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INGARDEN-TAMASSY CHANGE OF m-TH ROOT FINSLER METRICS

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ABSTRACT. In this paper, A new change of the Finsler metrics which is named Ingarden-Tamássy change, in the class of m-th root metrics has been presented. Locally projectively flat Finsler metrics obtained by Ingarden-Tamássy change of m-th root metrics have been characterized. Finally a necessary and sufficient condition under which these metrics are locally dually flat has been proved.

Keywords: Locally dually flat metric, projectively flat metric, m-th root metric.[1]

1. INTRODUCTION

Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. We put $F = \sqrt[m]{A}$, where $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices. Then F is called an m-th root Finsler metric. Recent works show that the theory of m-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory (see [1][2][3][6][9][10][11][14][16][17][18][19][20]). For quartic metrics, a study of the geodesics and of the related geometrical objects is made by S. Lebedev [7]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and N. Brinzei in [4], [5]. Tensorial connections for such spaces have been recently studied by L. Tamassy [15]. B. Li and Z. Shen study locally projectively flat fourth root metrics under some irreducibility condition [8]. Y. Yu and Y. You show that an m-th root Einstein Finsler metric is Ricci-flat [26].

In [16], Tayebi-Najafi characterize locally dually flat and Antonelli m-th root Finsler metrics. They show that every m-th root Finsler metric of isotropic mean Berwald curvature reduces to a weakly Berwald metric. In [17], they prove that every m-th root Finsler metric of isotropic Landsberg metric reduces to a Landsberg metric. Then, they show that every m-th root Finsler metric with almost vanishing H-curvature satisfies $\mathbf{H} = 0$. Recently, Tayebi-Nankali-Peyghan define some non-Riemannian curvature properties for Cartan spaces and consider Cartan space with the m-th root metric [18]. They prove that every m-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively. In [20], Tayebi-Peyghan-Shahbazi Nia characterize locally dually flat generalized m-th root Finsler metrics. They find a condition under which a generalized m-th root metric is projectively related to a m-th root metric.

In [19], Tayebi-Nankali-Peyghan find necessary and sufficient condition under which conformal β-change of an m-th root metric be locally dually flat. In [25], Tayebi-Tabatabaeifar-Peyghan consider Kropina change of m-th root Finsler metrics and find necessary and sufficient condition under which the Kropina change of an m-th root Finsler metric be locally dually flat. For the Kropina metric and its properties, see [23]. In [24], Tayebi-Shahbazi Nia-Peyghan consider Randers change of m-th root Finsler metrics and find necessary and sufficient condition under which the Randers change of an m-th root metric be locally dually flat or locally projectively flat. This motivates us to consider the other changes of m-th root metrics.

Let (M, F) and $(M, \bar{F})$ are Finsler manifolds, where fundamental metric function $\bar{F}$ is obtained from F by the relation $F(x, y) \to \bar{F}(x, y) = f(F, \beta)$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form on M and $f = f(F, \beta)$ is a positively homogeneous function of F and β of degree one. This change of Finsler metric function has been called a β-change. If $||\beta||_F := \sup_{F(x,y)=1} |\beta| < 1$, then $\bar{F}$ is again a Finsler metric.

There is a special case of β-change, namely

$$\bar{F}(x, y) = F + \frac{\beta^2}{F}, \tag{1}$$

where $\beta(x, y) = b_i(x)y^i$ is a non-zero 1-form on a manifold M. If F reduces to a Riemannian metric α, then $\bar{F}$ reduces to the Ingarden-Tamássy metric $F = \alpha + \frac{\beta^2}{\alpha}$. Due to this reason, the transformation (1) is called the *Ingarden-Tamássy change* of Finsler metrics. The Ingarden-Tamássy metric is an important metric in Finsler geometry.

A Finsler metric is said to be locally projectively flat if at any point there is a local coordinate system in which the geodesics are straight lines as point sets. A Finsler metric F on an open domain $U \subset \mathbb{R}^n$ is locally projectively flat if and only if $G^i = Py^i$, where $P = P(x, y)$ is a scalar function on TM_0 satisfying $P(x, \lambda y) = \lambda P(x, y)$ for all $\lambda > 0$.

