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PROPERTIES INVOLVING PEDAL TRIANGLES

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ABSTRACT. We deduce the area of a pedal triangle in various forms in barycentric coordinates and we establish a relation between the area of the pedal triangles of isogonal conjugates.

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1. INTRODUCTION

In this paper we deduce the area of a pedal triangle in various forms. We done the condition that a triangle inscribed in the triangle of reference ABC to be a pedal triangle. We will demonstrate that the symmetric points of the vertex of a pedal triangle with respect to the midpoints of the sides of ABC triangle form a pedal triangle. Finally we establish a relation between the area of the pedal triangles of isogonal conjugates.

We use the barycentric coordinates ([2], [3]) with respect to the ABC triangle: $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$. Let a,b,c be the length of sidelines BC, CA, AB and R the circumradius of the ABC triangle. In the plan of the triangle ABC we consider a varying point M (Figure 1), with barycentric coordinates $M = (u : v : w)$ so that $uvw \neq 0$ and note with the symbol μ_M the sum of coordinates of $M : \mu_M = u + v + w$. Let s be the semiperimeter $(2s = a + b + c)$, σ the area, $S = 2\sigma = \dfrac{abc}{4R}$ the twice of area of ABC triangle, $S_A = bc\cos A = \dfrac{1}{2}(-a^2 + b^2 + c^2)$, $S_B = ca\cos B = \dfrac{1}{2}(a^2 - b^2 + c^2)$, $S_C = ab\cos C = \dfrac{1}{2}(a^2 + b^2 - c^2)$ so that $S_A S_B S_C \neq 0$. More we introduce the following notation: $S_\theta \cdot S_\vartheta = S_{\theta\vartheta}$.

Definition 1. Let M_a, M_b, M_c be the orthogonal projections of the M point on the sidelines BC, CA, AB. The $M_a M_b M_c$ triangle is called the *pedal triangle of M* [1] (Figure 1).

I would like to dedicate this article to Professor Béla Orbán, on the occasion of his 85th birthday.

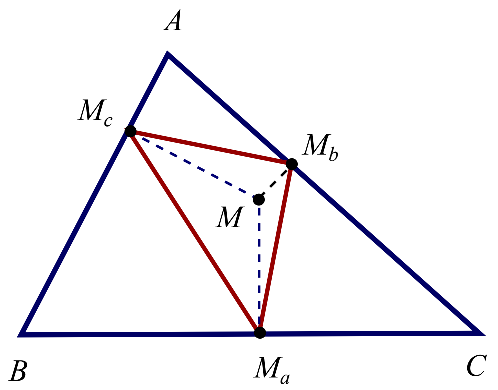

Figure 1

Note with the symbol $\sigma[A_1A_2\ldots A_n]$ the area of a polygon $A_1A_2\ldots A_n$.
In this paper we use the following conditioned trigonometric identities [2]:

$$b^2c^2 - S_A^2 = c^2a^2 - S_B^2 = a^2b^2 - S_C^2 = S^2, \tag{1.1}$$

$$a^2S_A + S_{BC} = b^2S_B + S_{CA} = c^2S_C + S_{AB} = S^2, \tag{1.2}$$

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C), \tag{1.3}$$

$$S_{BC} + S_{CA} + S_{AB} = S^2, \tag{1.4}$$

$$a^2S_A + b^2S_B + c^2S_C = 2S^2, \tag{1.5}$$

$$2(a^2b^2c^2 + S_AS_BS_C) = (a^2 + b^2 + c^2)S^2, \tag{1.6}$$

$$2(S_A + S_B + S_C) = a^2 + b^2 + c^2. \tag{1.7}$$

The demonstration of these relations are the following:

$$b^2c^2 - S_A^2 = b^2c^2 - b^2c^2\cos^2 A = b^2c^2\sin^2 A = \frac{a^2b^2c^2}{4R^2} = S^2,$$

$$a^2S_A + S_{BC} = a^2bc(\cos A + \cos B\cos C) = a^2bc \cdot \sin B \sin C = \frac{a^2b^2c^2}{4R^2} = S^2,$$

