THE INSCRIBED SQUARE OF THE ARBELOS

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ABSTRACT. We consider an inscribed square of the arbelos, which is determined uniquely and gives several Archimedean circles of the arbelos.

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1. INTRODUCTION

The arbelos is a plane figure surrounded by three mutually touching semicircles which have collinear centers and are elected on the same side of the line passing through the centers of the semicircles. Circles or semicircles having common radius equal to the half the harmonic mean of the radii of the two inner semicircles are said to be Archimedean, which are one of the main topics on the arbelos.

In this article we consider an inscribed square of the arbelos, one of whose vertices coincides with the point of tangency of the two inner semicircles. The square is determined uniquely and is constructed in a simple way and yields several Archimedean circles of the arbelos.

2. THE INSCRIBED SQUARE OF THE ARBELOS

In this section we construct the inscribed square of the arbelos, and consider several properties related to this. Let \( O \) be a point on the segment \( AB \), and let us consider an arbelos formed by the three semicircles with diameters \( AO, BO \) and \( AB \), which are denoted by \( a, \beta \) and \( \gamma \), respectively. Let \( a = |AO|/2 \) and \( b = |BO|/2 \). Circles or semicircles of radius \( ab/(a + b) \) are said to be Archimedean, and the common radius is denoted by \( r_A \). We use a rectangular coordinate system with origin \( O \) such that the points \( A \) and \( B \) have coordinates \((2a,0)\) and \((-2b,0)\) respectively, where we assume that the semicircles \( a, \beta \) and \( \gamma \) are constructed in the region \( y > 0 \). For two points \( P \) and \( Q \), \( P(Q) \) denotes the circle with center \( P \) passing through \( Q \), and \( (PQ) \) denotes the circle with a diameter \( PQ \). However if their centers lie on the line \( AB \), we consider them as semicircles with diameters lying on the line \( AB \) and constructed in the region \( y > 0 \). The center of a circle or a semicircle \( \delta \) is denoted by \( O_{\delta} \). We call the radical axis of \( a \) and \( \beta \) the axis of the arbelos, which overlaps with the \( y \)-axis.

Proposition 2.1. For a point \( Q \) on the semicircle \( \gamma \), if \( P \) is the point of intersection of \( AQ \) and \( a \) and \( R \) is the point of intersection of \( BQ \) and \( \beta \), then \( OPQR \) is a square if and only if \( |AQ|/|BQ| = a/b \).
Proof. Since $OPQR$ is a rectangle, it is a square if and only if $|PQ| = |RQ|$. But $|PQ| = |AQ| - |AP| = |AQ| - |AQ|a/(a + b) = |AQ|b/(a + b)$. Similarly $|QR| = |BQ|a/(a + b)$. Therefore $OPQR$ is a square if and only if $b|AQ| = a|BQ|$. Since the right triangle $AQB$ is uniquely determined by the value $|AQ|/|BQ|$, the proposition shows that the square $OPQR$ is determined uniquely for the arbelos.

Let $E$ be the external common tangent of $\alpha$ and $\beta$. If $a \neq b$, let $E$ be the point of intersection of the lines $AB$ and $E$, and let $\delta = E(O)$ and $\varepsilon = (EO)$ (see Figure 1). If $a = b$, let $\delta$ and $\varepsilon$ be the axis. In any case, let $Q$ be the point of intersection of $\gamma$ and $\delta$, and let $C$ be the point of intersection of $(O, O)$ and $\varepsilon$. Let $j = ab/(a^2 + b^2)$. Let us assume $a \neq b$, and let $q = ab/(b - a)$. The point $E$ is the external center of similitude of $\alpha$ and $\beta$ and divides the segment $O_\alpha O_\beta$ in the ratio $a : b$ externally. Therefore $E$ has coordinates $(2q, 0)$. The semicircles $\gamma$ and $\delta$ are expressed by the equations $(x - 2a)(x + 2b) + y^2 = 0$ and $(x - 4q)x + y^2 = 0$, respectively. Hence the point $Q$ has coordinates

$$(2j(b - a), 2j(a + b)).$$

Notice that the coordinates of $Q$ is also expressed by (2.1) in the case $a = b$.