Let F be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Put

$$A_i := \frac{\partial A}{\partial y^i}, \quad \text{and} \quad A_{ij} := \frac{\partial^2 A}{\partial y^i \partial y^j}.$$

Suppose that the matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold

$$y^i A_i = mA, \;\; y^i A_{ij} = (m-1)A_j, \;\; y_i = \frac{1}{m} A^{\frac{2}{m}-1} A_i, \quad A^{ij} A_{jk} = \delta^i_k,$$
$$A^{ij} A_i = \frac{1}{m-1} y^j, \;\; A_i A_j A^{ij} = \frac{m}{m-1} A, \quad A_0 := A_{x^m} y^m, \quad A_{0l} := A_{x^m y^l} y^m.$$

Let us put

$$\beta_{x^l} := \frac{\partial \beta}{\partial x^l} = (b_i)_{x^l} y^i, \quad \beta_0 := \beta_{x^l} y^i, \quad \beta_{0l} := \beta_{x^k y^l} y^k, \quad \beta_l := \frac{\partial \beta}{\partial y^l} = b_l.$$

Then, we prove the following.

Theorem 1.1. *Let $F = \sqrt[m]{A}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that $\bar{F} = F + \beta^2/F$ be the Ingarden-Tamássy change of F, where β is a non-zero 1-form on M. Then $\bar{F}$ is locally projectively flat if and only if*

$$\beta(\beta_{x^l} - \beta_{0l}) = b_l \beta_0 \tag{2}$$

and one of the following holds:

: *(i)* $\beta A_l = mAb_l$;

: *(ii) F reduces to a Berwald-Moór metric* $F = \sqrt[m]{y^{i_1}y^{i_2}\dots y^{i_m}}$.

A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the form $G^i = -\frac{1}{2}g^{ij}H_{y^j}$, where $H = H(x,y)$ is a positively homogeneous scalar function on $TM_0 = TM \setminus \{0\}$ (see [12][21][22]). If F is a locally dually flat metric, then at any point there is a standard coordinate system (x^i, y^i) in TM such that $L = F^2(x,y)$ satisfies

$$L_{x^k y^l}y^k = 2L_{x^l}.$$

It is easy to see that every locally Minkowskian metric satisfies in the above equation, hence is locally dually flat.

Theorem 1.2. *Let* $F = \sqrt[m]{A}$ $(m > 2)$, *be an m-th root Finsler metric on an open subset* $U \subset \mathbb{R}^n$. *Suppose that* $\bar{F} = F + \beta^2/F$ *be the Ingarden-Tamásy change of F, where* β *is a non-zero 1-form on M. Then* $\bar{F}$ *is a locally dually flat Finsler metric if and only if* $b_i = constant$ *and the following holds*

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big]. \tag{3}$$

2. Proof of the Theorem 1.1

Let F be a Finsler metric on a manifold M. The geodesics of F are characterized locally by the equations $\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$, where

$$G^i = \frac{1}{4}g^{ik}\left\{2\frac{\partial g_{pk}}{\partial x^q} - \frac{\partial g_{pq}}{\partial x^k}\right\}y^p y^q$$

are coefficients of the spray associated with F. A Finsler metric $F(x,y)$ on an open domain $U \subset \mathbb{R}^n$ is said to be locally projectively flat if its geodesic coefficients G^i are in the form

$$G^i(x,y) = P(x,y)y^i,$$

where $P : TU = U \times \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous with degree one, $P(x, \lambda y) = \lambda P(x,y)$, $\lambda > 0$. We call $P(x,y)$ the projective factor of F. In this section, we are going to prove the Theorem 1.1. The following lemma plays an important role.

Lemma 2.1. ([13]) Let $F(x,y)$ be a Finsler metric on an open subset $U \subset \mathbb{R}^n$. $F(x,y)$ is projective on U if and only if it satisfies

$$F_{x^k y^l}y^k = F_{x^l}. \tag{4}$$

In this case, the projective factor $P(x,y)$ is given by $P = \frac{F_{x^k}y^k}{2F}$.

To prove the Theorem 1.1, we need the following.

