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THE INSCRIBED SQUARE OF THE ARBELOS

HIROSHI OKUMURA

ABSTRACT. We consider an inscribed square of the arbelos, which is determined uniquely and gives several Archimedean circles of the arbelos.

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1. INTRODUCTION

The arbelos is a plane figure surrounded by three mutually touching semicircles which have collinear centers and are elected on the same side of the line passing through the centers of the semicircles. Circles or semicircles having common radius equal to the half the harmonic mean of the radii of the two inner semicircles are said to be Archimedean, which are one of the main topics on the arbelos.
In this article we consider an inscribed square of the arbelos, one of whose vertices coincides with the point of tangency of the two inner semicircles. The square is determined uniquely and is constructed in a simple way and yields several Archimedean circles of the arbelos.

2. THE INSCRIBED SQUARE OF THE ARBELOS

In this section we construct the inscribed square of the arbelos, and consider several properties related to this. Let O be a point on the segment AB, and let us consider an arbelos formed by the three semicircles with diameters AO, BO and AB, which are denoted by α, β and γ, respectively. Let $a = |AO|/2$ and $b = |BO|/2$. Circles or semicircles of radius $ab/(a+b)$ are said to be Archimedean, and the common radius is denoted by r_{A}. We use a rectangular coordinate system with origin O such that the points A and B have coordinates $(2a, 0)$ and $(-2b, 0)$ respectively, where we assume that the semicircles α, β and γ are constructed in the region $y > 0$. For two points P and Q, $P(Q)$ denotes the circle with center P passing through Q, and (PQ) denotes the circle with a diameter PQ. However if their centers lie on the line AB, we consider them as semicircles with diameters lying on the line AB and constructed in the region $y > 0$. The center of a circle or a semicircle δ is denoted by O_δ. We call the radical axis of α and β the axis of the arbelos, which overlaps with the y-axis.

Proposition 2.1. *For a point Q on the semicircle γ, if P is the point of intersection of AQ and α and R is the point of intersection of BQ and β, then $OPQR$ is a square if and only if $|AQ|/|BQ| = a/b$.*

Proof. Since $OPQR$ is a rectangle, it is a square if and only if $|PQ| = |RQ|$. But $|PQ| = |AQ| - |AP| = |AQ| - |AQ|a/(a+b) = |AQ|b/(a+b)$. Similarly $|QR| = |BQ|a/(a+b)$. Therefore $OPQR$ is a square if and only if $b|AQ| = a|BQ|$.
Since the right triangle AQB is uniquely determined by the value $|AQ|/|BQ|$, the proposition shows that the square $OPQR$ is determined uniquely for the arbelos.
Let $\mathcal{E}$ be the external common tangent of α and β. If $a \neq b$, let E be the point of intersection of the lines AB and $\mathcal{E}$, and let $\delta = E(O)$ and $\varepsilon = (EO)$ (see Figure 1). If $a = b$, let δ and ε be the axis. In any case, let Q be the point of intersection of γ and δ, and let C be the point of intersection of $(O_\alpha O_\beta)$ and ε. Let $j = ab/(a^2+b^2)$. Let us assume $a \neq b$, and let $q = ab/(b-a)$. The point E is the external center of similitude of α and β and divides the segment $O_\alpha O_\beta$ in the ratio $a : b$ externally. Therefore E has coordinates $(2q, 0)$. The semicircles γ and δ are expressed by the equations $(x-2a)(x+2b)+y^2 = 0$ and $(x-4q)x+y^2 = 0$, respectively. Hence the point Q has coordinates

$$(2j(b-a), 2j(a+b)). \tag{2.1}$$

Notice that the coordinates of Q is also expressed by (2.1) in the case $a = b$.

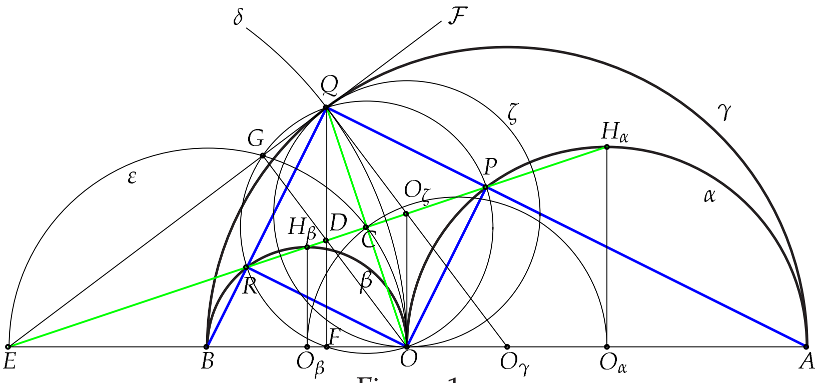

Figure 1.