$$\sin^2 A + \sin^2 B + \sin^2 C = \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + 1 - \cos^2 C$$
$$2 - 2\cos(A-B)\cos(A+B) - \cos^2 C = 2 + 2\cos C[\cos(A-B) - \cos C]$$
$$= 2 + 2\cos C \sin\frac{\pi - 2A}{2}\sin\frac{\pi - 2B}{2} = 2 + 2\cos A\cos B\cos C,$$

$$S_{BC} + S_{CA} + S_{AB} = abc(a\cos B\cos C + b\cos C\cos A + c\cos A\cos B)$$
$$= abc(c\cos C + c\cos A\cos B) = abc^2\sin A\sin B = \frac{a^2b^2c^2}{4R^2} = S^2,$$

$$2(a^2b^2c^2 + S_AS_BS_C) = 2a^2b^2c^2(1 + \cos A\cos B\cos C) = (a^2 + b^2 + c^2)S^2,$$

$$2(S_A + S_B + S_C) = -a^2 + b^2 + c^2 + a^2 - b^2 + c^2 + a^2 + b^2 - c^2 = a^2 + b^2 + c^2.$$

2. VARIOUS FORMS OF THE AREA OF PEDAL TRIANGLES IN BARYCENTRIC COORDINATES

Theorem 1. *Let ABC be a triangle and $M = (u : v : w)$ a point with barycentric coordinates with respect to the triangle ABC. The area of the pedal triangle $M_aM_bM_c$ in barycentric coordinates is*

$$\begin{aligned}\sigma[M_aM_bM_c] &= \frac{4\sigma^3}{a^2b^2c^2\mu_M^2}\,|a^2vw + b^2wu + c^2uv| \\ &= \frac{\sigma}{4R^2\mu_M^2}\,|a^2vw + b^2wu + c^2uv|. \qquad (2.1)\end{aligned}$$

Proof. The equation of the line, which passes through the point $(x' : y' : z')$ and is perpendicular to the line $lx + my + nz = 0$ is $\begin{vmatrix} x & y & z \\ x' & y' & z' \\ \mu_a & \mu_b & \mu_c \end{vmatrix} = 0$, where $\mu_a = la^2 - mS_C - nS_B$, $\mu_b = mb^2 - nS_A - lS_C$, $\mu_c = nc^2 - lS_B - mS_A$ (see [2]).
The equations of the line MM_a, MM_b and MM_c are

$$\begin{vmatrix} x & y & z \\ u & v & w \\ -a^2 & S_C & S_B \end{vmatrix} = 0 \Leftrightarrow (vS_B - wS_C)x - (wa^2 + uS_B)y + (va^2 + uS_C)z = 0,$$

$$\begin{vmatrix} x & y & z \\ u & v & w \\ S_C & -b^2 & S_A \end{vmatrix} = 0 \Leftrightarrow (wb^2 + vS_A)x + (wS_C - uS_A)y - (ub^2 + vS_C)z = 0,$$

$$\begin{vmatrix} x & y & z \\ u & v & w \\ S_B & S_A & -c^2 \end{vmatrix} = 0 \Leftrightarrow -(vc^2 + wS_A)x + (uc^2 + wS_B)y + (uS_A - vS_B)z = 0.$$

The barycentric coordinates of the M_a, M_b, M_c points are

$$M_a = (0 : va^2 + uS_C : wa^2 + uS_B),$$
$$M_b = (ub^2 + vS_C : 0 : wb^2 + vS_A),$$
$$M_c = (uc^2 + wS_B : vc^2 + wS_A : 0).$$