Let $P$ be the point of intersection of $\alpha$ and the segment $AQ$, and let $R$ be the point of intersection of $\beta$ and the segment $BQ$. The points $P$ and $R$ have coordinates

$$(2jb, 2ja) \text{ and } (-2ja, 2jb),$$

respectively by (2.1). Let $H_\alpha$ (resp. $H_\beta$) be the farthest point on $\alpha$ (resp. $\beta$) from $AB$. The circle touching $AB$ at $O$ and also touching $\gamma$ internally has radius $2r_\alpha$ [4], which is denoted by $\zeta$. Let $F$ be the foot of perpendicular from $Q$ to $AB$, and let $D$ be the point of intersection of the lines $PR$ and $QF$, and let $G$ be the reflection of $F$ in the line $PR$. Let $F$ be the tangent of $\gamma$ at $Q$.

Theorem 1. The following statements hold.

(i) $OPQR$ is a square with center $C$.

(ii) The point $C$ divides the segment $H_\alpha H_\beta$ in the ratio $a^2 : b^2$ internally.

(iii) The inscribed circle of the arbelos and the circle $\zeta$ and the line $F$ touch $\gamma$ at the point $Q$. If $a \neq b$, then $F$ passes through the point $E$.

(iv) The lines $AB$ and $F$ are symmetric in the line $PR$.

(v) If $a \neq b$, the points $O_\alpha$ and $E$ are harmonic conjugates with respect to $H_\alpha$ and $H_\beta$. Also the points $D$ and $E$ are harmonic conjugates with respect to $P$ and $R$. 

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(vi) The circumscribed circle of OPQR, the semicircle (or line) \( \varepsilon \) and the line \( \mathcal{F} \) intersect at the point \( G \). The point \( F \) lies on the circumscribed circle of OPQR.

**Proof.** From (2.2), we get \(|OP| = |OR| = 2\sqrt{a^2 + b^2}|. One the other hand the homothety with center \( O \) and ratio \( 1/2 \) carries \( \gamma \) and \( \delta \) into \((O_aO_{\beta})\) and \( \varepsilon \), respectively. Therefore \( C \) is the midpoint of \( OQ \). This proves (i). Since the point \( C \) has coordinates \((j(b - a), j(a + b))\) by (2.1),

\[
C = \frac{b^2}{a^2 + b^2} H_a + \frac{a^2}{a^2 + b^2} H_{\beta}
\]  

(2.3)

holds. This proves (ii). The part (iii) is true in the case \( a = b \). Let us assume \( a \neq b \). If we consider \( \delta \) as a complete circle, the inversion in \( \delta \) fixes the semicircle \( \gamma \), the line \( AB \) and the point \( O \). Hence it also fixes the circle \( \zeta \). But \( Q \) is the unique fixed point on \( \gamma \) by the inversion. Therefore \( \gamma \) and \( \zeta \) touch at \( Q \). The rest of (iii) is proved similarly. The part (iv) follows from the fact that the lines \( \mathcal{F} \) and \( AB \) are the tangents of \( \zeta \) from \( E \) in the case \( a \neq b \). The point \( O_{\zeta} \) has coordinates \((0, 2r_{\lambda})\), and we get

\[
O_{\zeta} = \frac{b}{a + b} H_a + \frac{a}{a + b} H_{\beta}.
\]

Since \( E \) divides \( H_aH_{\beta} \) in the ratio \( a : b \) externally, the first half of (v) is proved. Let \( D' \) be the point dividing \( PR \) in the ratio \( a : b \) internally. Then \( D' \) has coordinates

\[
(2j(b - a), 4jr_{\lambda})
\]

(2.4)

by (2.2). Therefore \( D' \) lies on \( QE \), i.e., \( D' = D \). Since \( E \) divides \( PR \) in the ratio \( a : b \) externally, (v) is proved. The part (vi) is true if \( a = b \). Let us assume \( a \neq b \). Let \( G' \) be the point of intersection of \( \varepsilon \) and the circumscribed circle of OPQR. Since \( \angle OG'Q \) and \( \angle QOE \) are right angles, \( G' \) lies on \( E Q = \mathcal{F} \). But \( EQ \) is an isosceles triangle with \(|EO| = |EQ|\) i.e., the triangles \( FQO \) and \( G'QQ \) are congruent. Therefore \( G' \) is the reflection of \( F \) in \( PR \), i.e., \( G' = G \). The rest of (vi) is obvious.