Let P be the point of intersection of α and the segment AQ, and let R be the point of intersection of β and the segment BQ. The points P and R have coordinates

$$(2jb, 2ja) \text{ and } (-2ja, 2jb), \tag{2.2}$$

respectively by (2.1). Let H_α (resp. H_β) be the farthest point on α (resp. β) from AB. The circle touching AB at O and also touching γ internally has radius $2r_A$ [4], which is denoted by ζ. Let F be the foot of perpendicular from Q to AB, and let D be the point of intersection of the lines PR and QF, and let G be the reflection of F in the line PR. Let $\mathcal{F}$ be the tangent of γ at Q.

Theorem 1. *The following statements hold.*
(i) *$OPQR$ is a square with center C.*
(ii) *The point C divides the segment $H_\alpha H_\beta$ in the ratio $a^2 : b^2$ internally.*
(iii) *The inscribed circle of the arbelos and the circle ζ and the line $\mathcal{F}$ touch γ at the point Q. If $a \neq b$, then $\mathcal{F}$ passes through the point E.*
(iv) *The lines AB and $\mathcal{F}$ are symmetric in the line PR.*
(v) *If $a \neq b$, the points O_ζ and E are harmonic conjugates with respect to H_α and H_β. Also the points D and E are harmonic conjugates with respect to P and R.*

(vi) *The circumscribed circle of* $OPQR$*, the semicircle* (*or line*) ε *and the line* $\mathcal{F}$ *intersect at the point* G*. The point* F *lies on the circumscribed circle of* $OPQR$*.*

Proof. From (2.2), we get $|OP| = |OR| = 2j\sqrt{a^2+b^2}$. One the other hand the homothety with center O and ratio $1/2$ carries γ and δ into $(O_\alpha O_\beta)$ and ε, respectively. Therefore C is the midpoint of OQ. This proves (i). Since the point C has coordinates $(j(b-a), j(a+b))$ by (2.1),

$$C = \frac{b^2}{a^2+b^2}H_\alpha + \frac{a^2}{a^2+b^2}H_\beta \tag{2.3}$$

holds. This proves (ii). The part (iii) is true in the case $a = b$. Let us assume $a \neq b$. If we consider δ as a complete circle, the inversion in δ fixes the semicircle γ, the line AB and the point O. Hence it also fixes the circle ζ. But Q is the unique fixed point on γ by the inversion. Therefore γ and ζ touch at Q. The rest of (iii) is proved similarly. The part (iv) follows from the fact that the lines $\mathcal{F}$ and AB are the tangents of ζ from E in the case $a \neq b$. The point O_ζ has coordinates $(0, 2r_\mathrm{A})$, and we get

$$O_\zeta = \frac{b}{a+b}H_\alpha + \frac{a}{a+b}H_\beta.$$

Since E divides $H_\alpha H_\beta$ in the ratio $a : b$ externally, the first half of (v) is proved. Let D' be the point dividing PR in the ratio $a : b$ internally. Then D' has coordinates

$$(2j(b-a), 4jr_\mathrm{A}) \tag{2.4}$$

by (2.2). Therefore D' lies on QF, i.e., $D' = D$. Since E divides PR in the ratio $a : b$ externally, (v) is proved. The part (vi) is true if $a = b$. Let us assume $a \neq b$. Let G' be the point of intersection of ε and the circumscribed circle of $OPQR$. Since $\angle OG'E$ and $\angle OG'Q$ are right angles, G' lies on $EQ = \mathcal{F}$. But EOQ is an isosceles triangle with $|EO| = |EQ|$, i.e., the triangles FQO and $G'OQ$ are congruent. Therefore G' is the reflection of F in PR, i.e., $G' = G$. The rest of (vi) is obvious.
The points P, R, C, D, O_ζ, H_α and H_β are collinear by the theorem. Hence the square $OPQR$ can easily be constructed from the points H_α and H_β, i.e., the line $H_\alpha H_\beta$ intersects α and β at the points P and Q, respectively. The fact that the square is constructed from the point of tangency of γ and the inscribed circle of the arbelos is mentioned at the Web site [2].

3. Quadruplets of Archimedean circles

In this section we consider two quadruplets of Archimedean circles passing through the point C, which are obtained from the square $OPQR$. Let P' (resp. R') be the point of intersection of the line OQ and the line passing through P (resp. R) parallel to AB (see Figure 2).

