the spray associated to (M, F). In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy

$$\ddot{c}^i + 2G^i(\dot{c}) = 0.$$

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j},$$

where $H = H(x, y)$ satisfying

$$H(x, \lambda y) = \lambda^3H(x, y), \quad \forall\lambda > 0.$$

Such a coordinate system is called an adapted coordinate system. In [8], Shen proved that the Finsler metric F on an open subset $U \subset R^n$ is dually flat if and only if it satisfies

$$(F^2)_{x^ky^l}y^k = 2(F^2)_{x^l}. \tag{6}$$

In this case, $H = -\frac{1}{6}[F^2]_{x^m}y^m$.

3. Proof of the Theorem 1

To prove Theorem 1, we need the following.

Lemma 1. *Suppose that the following equation holds*

$$\Phi A^{\frac{2}{m}-2}+\Psi A^{\frac{1}{m}-2}+\Theta A^{\frac{-1}{m}-2}+\Lambda A^{\frac{-2}{m}-2}+\Omega=0,$$

where Φ,Ψ,Θ are polynomials in y and $m>2$. Then

$$\Phi=\Psi=\Theta=\Lambda=\Omega=0.$$

Proof of Theorem 1: Let $\bar{F}$ be a locally dually flat metric. We have

$$\begin{aligned}(\bar{F}^2)_{x^k} &= c_1^2\frac{2}{m}A^{\frac{2}{m}-1}A_{x^k}+2c_2^2\beta\beta_{x^k}+4c_3^2\beta^3\beta_{x^k}A^{-\frac{2}{m}}-c_3^2\frac{2}{m}\beta^4A^{-\frac{2}{m}-1}A_{x^k}+4c_1c_3\beta\beta_{x^k}\\ &+ 2c_1c_2\Big(\beta_{x^k}A^{\frac{1}{m}}+\frac{1}{m}\beta A^{\frac{1}{m}-1}A_{x^k}\Big)+2c_2c_3\Big(3\beta^2\beta_{x^k}A^{-\frac{1}{m}}-\frac{1}{m}\beta^3A^{-\frac{1}{m}-1}A_{x^k}\Big)\end{aligned}$$

Then

$$\begin{aligned}[\bar{F}^2]_{x^ky^l}y^k &= \frac{2}{m}c_1^2\Big[A^{\frac{2}{m}-1}A_{0l}+\Big(\frac{2}{m}-1\Big)A^{\frac{2}{m}-2}A_0A_l\Big]+2c_2^2(\beta\beta_{0l}+\beta_0\beta_l)\\ &+2c_3^2\Bigg[(2\beta^3\beta_{0l}+6\beta^2\beta_0\beta_l)A^{-\frac{2}{m}}-\frac{1}{m}\beta^3(4\beta_0A_l+\beta A_{0l}+4\beta_lA_0)A^{-\frac{2}{m}-1}\\ &+\frac{1}{m}\Big(\frac{2}{m}+1\Big)A^{-\frac{2}{m}-2}\beta^4A_0A_l\Bigg]+4c_1c_3(\beta_0\beta_l+\beta\beta_{0l})\\ &+2c_1c_2\Big[\beta_{0l}A^{\frac{1}{m}}+\frac{1}{m}A^{\frac{1}{m}-1}A_l\beta_0\Big]+\frac{2}{m}c_1c_2\Bigg[(\beta_lA_0+\beta A_{0l})A^{\frac{1}{m}-1}\\ &+(\frac{1}{m}-1)A^{\frac{1}{m}-2}\beta A_0A_l\Bigg]+2c_2c_3\Bigg[3\beta(2\beta_0\beta_l+\beta\beta_{0l})A^{-\frac{1}{m}}\\ &-\frac{1}{m}\beta^2(3\beta_0A_l+3\beta_lA_0+\beta A_{0l})A^{-\frac{1}{m}-1}+\Big(\frac{1}{m}\Big)\Big(\frac{1}{m}+1\Big)\beta^3A^{-\frac{1}{m}-2}A_0A_l\Bigg]\end{aligned}$$

Thus, we get

$$\begin{aligned}
&\frac{2}{m}c_1^2A^{\frac{2}{m}-2}\Big[AA_{0l}+(\frac{2}{m}-1)A_0A_l-2AA_{x^l}\Big]\\
+\ &\frac{2c_1c_2}{m}A^{\frac{1}{m}-2}\Big[mA^2(\beta_{0l}-2\beta_{x^l})+A(A_l\beta_0+\beta_lA_0+\beta A_{0l}-2\beta A_{x^l})\\
+\ &\Big(\frac{1}{m}-1\Big)\beta A_0A_l\Big]+2(c_2^2+2c_1c_3)(\beta\beta_{0l}+\beta_0\beta_l-2\beta\beta_{x^l})\\
+\ &\frac{2c_2c_3}{m}A^{-\frac{1}{m}-2}\Big[3\beta(2\beta_0\beta_l+\beta\beta_{0l}-2\beta\beta_{x^l})A^2-\beta^2(3\beta_0A_l+3\beta_lA_0+\beta A_{0l}+\beta A_{x^l})A\\
+\ &\Big(\frac{1}{m}+1\Big)\beta^3A_0A_l\Big]+\frac{2c_3^2}{m}A^{-\frac{2}{m}-2}\beta^2\Big[2mA^2(\beta\beta_{0l}+3\beta_0\beta_l-2\beta\beta_{x^l})\\
-\ &\beta A(4\beta_0A_l+\beta A_{0l}+4\beta_lA_0-2\beta A_{x^l})+\Big(\frac{2}{m}+1\Big)\beta^2A_0A_l\Big]=0.
\end{aligned}$$