Consequently, the absolute barycentric coordinates of the M_a, M_b, M_c points are

$$M_a = \left(0, \frac{va^2 + uS_C}{a^2\mu_M}, \frac{wa^2 + uS_B}{a^2\mu_M}\right) = \left(0, \frac{v}{\mu_M} + \frac{uS_C}{a^2\mu_M}, \frac{w}{\mu_M} + \frac{uS_B}{a^2\mu_M}\right),$$

$$M_b = \left(\frac{ub^2 + vS_C}{b^2\mu_M}, 0, \frac{wb^2 + vS_A}{b^2\mu_M}\right) = \left(\frac{u}{\mu_M} + \frac{vS_C}{b^2\mu_M}, 0, \frac{w}{\mu_M} + \frac{vS_A}{b^2\mu_M}\right),$$

$$M_c = \left(\frac{uc^2 + wS_B}{c^2\mu_M}, \frac{vc^2 + wS_A}{c^2\mu_M}, 0\right) = \left(\frac{u}{\mu_M} + \frac{wS_B}{c^2\mu_M}, \frac{v}{\mu_M} + \frac{wS_A}{c^2\mu_M}, 0\right).$$

In absolute barycentric coordinates the area of the triangle determined by the points $M_1 = (u_1, v_1, w_1)$, $M_2 = (u_2, v_2, w_2)$, $M_3 = (u_3, v_3, w_3)$ is $\sigma[M_1M_2M_3] = |\Delta|\sigma$, where $\Delta = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$ (see [2]). The area of $M_aM_bM_c$ triangle is $\sigma[M_aM_bM_c] = |\Delta|\sigma$, where

$$\begin{aligned}
\Delta &= \frac{1}{a^2b^2c^2\mu_M^3} \begin{vmatrix} 0 & va^2 + uS_C & wa^2 + uS_B \\ ub^2 + vS_C & 0 & wb^2 + vS_A \\ uc^2 + wS_B & vc^2 + wS_A & 0 \end{vmatrix} \\
&= \frac{1}{a^2b^2c^2\mu_M^2} \begin{vmatrix} a^2 & va^2 + uS_C & wa^2 + uS_B \\ b^2 & 0 & wb^2 + vS_A \\ c^2 & vc^2 + wS_A & 0 \end{vmatrix} \\
&= \frac{1}{a^2b^2c^2\mu_M^2} [-a^2(wb^2+vS_A)(vc^2 + wS_A) + b^2(vc^2+wS_A)(wa^2+uS_B) \\
&\qquad + c^2(va^2 + uS_C)(wb^2 + vS_A)] \\
&= \frac{1}{a^2b^2c^2\mu_M^2} [a^2vw(b^2c^2 - S_A^2) + b^2wu(c^2S_C + S_{AB}) + c^2uv(b^2S_B + S_{CA})] \\
&= \frac{S^2}{a^2b^2c^2\mu_M^2}(a^2vw + b^2wu + c^2uv) = \frac{4\sigma^2}{a^2b^2c^2\mu_M^2}(a^2vw + b^2wu + c^2uv).
\end{aligned}$$

So

$$\begin{aligned}
\sigma[M_aM_bM_c] &= \frac{4\sigma^3}{a^2b^2c^2\mu_M^2} \, |a^2vw + b^2wu + c^2uv| \\
&= \frac{\sigma}{4R^2\mu_M^2} \, |a^2vw + b^2wu + c^2uv|.
\end{aligned}$$

Remark 1. An other form of the area of the pedal triangle is the following:

$$\begin{aligned}
\sigma[M_aM_bM_c] &= \frac{\sigma}{4R^2\mu_M^2} \, |a^2vw + b^2wu + c^2uv| \\
&= \frac{\sigma}{\mu_M^2} \, |vw \cdot \sin^2 A + wu \cdot \sin^2 B + uv \cdot \sin^2 C| \qquad (2.2)
\end{aligned}$$