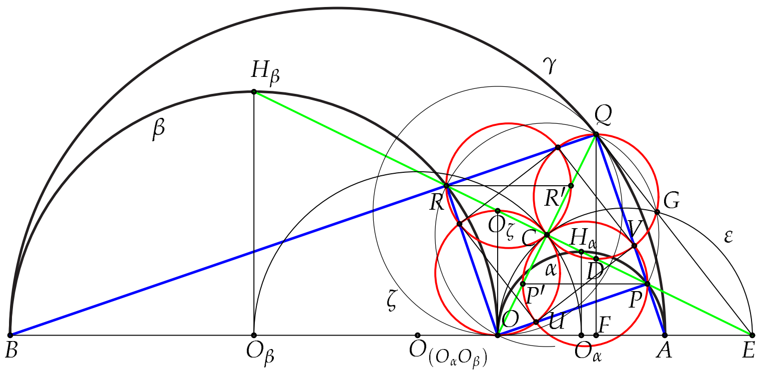

Figure 2.

Theorem 2. *The following statements are true.*
(i) *The circles (OO_ζ), (PP'), (QD) and (RR') are Archimedean.*
(ii) *The circles (OO_ζ), (QD) and the semicircle $(O_\alpha O_\beta)$ touch at the point C, also the circles (PP'), (RR') and the semicircle (or line) ε touch at C. The former three are orthogonal to the latter.*
(iii) *The circle (QD) passes through the point G.*
(iv) *The point of intersection of (OO_ζ) and (PP') different from C divides the segment OP in the ratio $a : b$ internally. Similar facts are also true for the other pairs of the Archimedean circles passing through C and the corresponding sides of $OPQR$.*
(v) *The points of intersection of (OO_ζ), (PP'), (QD) and (RR') different from C form vertices of a square of side length $2r_A$.*

Proof. Since the circle ζ has radius $2r_A$ and $\angle OCO_\zeta$ is a right angle, the circle (OO_ζ) is Archimedean and passes through C. Rotating (OO_ζ) about C through $90°$, $180°$ and $270°$, we get the Archimedean circles (PP'), (QD) and (RR'). This proves (i). The part (ii) holds in the case $a = b$. Let us assume $a \neq b$. The point C is the homothety center of the triangles CPP' and CEO, i.e., it is the homothety center of ε and the upper half part of (PP'). Hence C is the point of tangency of (PP') and ε. But (PP') and (RR') also touch at C. Therefore (PP'), (RR') and ε touch at C. On the other hand, $|O_\varepsilon O_\alpha||O_\varepsilon O_\beta| = |(q-a)(q+b)| = q^2$, i.e., the point O_β is the inverse of O_α in the circle ε. Therefore $(O_\alpha O_\beta)$ and ε are orthogonal. While $(O_\alpha O_\beta)$ and ε intersect at C. Hence the rest of (ii) is now obvious. The reflection of the segment DF in PR is DG. Therefore $\angle DGQ$ is a right angle. This proves (iii). The circles (OO_ζ) and (PP') are expressed by the equations $x^2 + (y - 2r_A)y = 0$ and $(x - (2jb - r_A))^2 + (y - 2ja)^2 = r_A^2$ by (2.2). Let U be the point of intersection of (OO_ζ) and (PP') different from C. Then U has coordinates $(2r_A j, 2r_A ja/b)$, i.e., $U = (b/(a+b))O + (a/(a+b))P$. This proves (iv). Let V be the point of intersection of (QD) and (PP') different from C. Then V is obtained by rotating U through $90°$ about C. Hence UV is a diameter of (PP'). This proves (v).
For a point S on γ distinct from A and B, if U is the point of intersection of α and the segment AS and the line parallel to AB passing through U intersects OS at a point T, then the circle (TU) is Archimedean [3]. Therefore the Archimedean circle (PP') (as well as (RR')) is also obtained by this fact. The circle (OO_ζ) coincides with the Bankoff triplet circle, which is denoted by W_3 in [1].
The square $OPQR$ gives one more quadruplet of Archimedean circles (see Figure 3).

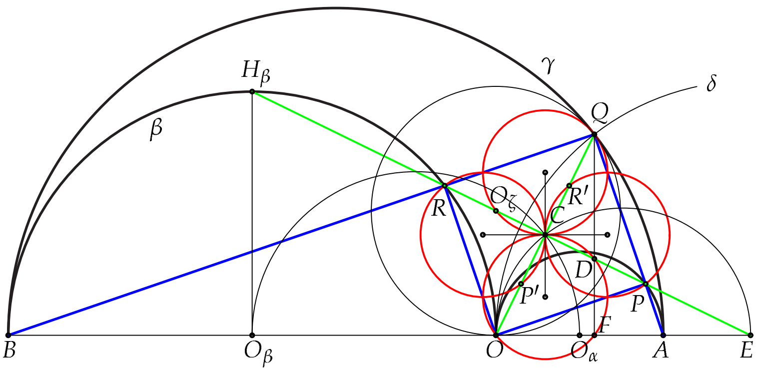

Figure 3.