The points \( P, R, C, D, O_{\zeta}, H_a \) and \( H_{\beta} \) are collinear by the theorem. Hence the square \( OPQR \) can easily be constructed from the points \( H_a \) and \( H_{\beta} \), i.e., the line \( H_aH_{\beta} \) intersects \( a \) and \( b \) at the points \( P \) and \( Q \), respectively. The fact that the square is constructed from the point of tangency of \( \gamma \) and the inscribed circle of the arbelos is mentioned at the website [2].

### 3. Quadruplets of Archimedean Circles

In this section we consider two quadruplets of Archimedean circles passing through the point \( C \), which are obtained from the square \( OPQR \). Let \( P' \) (resp. \( R' \)) be the point of intersection of the line \( OQ \) and the line passing through \( P \) (resp. \( R \)) parallel to \( AB \) (see Figure 2).
Theorem 2. The following statements are true.
(i) The circles \((OO_\gamma), (PP'), (QD)\) and \((RR')\) are Archimedean.
(ii) The circles \((OO_\gamma), (QD)\) and the semicircle \((O_aO_b)\) touch at the point \(C\), also the circles \((PP'), (RR')\) and the semicircle (or line) \(e\) touch at \(C\). The former three are orthogonal to the latter.
(iii) The circle \((QD)\) passes through the point \(G\).
(iv) The point of intersection of \((OO_\gamma)\) and \((PP')\) different from \(C\) divides the segment \(OP\) in the ratio \(a : b\) internally. Similar facts are also true for the other pairs of the Archimedean circles passing through \(C\) and the corresponding sides of \(OPQR\).
(v) The points of intersection of \((OO_\gamma), (PP'), (QD)\) and \((RR')\) different from \(C\) form vertices of a square of side length \(2r_\lambda\).

Proof. Since the circle \(\zeta\) has radius \(2r_\lambda\) and \(\angle OCO_\gamma\) is a right angle, the circle \((OO_\gamma)\) is Archimedean and passes through \(C\). Rotating \((OO_\gamma)\) about \(C\) through \(90^\circ, 180^\circ\) and \(270^\circ\), we get the Archimedean circles \((PP'), (QD)\) and \((RR')\). This proves (i). The part (ii) holds in the case \(a = b\). Let us assume \(a \neq b\). The point \(C\) is the homothety center of the triangles \(CPP'\) and \(CEO\), i.e., it is the homothety center of \(e\) and the upper half part of \((PP')\). Hence \(C\) is the point of tangency of \((PP')\) and \(e\). But \((PP')\) and \((RR')\) also touch at \(C\). Therefore \((PP'), (RR')\) and \(e\) touch at \(C\). On the other hand, \(|O_aO_\alpha||O_\alphaO_\beta| = |(q - a)(q + b)| = q^2\), i.e., the point \(O_\beta\) is the inverse of \(O_a\) in the circle \(e\). Therefore \((O_aO_\beta)\) and \(e\) are orthogonal. While \((O_aO_\beta)\) and \(e\) intersect at \(C\). Hence the rest of (ii) is now obvious. The reflection of the segment \(DF\) in \(PR\) is \(DG\). Therefore \(\angle DQ\) is a right angle. This proves (iii). The circles \((OO_\gamma)\) and \((PP')\) are expressed by the equations \(x^2 + (y - 2r_\lambda)y = 0\) and \((x - (2jb - r_\lambda))^2 + (y - 2ja)^2 = r^2_\lambda\) by (2.2). Let \(U\) be the point of intersection of \((OO_\gamma)\) and \((PP')\) different from \(C\). Then \(U\) has coordinates \((2r_\lambda j, 2r_\lambda ja/b)\), i.e., \(U = (b/(a + b))O + (a/(a + b))P\). This proves (iv). Let \(V\) be the point of intersection of \((QD)\) and \((PP')\) different from \(C\). Then \(V\) is obtained by rotating \(U\) through \(90^\circ\) about \(C\). Hence \(UV\) is a diameter of \((PP')\). This proves (v).