By Lemma 1, we have

$$\Big(\frac{2}{m}-1\Big)A_lA_0+AA_{0l}=2AA_{x^k}, \tag{7}$$

$$mA^2(\beta_{0l}-2\beta_{x^l})+A(A_l\beta_0+\beta_lA_0+\beta A_{0l}-2\beta A_{x^l})+\Big(\frac{1}{m}-1\Big)\beta A_0A_l=0, \tag{8}$$

$$2(c_2^2+2c_1c_3)(\beta\beta_{0l}+\beta_0\beta_l-2\beta\beta_{x^l})=0, \tag{9}$$

$$\begin{aligned}
\frac{2c_2c_3}{m}&\Big[3\beta(2\beta_0\beta_l+\beta\beta_{0l}-2\beta\beta_{x^l})A^2-\beta^2(3\beta_0A_l+3\beta_lA_0+\beta A_{0l}+\beta A_{x^l})A\Big]\\
&+\frac{2c_2c_3}{m}\Big[\Big(\frac{1}{m}+1\Big)\beta^3A_0A_l\Big]=0,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\frac{2c_3^2}{m}&\Big[2mA^2(\beta\beta_{0l}+3\beta_0\beta_l-2\beta\beta_{x^l})-\beta A(4\beta_0A_l+\beta A_{0l}+4\beta_lA_0-2\beta A_{x^l})\Big]\\
&+\frac{2c_3^2}{m}\Big[\Big(\frac{2}{m}+1\Big)\beta^2A_0A_l\Big]=0.
\end{aligned} \tag{11}$$

By (10) and (11) it follows that $c_2=c_3=0$ or the following holds

$$\begin{aligned}
3\beta(2\beta_0\beta_l+\beta\beta_{0l}-2\beta\beta_{x^l})A^2-\beta^2(3\beta_0A_l+3\beta_lA_0+\beta A_{0l}+\beta A_{x^l})A+\\
\Big(\frac{1}{m}+1\Big)\beta^3A_0A_l=0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
2mA^2(\beta\beta_{0l}+3\beta_0\beta_l-2\beta\beta_{x^l})-\beta A(4\beta_0A_l+\beta A_{0l}+4\beta_lA_0-2\beta A_{x^l})\\
+\Big(\frac{2}{m}+1\Big)\beta^2A_0A_l=0.
\end{aligned} \tag{13}$$

In the first case, we get

$$\bar{F}=c_1F.$$

In this case, it is easy to see that $\bar{F}$ is locally dually flat if and only if F is locally dually flat. Now, let us suppose that $c_2 \neq 0$ and $c_3 \neq 0$. Then (12) and (13) reduce to following

$$3m(2\beta_0\beta_l + \beta\beta_{0l} - 2\beta\beta_{x^l})A^2 - m\beta(3\beta_0 A_l + 3\beta_l A_0 + \beta A_{0l} + \beta A_{x^l})A$$

$$+(1+m)\beta^2 A_0 A_l = 0, \tag{14}$$

$$2mA^2(\beta\beta_{0l} + 3\beta_0\beta_l - 2\beta\beta_{x^l}) - \beta A(4\beta_0 A_l + \beta A_{0l} + 4\beta_l A_0 - 2\beta A_{x^l})$$

$$+\Big(\frac{2}{m}+1\Big)\beta^2 A_0 A_l = 0. \tag{15}$$

One can rewrite (7) as follows

$$A(2A_{x^l} - A_{0l}) = \Big(\frac{2}{m}-1\Big)A_l A_0. \tag{16}$$

By (16), we have $A_l A_0 | A$. Irreducibility of A and $deg(A_l) = m-1$ imply that there exists a 1-form $\theta = \theta_l y^l$ on U such that

$$A_0 = \theta A. \tag{17}$$

Plugging (17) into (16), we get

$$A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \tag{18}$$

Substituting (17) and (18) into (16) yields (2).
Contracting (8) with y^l yields

$$mA\beta_0 = -\beta A_0, \tag{19}$$

By putting (17) in (19), we get

$$m\beta_0 = -\theta\beta, \tag{20}$$

Substituting (20) into (9) yields (3). The converse is a direct computation. This completes the proof. □

Corollary 1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset R^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following hold*

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big], \tag{21}$$

$$\beta_{x^l} = \frac{-1}{3m}\Big[(\theta\beta)_l + \theta\beta_l\Big]. \tag{22}$$

Proof. By putting $c_1 = c_2 = 1$ and $c_3 = 0$ in Theorem 1, we get the proof. □

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