Theorem 3. *The following statements are true.*
(i) *The circles (QO_ζ), (PR'), (OD) and (RP') are Archimedean.*
(ii) *The circles (QO_ζ) and (OD) touch at C and the segment joining their centers is perpendicular to AB. The circles (PR') and (RP') touch at C and the segment joining their centers is parallel to AB.*
(iii) *The circle (OD) passes through the point F.*
(iv) *The point of intersection of (QO_ζ) and (PR') different from C divides the segment QP in the ratio $a : b$ internally. Similar facts are also true for the other pairs of the circles (QO_ζ), (PR'), (OD) and (RP') and the corresponding sides of $OPQR$.*
(v) *The points of intersections of (QO_ζ), (PR'), (OD) and (RP') different from C form vertices of a square of side length $2r_{\mathrm{A}}$.*

Proof. Since Q and R' are reflections of O and P' in the line PR, respectively, the part (i) follows from (i) of Theorem 2. The centers of (OO_ζ) and (QO_ζ) are symmetric in PR. Hence $CO_{(OO_\zeta)}O_\zeta O_{(QO_\zeta)}$ is a rhombus, while the segment $O_{(OO_\zeta)}O_\zeta$ is perpendicular to AB. Therefore $CO_{(QO_\zeta)}$ is perpendicular to AB. This proves the first part of (ii). The rest of (ii) is now obvious. The rest of the theorem follows from (iii), (iv) and (v) of Theorem 2.

4. Some other Archimedean circles

In this section we consider some other Archimedean circles related to the square $OPQR$ (see Figure 4). Let $\mathcal{C}_1^\alpha$ be the circle touching AB at the point O_α passing through the midpoint of AQ. Similarly the circle $\mathcal{C}_1^\beta$ is defined. Let $\mathcal{C}_2$ be the circle touching AB at the point F and passing through C. If $a \neq b$, let H be the point of intersection of PR and the perpendicular to AB passing through O_ε.

Theorem 4. *The following statements hold.*
(i) *The circle $\mathcal{C}_1^\alpha$ is Archimedean, and one of the endpoints of its diameter parallel to AB lies on AQ, which divides AQ in the ratio $(a + b - 2r_{\mathrm{A}}) : (a + b + 2r_{\mathrm{A}})$ internally. Similar facts also hold for $\mathcal{C}_1^\beta$ and BQ.*
(ii) *The circle $\mathcal{C}_2$ is Archimedean.*
(iii) *If $a \neq b$, then H is the midpoint of EO_ζ, and the semicircle $O_\varepsilon(H)$ is Archimedean and touches the line $\mathcal{E}$ and intersects PR at H and the midpoint of DE. The circle $H(O_\varepsilon)$ is Archimedean and touches EQ at its midpoint. Also the circle with center at the midpoint of EQ and passing through the points of intersection of PR and $O_\varepsilon(H)$ is Archimedean.*

(iv) *If $a \neq b$, the points H and C are harmonic conjugates with respect to H_α and H_β.*

Proof. Let M be the midpoint of AQ. Then $MCOO_\alpha$ is a parallelogram. Therefore the translation $O \longmapsto O_\alpha$ carries C into M. Hence it carries (OO_ζ) into $\mathcal{C}_1^\alpha$, i.e., $\mathcal{C}_1^\alpha$ is Archimedean. While we get

$$(a + r_A, r_A) = \frac{a + b + 2r_A}{2(a + b)} A + \frac{a + b - 2r_A}{2(a + b)} Q.$$

This proves (i). The circle $\mathcal{C}_2$ is the reflection of (OO_ζ) in the perpendicular to AB from C. This proves (ii). Let us assume $a \neq b$. Since the distance between O and $\mathcal{E}$ is $2r_A$ [1], the semicircle $O(O_\zeta)$ of radius $2r_A$ touches $\mathcal{E}$, and intersects PR at O_ζ and D. But the homothety with center E and ratio $1/2$ carries $O(O_\zeta)$ into $O_\varepsilon(H)$. Therefore H is the midpoint of EO_ζ, and $O_\varepsilon(H)$ is Archimedean and touches $\mathcal{E}$ and intersects PR at H and the midpoint of DE. The rest of (iii) is obvious. The coordinates of H are (q, r_A) by (iii), and we get

$$H = \frac{-b^2}{a^2 - b^2} H_\alpha + \frac{a^2}{a^2 - b^2} H_\beta.$$

This proves (iv) by (2.3).