Remark 2. Let $x_M = \dfrac{u}{\mu_M} = \dfrac{u}{u+v+w}$, $y_M = \dfrac{v}{\mu_M} = \dfrac{v}{u+v+w}$, $z_M = \dfrac{w}{\mu_M} = \dfrac{w}{u+v+w}$ be the absolute barycentric coordinates of the M point. In these coordinates the area of the pedal triangle $M_aM_bM_c$ is

$$\begin{aligned}
\sigma[M_aM_bM_c] &= \frac{4\sigma^3}{a^2b^2c^2} \, |a^2y_Mz_M + b^2z_Mx_M + c^2x_My_M| \\
&= \frac{\sigma}{4R^2} \, |a^2y_Mz_M + b^2z_Mx_M + c^2x_My_M| \\
&= \sigma|y_Mz_M \cdot \sin^2 A + z_Mz_M \cdot \sin^2 B + x_My_M \cdot \sin^2 C|. \qquad (2.3)
\end{aligned}$$

If X_i is a triangle center [3], note the orthogonal projections of X_i on the sidelines BC, CA, AB with X_a^i, X_b^i, X_c^i, respectively. So the pedal triangle of X_i is $X_a^i X_b^i X_c^i$. Now we calculate the area of pedal triangle of X_i for $i \in \{1, 2, 3, 4\}$.

2.1. The point $X_1 = (a : b : c)$ is the *incenter* of the ABC triangle. Its pedal triangle is called the *intouch triangle*:

$$\sigma[X_a^1 X_b^1 X_c^1] = \frac{4\sigma^3}{a^2b^2c^2 4s^2} |a^2bc + b^2ca + c^2ab| = \frac{2\sigma^3}{abcs} = \frac{(s-a)(s-b)(s-c)}{2R}.$$

2.2. The point $X_2 = (1 : 1 : 1)$ is the *centroid* of ABC triangle. The area of its pedal triangle is

$$\begin{aligned}\sigma[X_a^2 X_b^2 X_c^2] &= \frac{\sigma}{4R^2} \cdot \frac{1}{9} |a^2 + b^2 + c^2| = \frac{\sigma}{9} | \sin^2 A + \sin^2 B + \sin^2 C| \\ &= \frac{S}{9} (1 + \cos A \cos B \cos C).\end{aligned}$$

2.3. The point $X_3 = (a^2 S_A : b^2 S_B : c^2 S_C)$ is the *circumcenter* of ABC triangle. Its pedal triangle is called the *medial triangle*:

$$\sigma[X_a^3 X_b^3 X_c^3] = \frac{4\sigma^3}{a^2b^2c^2} \cdot \frac{a^2b^2c^2}{4S^4} |S_{BC} + S_{CA} + S_{AB}| = \frac{\sigma^3}{S^2} = \frac{\sigma}{4}.$$

2.4. The point $X_4 = (S_{BC} : S_{CA} : S_{AB})$ is the *orthocenter* of ABC triangle. Its pedal triangle is called the *orthic triangle*:

$$\begin{aligned}\sigma[X_a^4 X_b^4 X_c^4] &= \frac{4\sigma^3}{a^2b^2c^2} \cdot \frac{|S_A S_B S_C|}{S^4} |a^2 S_A + b^2 S_B + c^2 S_C| \\ &= \frac{8\sigma^3}{a^2b^2c^2} \cdot \frac{a^2b^2c^2 | \cos A \cos B \cos C|}{S^2} \\ &= S| \cos A \cos B \cos C|.\end{aligned}$$

Remark 3. If ABC is an acute-angled triangle, then comparing the areas of pedal triangles $X_a^2 X_b^2 X_c^2$ and $X_a^4 X_b^4 X_c^4$, we obtain

$$2\sigma[ABC] = 9\sigma[X_a^2 X_b^2 X_c^2] - \sigma[X_a^4 X_b^4 X_c^4]. \tag{2.4}$$

3. Conditions needed in order that an inscribed triangle in a triangle to be a pedal triangle

A $T_a T_b T_c$ triangle is inscribed in an ABC triangle if T_a lies on BC, T_b lies on CA and T_c lies on AB. Consider the T_a, T_b, T_c points with absolute barycentric coordinates: $T_a = (0, \alpha, 1-\alpha)$, $T_b = (1-\beta, 0, \beta)$, $T_c = (\gamma, 1-\gamma, 0)$ (Figure 2). What is the condition needed so that $T_a T_b T_c$ triangle to be a pedal triangle?