For a point \(S\) on \(\gamma\) distinct from \(A\) and \(B\), if \(U\) is the point of intersection of \(a\) and the segment \(AS\) and the line parallel to \(AB\) passing through \(U\) intersects \(OS\) at a point \(T\), then the circle \((TU)\) is Archimedean [3]. Therefore the Archimedean circle \((PP')\) (as well as \((RR')\)) is also obtained by this fact. The circle \((OO_\gamma)\) coincides with the Bankoff triplet circle, which is denoted by \(W_3\) in [1].

The square \(OPQR\) gives one more quadruplet of Archimedean circles (see Figure 3).
Theorem 3. The following statements are true.
(i) The circles \((QO_\Omega), (PR'), (OD)\) and \((RP')\) are Archimedean.
(ii) The circles \((QO_\Omega)\) and \((OD)\) touch at \(C\) and the segment joining their centers is perpendicular to \(AB\). The circles \((PR')\) and \((RP')\) touch at \(C\) and the segment joining their centers is parallel to \(AB\).
(iii) The circle \((OD)\) passes through the point \(F\).
(iv) The point of intersection of \((QO_\Omega)\) and \((PR')\) different from \(C\) divides the segment \(QP\) in the ratio \(a : b\) internally. Similar facts are also true for the other pairs of the circles \((QO_\Omega), (PR'), (OD)\) and \((RP')\) and the corresponding sides of \(OPQR\).
(v) The points of intersections of \((QO_\Omega), (PR'), (OD)\) and \((RP')\) different from \(C\) form vertices of a square of side length \(2r_\lambda\).

Proof. Since \(Q\) and \(R'\) are reflections of \(O\) and \(P'\) in the line \(PR\), respectively, the part (i) follows from (i) of Theorem 2. The centers of \((QO_\Omega)\) and \((QO_\Omega)\) are symmetric in \(PR\). Hence \(CO_{(QO_\Omega)}O_{(QO_\Omega)}\) is a rhombus, while the segment \(O_{(QO_\Omega)}O_{(QO_\Omega)}\) is perpendicular to \(AB\). Therefore \(CO_{(QO_\Omega)}\) is perpendicular to \(AB\). This proves the first part of (ii). The rest of (ii) is now obvious. The rest of the theorem follows from (iii), (iv) and (v) of Theorem 2.

4. Some other Archimedean circles

In this section we consider some other Archimedean circles related to the square \(OPQR\) (see Figure 4). Let \(C_1^a\) be the circle touching \(AB\) at the point \(O_\alpha\) passing through the midpoint of \(AQ\). Similarly the circle \(C_1^b\) is defined. Let \(C_2\) be the circle touching \(AB\) at the point \(F\) and passing through \(C\). If \(a \neq b\), let \(H\) be the point of intersection of \(PR\) and the perpendicular to \(AB\) passing through \(O_c\).

Theorem 4. The following statements hold.
(i) The circle \(C_1^a\) is Archimedean, and one of the endpoints of its diameter parallel to \(AB\) lies on \(AQ\), which divides \(AQ\) in the ratio \((a + b - 2r_\lambda) : (a + b + 2r_\lambda)\) internally. Similar facts also hold for \(C_1^b\) and \(BQ\).
(ii) The circle \(C_2\) is Archimedean.
(iii) If \(a \neq b\), then \(H\) is the midpoint of \(EO_\Gamma\) and the semicircle \(O_\lambda(H)\) is Archimedean and touches the line \(E\) and intersects \(PR\) at \(H\) and the midpoint of \(DE\). The circle \(H(O_\lambda)\) is Archimedean and touches \(EQ\) at its midpoint. Also the circle with center at the midpoint of \(EQ\) and passing through the points of intersection of \(PR\) and \(O_\lambda(H)\) is Archimedean.
(iv) If $a \neq b$, the points $H$ and $C$ are harmonic conjugates with respect to $H_A$ and $H_B$.