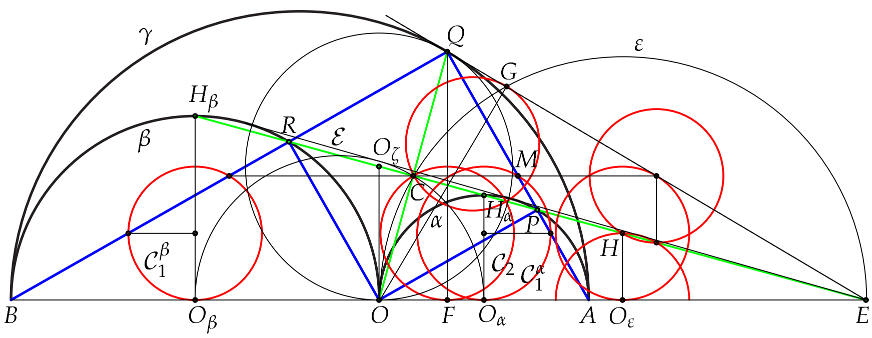

Figure 4.

5. Special case

In section 3, we have considered two quadruplets of Archimedean circles passing through the point C. In this section we consider the case in which the centers of the eight Archimedean circles form vertices of a regular octagon (see Figure 5). Let J be the point of intersection of the lines BQ and $H_\beta O_\beta$.

Theorem 5. *If $a < b$, the following statements are equivalent.*
(i) *The centers of the eight circles of the two quadruplets of Archimedean circles form vertices of a regular octagon.*
(ii) *The points F and O_α coincide.*
(iii) *The points D and H_α coincide.*
(iv) *The semicircle (AE) is Archimedean.*
(v) *The circle (JH_β) is Archimedean.*
(vi) *The point R lies on the semicircle $(O_\alpha O_\beta)$.*
(vii) *The semicircles γ and δ are congruent.*
(viii) *The distance between P and AB equals r_A.*
(ix) *The distance between R and the axis equals r_A.*

(xx) $b = \left(1 + \sqrt{2}\right) a$.

Proof. By (ii) of Theorem 3, the part (i) holds if and only if the circles (PP') and (OO_ζ) are symmetric in the perpendicular from C to AB. Therefore (i) and (viii) are equivalent. Since (viii) holds if and only if $2ja = r_{\mathrm{A}}$, (viii) is also equivalent to (xx). Similarly we can show that each of the other parts is equivalent to (xx), where recall that D has coordinates (2.4) to consider (iii).
The proof of the following facts is also accomplished by the straightforward calculation. Let us assume (i). Then R is the midpoint of JQ and DH_β. Hence the triangles RQD and RJH_β are congruent isosceles triangles. Therefore the circles (QD) and (JH_β) are symmetric in the perpendicular from R to AB. While R is the farthest point on $(O_\alpha O_\beta)$ from AB, and (QD) touches $(O_\alpha O_\beta)$ at the point C by (ii) of Theorem 2, and C is the midpoint of EH_β. Therefore (JH_β) also touches $(O_\alpha O_\beta)$ at the midpoint of BQ. Reflecting the circle (OD) in the perpendicular to AB from R, we get the Archimedean circle passing through the midpoint of BQ and the points J and O_β. The circle is also the reflection of (JH_β) in BQ.

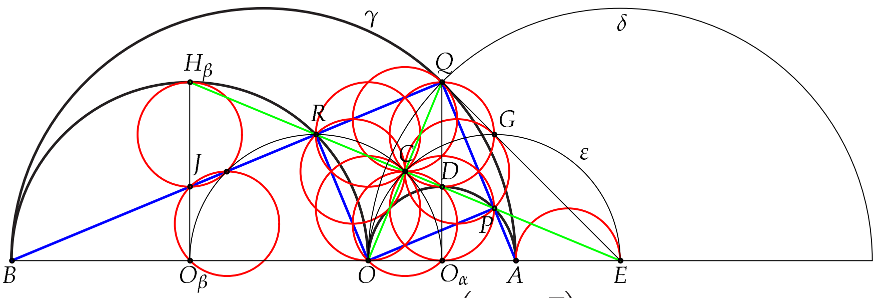

Figure 5: $b = \left(1 + \sqrt{2}\right) a$

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DEPARTMENT OF MATHEMATICS
YAMATO UNIVERSITY
2-5-1 KATAYAMA SUITA OSAKA 564-0082, JAPAN
E-mail address: `okumura.hiroshi@yamato-u.ac.jp`