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ON (α, β)-METRICS OF ISOTROPIC BERWALD CURVATURE

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ABSTRACT. The class of (α, β)-metrics is an important class of Finsler metrics which contains well-known metrics such as Funk, Berwald and Matsumoto metrics. Let $F = \alpha\phi(s)$ be an (α, β)-metric of non-Randers type. We prove that F is of scalar flag curvature with isotropic S-curvature if and only if it has isotropic Berwald curvature with almost isotropic flag curvature. In this case, F must be locally Minkowskian.

Keywords: (α, β)-metric, isotropic Berwald curvature, flag curvature.[1]

1. INTRODUCTION

Let (M, F) be a Finsler manifold. The geodesic curves of F on M are determined by $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i{}_{jkl} = c\Big\{F_{y^j y^k}\delta^i{}_l + F_{y^k y^l}\delta^i{}_j + F_{y^l y^j}\delta^i{}_k + F_{y^j y^k y^l}y^i\Big\}, \tag{1}$$

for some scalar function $c = c(x)$ on M. Berwald metrics are trivially isotropic Berwald metrics with $c = 0$. Funk metrics are also non-trivial isotropic Berwald metrics. Recently studies show that the notion of Finsler metrics with isotropic Berwald curvature deserve more attentions. For example, Tayebi-Rafie Rad show that every isotropic Berwald metric has isotropic S-curvature [14]. Recently, Tayebi-Najafi investigate isotropic Berwald metric of scalar flag curvature and prove these Finsler metrics are of Randers type [13]. In [7], Mo-Guo-Liu show that every spherically symmetric Finsler metric of isotropic Berwald curvature is a Randers metric.

In order to find explicit examples of isotropic Berwald metrics, we consider (α, β)-metrics [11][12]. This class of metrics was first introduced by Matsumoto which appearing iteratively in formulating Physics, Mechanics, Biology and Ecology, etc [8]. An (α, β)-metric is a Finsler metric of the form $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. For example, $\phi = c_1\sqrt{1 + c_2s^2} + c_3s$ is called Randers type metric, where $c_1 > 0$, c_2 and c_3 are constant. For a Finsler manifold (M, F), the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags P associated with any fixed flagpole y. F is called of almost isotropic flag curvature if

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \tag{2}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. One of the important problems in Finsler geometry is to characterize Finsler manifolds of almost isotropic flag curvature [9].
Among the non-Riemannian quantities, the S-curvature $\mathbf{S} = \mathbf{S}(x,y)$ is closely related to the flag curvature which constructed by Shen for given comparison theorems on Finsler manifolds [10]. A Finsler metric F is called of isotropic S-curvature if

$$\mathbf{S} = (n+1)cF, \tag{3}$$

for some scalar function $c = c(x)$ on M.
Here, we are going to characterize the isotropic Berwald (α,β)-metrics of scalar flag curvature. More precisely, we prove the following.

Theorem 1. *Let $F := \alpha\phi(s)$ be an (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is not a Finsler metric of Randers type. Then F is of scaler flag curvature with isotropic S-curvature (3), if and only if it has isotropic Berwald curvature (1) with almost isotropic flag curvature (4). In this case, F must be locally Minkowskian.*

2. Preliminary

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of M and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle. A Finsler metric on M is a function $F : TM \to [0,\infty)$ which has the following properties:
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) for each $y \in T_xM$, the following quadratic form g_y on T_xM is positive definite,

$$g_y(u,v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y+su+tv)\right]|_{s,t=0}, \ \ u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\left[g_{y+tw}(u,v)\right]|_{t=0}, \ \ u,v,w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that **C=0** if and only if F is Riemannian.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i},$$

where $G^i(y)$ are local functions on TM given by

$$G^i := \frac{1}{4}g^{il}\left\{\frac{\partial^2[F^2]}{\partial x^k \partial y^l}y^k - \frac{\partial[F^2]}{\partial x^l}\right\}, \ \ y \in T_xM.$$

G is called the associated spray to (M, F). The projection of an integral curve of **G** is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to R$ by

$$\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i}|_x, \quad \mathbf{E}_y(u, v) := E_{jk}(y)u^jv^k,$$

where

$$B^i{}_{jkl}(y) := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}(y), \quad E_{jk}(y) := \frac{1}{2}B^m_{jkm}(y),$$

$u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. **B** and **E** are called the Berwald curvature and mean Berwald curvature, respectively. A Finsler metric is called a Berwald metric and mean Berwald metric if $\mathbf{B} = 0$ or $\mathbf{E} = 0$, respectively.