**Proof.** Let $M$ be the midpoint of $AQ$. Then $MCOO_a$ is a parallelogram. Therefore the translation $O \mapsto O_A$ carries $C$ into $M$. Hence it carries $(OO_C)$ into $C_1^a$, i.e., $C_1^a$ is Archimedean. While we get

$$
(a + r_A, r_A) = \frac{a + b + 2r_A}{2(a + b)} A + \frac{a + b - 2r_A}{2(a + b)} Q.
$$

This proves (i). The circle $C_2$ is the reflection of $(OO_C)$ in the perpendicular to $AB$ from $C$. This proves (ii). Let us assume $a \neq b$. Since the distance between $O$ and $\mathcal{E}$ is $2r_A$ [1], the semicircle $O(O_C)$ of radius $2r_A$ touches $\mathcal{E}$, and intersects $PR$ at $O_C$ and $D$. But the homothety with center $E$ and ratio $1/2$ carries $O(O_C)$ into $O_E(H)$. Therefore $H$ is the midpoint of $EO_C$, and $O_E(H)$ is Archimedean and touches $\mathcal{E}$ and intersects $PR$ at $H$ and the midpoint of $DE$. The rest of (iii) is obvious. The coordinates of $H$ are $(q, r_A)$ by (iii), and we get

$$
H = \frac{-b^2}{a^2 - b^2} H_a + \frac{a^2}{a^2 - b^2} H_B.
$$

This proves (iv) by (2.3).

5. **Special case**

In section 3, we have considered two quadruplets of Archimedean circles passing through the point $C$. In this section we consider the case in which the centers of the eight Archimedean circles form vertices of a regular octagon (see Figure 5). Let $J$ be the point of intersection of the lines $BQ$ and $H_B O_B$.

**Theorem 5.** If $a < b$, the following statements are equivalent.

(i) The centers of the eight circles of the two quadruplets of Archimedean circles form vertices of a regular octagon.

(ii) The points $F$ and $O_A$ coincide.

(iii) The points $D$ and $H_A$ coincide.

(iv) The semicircle $(AE)$ is Archimedean.

(v) The circle $(JH_B)$ is Archimedean.

(vi) The point $R$ lies on the semicircle $(O_A O_B)$.

(vii) The semicircles $\gamma$ and $\delta$ are congruent.

(viii) The distance between $P$ and $AB$ equals $r_A$.

(ix) The distance between $R$ and the axis equals $r_A$. 

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\[(xx) \, b = \left(1 + \sqrt{2}\right) a.\]

**Proof.** By (ii) of Theorem 3, the part (i) holds if and only if the circles \((PP')\) and \((OO')\) are symmetric in the perpendicular from \(C\) to \(AB\). Therefore (i) and (viii) are equivalent. Since (viii) holds if and only if \(2ja = r_A\), (viii) is also equivalent to (xx). Similarly we can show that each of the other parts is equivalent to (xx), where recall that \(D\) has coordinates (2.4) to consider (ii).

The proof of the following facts is also accomplished by the straightforward calculation. Let us assume (i). Then \(R\) is the midpoint of \(JQ\) and \(DH_B\). Hence the triangles \(RQD\) and \(RJH_B\) are congruent isosceles triangles. Therefore the circles \((QD)\) and \((JH_B)\) are symmetric in the perpendicular from \(R\) to \(AB\). While \(R\) is the farthest point on \((O_sO_B)\) from \(AB\), and \((QD)\) touches \((O_sO_B)\) at the point \(C\) by (ii) of Theorem 2, and \(C\) is the midpoint of \(EH_B\). Therefore \((JH_B)\) also touches \((O_sO_B)\) at the midpoint of \(BQ\). Reflecting the circle \((OD)\) in the perpendicular to \(AB\) from \(R\), we get the Archimedean circle passing through the midpoint of \(BQ\) and the points \(J\) and \(O_B\). The circle is also the reflection of \((JH_B)\) in \(BQ\).

![Figure 5: \(b = \left(1 + \sqrt{2}\right) a\)](image)

**REFERENCES**


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