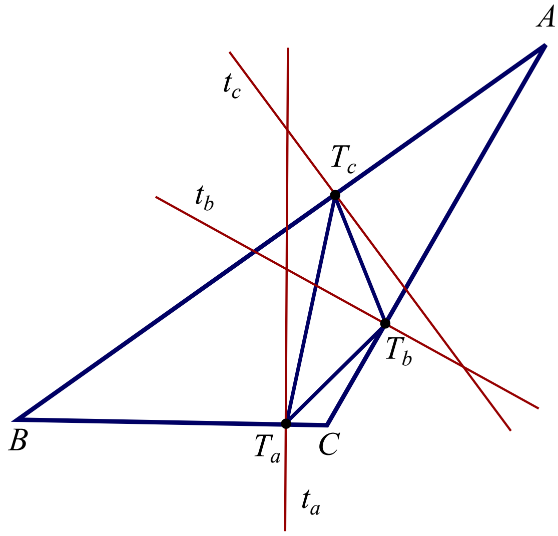

Figure 2

Theorem 2. *The $T_aT_bT_c$ triangle inscribed in a ABC triangle is a pedal triangle if and only if*

$$a^2\alpha + b^2\beta + c^2\gamma = \frac{1}{2}\left(a^2 + b^2 + c^2\right) = S_A + S_B + S_C. \tag{3.1}$$

Proof. Let t_a be the perpendicular to BC through T_a, t_b the perpendicular to CA through T_b, t_c the perpendicular to AB through T_c (Figure 2). The equations of these lines are

$$t_a : \begin{vmatrix} x & y & z \\ 0 & \alpha & 1-\alpha \\ -a^2 & S_C & S_B \end{vmatrix} = 0 \Leftrightarrow (a^2\alpha - S_C)x - a^2(1-\alpha)y + a^2\alpha z = 0,$$

$$t_b : \begin{vmatrix} x & y & z \\ 1-\beta & 0 & \beta \\ S_C & -b^2 & S_A \end{vmatrix} = 0 \Leftrightarrow b^2\beta x + (b^2\beta - S_A)y - b^2(1-\beta)z = 0.$$

$$t_c : \begin{vmatrix} x & y & z \\ \gamma & 1-\gamma & 0 \\ S_B & S_A & -c^2 \end{vmatrix} = 0 \Leftrightarrow -c^2(1-\gamma)x + c^2\gamma y + (c^2\gamma - S_B)z = 0.$$

The t_a, t_b, t_c lines are concurrent if and only if

$$\begin{vmatrix} a^2\alpha - S_C & -a^2(1-\alpha) & a^2\alpha \\ b^2\beta & b^2\beta - S_A & -b^2(1-\beta) \\ -c^2(1-\gamma) & c^2\gamma & c^2\gamma - S_B \end{vmatrix} = 0 \Leftrightarrow$$

$$\begin{vmatrix} S_B & -a^2(1-\alpha) & -a^2 \\ S_A & b^2\beta - S_A & S_C \\ -c^2 & c^2\gamma & S_B \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow (b^2\beta - S_A)S_B^2 - c^2a^2[\gamma S_A - (1-\alpha)S_C + (b^2\beta - S_A)]$$
$$-c^2\gamma S_B S_C + a^2(1-\alpha)S_A S_B = 0 \Leftrightarrow$$
$$(b^2\beta - S_A)S_B^2 - c^2a^2[\gamma S_A + \alpha S_C - b^2(1-\beta)]$$
$$-c^2\gamma S_B S_C + a^2(1-\alpha)S_A S_B = 0 \Leftrightarrow$$

$$a^2(c^2S_C + S_AS_B)\alpha + b^2(c^2a^2 - S_B^2)\beta + c^2(a^2S_A + S_BS_C)\gamma$$
$$= a^2b^2c^2 + S_AS_BS_C \Leftrightarrow$$
$$a^2S^2\alpha + b^2S^2\beta + c^2S^2\gamma = a^2b^2c^2 + S_AS_BS_C \Leftrightarrow$$
$$a^2\alpha + b^2\beta + c^2\gamma = \frac{1}{2}(a^2+b^2+c^2) \Leftrightarrow$$
$$a^2\alpha + b^2\beta + c^2\gamma = S_A + S_B + S_C.$$