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^iv^jw^k\frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\}.$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y \in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric [6]. It is remarkable that, the notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics (see [2] and [3]).

The Riemann curvature $\mathbf{K}_y = K^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, where

$$K^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald [4]. For a flag $P = \text{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}.$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, and is just the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0. If $\mathbf{K} = constant$, then F is said to be of constant flag curvature. F is called of almost isotropic flag curvature if

$$\mathbf{K} = \frac{3c_{x^m}y^m}{F} + \sigma, \tag{4}$$

where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M.

Let $F := \alpha\phi(s)$, $s = \beta/\alpha$, be an (α,β)-metric on a manifold M, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Define $b_{i|j}$ by

$$b_{i|j}\theta^j := db_i - b_j\theta_i^j,$$

where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\Big[(y^i) \in R^n \;\Big|\; F\Big(y^i\frac{\partial}{\partial x^i}|_x\Big) < 1\Big]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$. $\mathbf{S}$ said to be isotropic if there is a scalar functions $c = c(x)$ on M such that $\mathbf{S} = (n+1)cF$.

Now, let $\phi = \phi(s)$ be a positive C^∞ function on $(-b_0, b_0)$. For an (α,β)-metric $F = \alpha\phi(s)$, put

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

For a number $b \in [0, b_0)$, let us define

$$\Phi := -(Q - sQ')[n\Delta + 1 + sQ] - (b^2 - s^2)(1 + sQ)Q''$$

where

$$\Delta := 1 + sQ + (b^2 - s^2)Q'.$$

Lemma 1. ([1]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Suppose that $\phi \neq c_1\sqrt{1 + c_2s^2} + c_3s$ for any constant $c_1 > 0$, c_2 and c_3. Then F is of isotropic S-curvature if and only if one of the following holds
(a) β satisfies

$$r_{ij} = 0, \quad s_j = 0 \tag{5}$$

In this case, $\mathbf{S} = 0$.
(b) β satisfies

$$r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j), \quad s_j = 0, \tag{6}$$

where $\varepsilon = \varepsilon(x)$ is a scalar function, $b := \|\beta_x\|_\alpha$ and $\phi = \phi(s)$ satisfies

$$\Phi = -2(n+1)k\frac{\phi\Delta^2}{b^2 - s^2}, \tag{7}$$

where k is a constant. In this case, $\mathbf{S} = (n+1)cF$ with $c = k\varepsilon$.

It is remarkable that, it prove that the condition $\Phi = 0$ characterizes the Riemannian metrics among (α, β)-metrics. Hence, in the continue, we suppose that $\Phi \neq 0$.

3. The Proof of the Theorem 1

In this section, we are going to prove the Theorem 1.

The Proof of the Theorem 1: Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a non-Randers type (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F has isotropic Berwald curvature (1) with almost isotropic flag curvature (4). By Theorem 1.1 in [13], every isotropic Berwald metric has isotropic S-curvature. Then, by Theorem 1.1 in [9], F has almost isotropic flag curvature (4).
Conversely, suppose that F is of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ with isotropic S-curvature (3). It is sufficient to prove that F is a isotropic Berwald metric or equivalently the Berwald curvature of F is given by

$$B^i{}_{jkl} = c\Big\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\Big\}, \tag{8}$$

where $c = c(x)$ is a scalar function on M. It is well-known that, F is a isotropic Berwald metric if and only if it is a Douglas metric $\mathbf{D} = 0$ with isotropic mean Berwald curvature

$$E_{ij} = \frac{n+1}{2}cF^{-1}h_{ij}$$

Since every isotropic Berwald metric has isotropic mean Berwald curvature, then it is sufficient to prove that F is a Douglas metric $\mathbf{D} = 0$. By assumption, F has isotropic S-curvature. According to the Lemma 1, we have two cases as follows:

Case 1: Let (5) hold, i.e., $r_{ij} = 0$ and $s_j = 0$. In this case, $\mathbf{S} = 0$. By Lemma 5 in [5], β must be a closed 1-form, i.e.,

$$s_{ij} = 0.$$

Then β is parallel with respect to α and in this case F reduces to a Berwald metric. By Theorem 4 in [5], F is locally Minkowskian.