Lemma 2.2. *Let $F = A^{\frac{1}{m}}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the equation*

$$\Omega A^{\frac{-1}{m}} + \Phi A^{\frac{1}{m}-2} + \Xi A^{\frac{-1}{m}-2} = 0$$

holds, where Φ, Ω, Ξ are polynomials in y. Then

$$\Omega = \Phi = \Xi = 0.$$

Proof of the Theorem 1.1: The following hold

$$\bar{F}_{x^k} = \frac{1}{m} A_{x^k} A^{\frac{1}{m}-1} + 2\beta\beta_{x^k} A^{\frac{-1}{m}} - \frac{-1}{m} A_{x^k} \beta^2 A^{\frac{-1}{m}-1}, \tag{5}$$

$$\begin{aligned} \bar{F}_{x^k y^l} y^k = & \frac{1}{m} A_{0l} A^{\frac{1}{m}-1} + \frac{1}{m}\Big(\frac{1}{m} - 1\Big) A^{\frac{1}{m}-2} A_0 A_l + 2\beta\beta_{0l} A^{\frac{-1}{m}} \\ & + 2\beta_0 b_l A^{\frac{-1}{m}} - \frac{2}{m} A_l \beta_0 \beta A^{\frac{-1}{m}-1} + \frac{1}{m}\Big(\frac{1}{m} + 1\Big) A_0 A_l \beta^2 A^{\frac{-1}{m}-2} \\ & - \frac{1}{m} A_{0l} \beta^2 A^{\frac{-1}{m}-1} - 2\frac{1}{m} \beta b_l A_0 A^{\frac{-1}{m}-1}. \end{aligned} \tag{6}$$

Since $\bar{F}$ is projectively flat, then by the Lemma 2.1 we have

$$\bar{F}_{x^k y^l} y^k = \bar{F}_{x^l}. \tag{7}$$

By (5), (6) and (7), we get

$$\begin{aligned} & 2\Big[\beta\big(\beta_{0l} - \beta_{x^l}\big) + b_l \beta_0\Big] A^{\frac{-1}{m}} + \frac{1}{m}\Big[\Big(\frac{1}{m} - 1\Big) A_0 A_l + A A_{0l} - A A_{x^l}\Big] A^{\frac{1}{m}-2} \\ & -\frac{1}{m}\Big[\Big(\frac{-1}{m} - 1\Big) A_0 A_l \beta^2 + \beta^2 A\Big(A_{0l} - A_{x^l}\Big) + 2 A A_0 \beta b_l\Big] A^{\frac{-1}{m}-2} = 0. \end{aligned} \tag{8}$$

By the Lemma 2.2, (8) reduces to following

$$\beta(\beta_{0l} - \beta_{x^l}) = -b_l \beta_0, \tag{9}$$

$$m A (A_{0l} - A_{x^l}) = (m-1) A_0 A_l, \tag{10}$$

$$\Big(\frac{-1}{m} - 1\Big) A_0 A_l \beta^2 + \beta^2 A (A_{0l} - A_{x^l}) + 2 A A_0 \beta b_l = 0. \tag{11}$$

By (10) and (11), we obtain

$$\beta A_0 (\beta A_l - m A b_l) = 0. \tag{12}$$

Since $\beta \neq 0$, then by (12) we have two cases as follows:
(i) $A_0 = A_{x^l} y^l = 0$, or equivalently

$$\frac{\partial}{\partial x^l}\Big(a_{i_1 \ldots i_m}(x)\Big) = 0$$

that yields

$$a_{i_1 \ldots i_m} = constant.$$

Then

$$F = \sqrt[m]{y^{i_1} y^{i_2} \ldots y^{i_m}}$$

In this case, F reduces to a Berwald-Moór metric;

(ii) $\beta A_l = mAb_l$. This completes the proof. □

3. THE PROOF OF THE THEOREM 1.2

A Finsler metric $F = F(x,y)$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system (x^i, y^i) in TM such that $L = F^2(x,y)$ satisfies

$$L_{x^k y^l} y^k = 2L_{x^l}.$$

In this case, the coordinate (x^i) is called an adapted local coordinate system. It is easy to see that every locally Minkowskian metric satisfies in the above equation, hence is locally dually flat. In this section, we are going to prove the Theorem 1.2. To prove it, we need the following.

Lemma 3.1. *Let $F = A^{\frac{1}{m}}$ $(m > 2)$, be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$. Suppose that the equation*

$$\Xi A^{\frac{-2}{m}-2} + \Omega A^{\frac{-2}{m}} + \Phi A^{\frac{2}{m}-2} + \Upsilon = 0$$

holds, where Φ, Ω, Ξ are polynomials in y. Then

$$\Xi = \Omega = \Phi = \Upsilon = 0.$$

The Proof of Theorem 1.2: Let $\bar{F} = F + \beta^2/F$ be the Ingarden-Tamássy change of $F = A^{\frac{1}{m}}$, where β is a non-zero 1-form on M. The following hold