Case 2: Let (6) hold, i.e., $r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j)$ and $s_j = 0$, where $\varepsilon = \varepsilon(x)$ is a scalar function on M. We show that F can not be a Douglas metric and then this case is not hold. In contrary, suppose that F is a Douglas metric with isotropic S-curvature or equivalently it is a isotropic Berwald metric. Then the Berwald curvature of F is given by

$$B^i{}_{jkl} = c\Big\{F_{y^jy^k}\delta^i{}_l + F_{y^ky^l}\delta^i{}_j + F_{y^ly^j}\delta^i{}_k + F_{y^jy^ky^l}y^i\Big\}, \tag{9}$$

where $c = c(x)$ is a scalar function on M. By Theorem 1.1 in [13], every isotropic Berwald metric of scalar flag curvature on a manifold M of dimension $n \geq 3$ is a Randers metric $F = \alpha + \beta$. This is a contradiction with our assumption. □

By Theorem 1, we get the following.

Corollary 1. *Let $F := \alpha\phi(s)$ be an (α, β)-metric on a manifold M of dimension $n \geq 3$. Suppose that F is not a Finsler metric of Randers type. Then F is a Finsler metric of scalar flag curvature with vanishing S-curvature if and only if the flag curvature $\mathbf{K} = 0$ and F is a Berwald metric. In this case, F is a locally Minkowski metric.*

Now, we study two dimensional (α, β)-metrics. Every isotropic Berwald metric has isotropic S-curvature [13]. In [15], Yang proved that every two dimensional (α, β)-metric has vanishing S-curvature. Then, by Szabó rigidity theorem for Berwald surface, we get the following.

Corollary 2. *Let $F := \alpha\phi(s)$ be an (α, β)-metric with isotropic Berwald curvature on a 2-dimensional manifold M. Suppose that F is not a Finsler metric of Randers type. Then F is locally Minkowskian.*

The author conjecture that the Corollary 2 might be extended for the case that $dim(M) \geq 3$. But, we have not been able to prove it yet.

Conjecture: Every non-Randers type (α, β)-metric with isotropic Berwald curvature is a Berwald metric.

REFERENCES

[1] S. Bácsó, X. Cheng and Z. Shen, *Curvature properties of (α, β)-metrics, In "Finsler Geometry"*, Sapporo 2005-In Memory of Makoto Matsumoto, ed. S. Sabau and H. Shimada, Advanced Studies in Pure Mathematics 48, Mathematical Society of Japan, (2007), 73-110.

[2] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type, A generalization of notion of Berwald space*, Publ. Math. Debrecen. **51**(1997), 385-406.

[3] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type II, Projectively flat spaces*, Publ. Math. Debrecen. **53**(1998), 423-438.

[4] L. Berwald, *Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung*, Jber. Deutsch. Math.-Verein., **34**(1926), 213-220.

[5] X. Cheng, *On (α, β)-metrics of scalar flag curvature with constant S-curvature*, Acta Mathematica Sinica, English Series, **26**(9) (2010), 1701-1708.

[6] J. Douglas, *The general geometry of path*, Ann. Math. **29**(1927-28), 143-168.

[7] E. Guo, H. Liu and X. Mo, *On spherically symmetric Finsler metrics with isotropic Berwald curvature*, Int. J. Geom. Meth. Mod. Phys. **10**(10) (2013), 1350054 (13 pages).

[8] M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31**(1992), 43-84.

[9] B. Najafi, Z. Shen and A. Tayebi, *Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties*, Geom. Dedicata. **131**(2008), 87-97.

[10] Z. Shen, *Volume comparison and its applications in Riemann-Finsler geometry*, Advances. Math. **128** (1997), 306-328.

[11] Z. Shen, *On projectively flat (α, β)-metrics*, Canadian Mathematical Bulletin, **52**(1)(2009), 132-144.

[12] Z. Shen, *On a class of Landsberg metrics in Finsler Geometry*, Canadian Journal of Mathematics, **61**(6) (2009), 1357-1374.

[13] A. Tayebi and B. Najafi, *On isotropic Berwald metrics*, Ann. Polon. Math. **103**(2012), 109-121.

[14] A. Tayebi and M. Rafie. Rad, *S-curvature of isotropic Berwald metrics*, Sci. China. Series A: Mathematics. **51**(2008), 2198-2204.

[15] G. Yang, *A note on a class of Finsler metrics of isotropic S-curvature*, preprint. http://arxiv.org/abs/1310.3463.

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