$$\begin{aligned} \bar{F}^2 =& \; A^{\frac{2}{m}} + 2\beta^2 + \beta^4 A^{\frac{-2}{m}}, \\ (\bar{F}^2)_{x^k} =& \; \frac{2}{m} A^{\frac{2}{m}-1} A_{x^k} + 4\beta\beta_{x^k} + 4\beta^3 \beta_{x^k} A^{\frac{-2}{m}} - \frac{2}{m} A^{\frac{-2}{m}-1} A_{x^k} \beta^4, \end{aligned} \tag{13}$$

$$\begin{aligned} (\bar{F}^2)_{x^k y^l} y^k =& \; \frac{2}{m} A^{\frac{2}{m}-1} A_{0l} + \frac{2}{m}\Big(\frac{2}{m} - 1\Big) A^{\frac{2}{m}-2} A_0 A_l + 4b_l\beta_0 + 4\beta\beta_{0l} + 12\beta^2\beta_0 b_l A^{\frac{-2}{m}} \\ & + 4\beta^3 \beta_{0l} A^{\frac{-2}{m}} - \frac{8}{m}\beta^3\beta_0 A_l A^{\frac{-2}{m}-1} - \frac{8}{m}\beta^3 b_l A_0 A^{\frac{-2}{m}-1} - \frac{2}{m}\beta^4 A_{0l} A^{\frac{-2}{m}-1} \\ & - \frac{2}{m}\Big(\frac{-2}{m} - 1\Big) A^{\frac{-2}{m}-2} A_0 A_l \beta^4. \end{aligned} \tag{14}$$

By assumptions, $\bar{F}$ is locally dually flat metric. Then

$$(\bar{F}^2)_{x^k y^l} y^k = 2(\bar{F}^2)_{x^l}. \tag{15}$$

From (13), (14) and (15), we obtain

$$\begin{aligned} & -\frac{2}{m}\left[\beta^4\Big[\Big(\frac{-2}{m} - 1\Big) A_0 A_l + A(A_{0l} - 2A_{x^l})\Big] + 4A\beta^3(A_0\beta_l + A_l\beta_0)\right] A^{\frac{-2}{m}-2} \\ & 4\beta^2\left[\beta(\beta_{0l} - 2\beta_{x^l}) + 3\beta_l\beta_0\right] A^{\frac{-2}{m}} + \frac{2}{m}\left[\Big(\frac{2}{m} - 1\Big) A_0 A_l + AA_{0l} - 2AA_{x^l}\right] A^{\frac{2}{m}-2} \\ & + 4\beta(\beta_{0l} - 2\beta_{x^l}) + 4\beta_l\beta_0 = 0. \end{aligned} \tag{16}$$

According to the Lemma 3.1, (16) reduces to following

$$mA(A_{0l} - 2A_{x^l}) = (m-2)A_0 A_l, \tag{17}$$
$$\beta(\beta_{0l} - 2\beta_{x^l}) = -3\beta_l\beta_0, \tag{18}$$
$$\beta\Big[mA(A_{0l} - 2A_{x^l}) - (m+2)A_0A_l\Big] + 4mA(A_0\beta_l + A_l\beta_0) = 0, \tag{19}$$
$$\beta(\beta_{0l} - 2\beta_{x^l}) = -\beta_l\beta_0. \tag{20}$$

By (17), irreducibility of A and $deg(A_l) = m-1$, imply that there exists a 1-form $\theta = \theta_l y^l$ on U such that

$$A_0 = \theta A. \tag{21}$$

Taking a vertical derivative of (21) with respect y^l, implies that

$$A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \tag{22}$$

Substituting (21) and (22) in (17) yields (3).
On the other hand, by plugging (17) in (19), we obtain

$$mA(\beta_0 A_l + A_0\beta_l) = \beta A_0 A_l, \tag{23}$$

It follows that (by (21) and (23)),

$$\beta_0 = 0.$$

Thus

$$b_i = constant.$$

Putting (21) in (23) yields

$$m\beta_0 A_l = \theta(\beta A_l - mA\beta_l). \tag{24}$$

Now from (18) and (20), we get

$$\beta_0\beta_l = 0 \tag{25}$$

Then by (25),

$$\beta_l = 0, \quad or \quad \beta_0 = 0.$$

In the first case, we get

$$\beta = 0$$

which contradicts with assumptions. Then

$$\frac{\partial b_i}{\partial x^l} y^l y^i = 0,$$

which implies that b_i are constants. Thus in any case, b_i are constants. This completes the proof. □

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