Remark 4. The conditions (3.1) can be written in the following equivalent forms:

$$a^2\alpha + b^2\beta + c^2\gamma = a^2 + S_A = b^2 + S_B = c^2 + S_C,$$

or

$$a^2\alpha + b^2\beta + c^2\gamma = a^2 + bc\cos A = b^2 + ca\cos B = c^2 + ab\cos C,$$

or

$$a^2(2\alpha - 1) + b^2(2\beta - 1) + c^2(2\gamma - 1) = 0.$$

Introduce the following notation:

$$E(\alpha,\beta,\gamma) = a^2(2\alpha - 1) + b^2(2\beta - 1) + c^2(2\gamma - 1).$$

Now we suppose that the t_a, t_b, t_c lines are concurrent and let $T = t_a \cap t_b \cap t_c$. We determine the coordinates of the T point, solving the following system with Cramer rule:

$$\begin{cases} (a^2\alpha - S_C)x - a^2(1-\alpha)y = -a^2\alpha z \\ -c^2(1-\gamma)x + c^2\gamma y = -(c^2\gamma - S_B)z \end{cases}$$

$$\delta = \begin{vmatrix} a^2\alpha - S_C & -a^2(1-\alpha) \\ -c^2(1-\gamma) & c^2\gamma \end{vmatrix} = c^2(a^2\alpha + S_B\gamma - a^2).$$

$$\delta_x = \begin{vmatrix} -a^2\alpha z & -a^2(1-\alpha) \\ -(c^2\gamma - S_B)z & c^2\gamma \end{vmatrix} = -a^2(S_B\alpha + c^2\gamma - S_B)z,$$

$$\delta_y = \begin{vmatrix} a^2\alpha - S_C & -a^2\alpha z \\ -c^2(1-\gamma) & -(c^2\gamma - S_B)z \end{vmatrix} = -(a^2S_A\alpha - c^2S_C\gamma + S_BS_C)z.$$

So the barycentric coordinates of the T point are

$$T = [a^2(S_B\alpha + c^2\gamma - S_B) : a^2S_A\alpha - c^2S_C\gamma + S_BS_C : -c^2(a^2\alpha + S_B\gamma - a^2)].$$

Note briefly the coordinates of T with x_T, y_T, z_T. Their sum is:

$$x_T + y_T + z_T = a^2(S_B\alpha + c^2\gamma - S_B) + a^2S_A\alpha - c^2S_C\gamma + S_BS_C$$
$$- c^2(a^2\alpha + S_B\gamma - a^2) = c^2a^2 - S_B^2 = S^2.$$

Now we give these coordinates in another form:

$$x_T = a^2(S_B\alpha + c^2\gamma - S_B) = a^2\left(S_B\alpha + c^2\gamma - \frac{a^2\alpha + b^2\beta + c^2\gamma}{S_A + S_B + S_C}\,S_B\right)$$
$$= \frac{a^2}{S_A + S_B + S_C}\,[(S_B\alpha + c^2\gamma)(a^2 + S_A) - (a^2\alpha + b^2\beta + c^2\gamma)S_B]$$
$$= \frac{a^2}{S_A + S_B + S_C}\,(S_AS_B\alpha - b^2S_B\beta + b^2c^2\gamma),$$

$$y_T = a^2S_A\alpha - c^2S_C\gamma + S_BS_C = a^2S_A\alpha - c^2S_C\gamma + \frac{a^2\alpha + b^2\beta + c^2\gamma}{S_A + S_B + S_C}\,S_BS_C$$
$$= \frac{1}{S_A + S_B + S_C}\,[(a^2S_A\alpha - c^2S_C\gamma)(b^2 + S_B) + (a^2\alpha + b^2\beta + c^2\gamma)S_BS_C]$$
$$= \frac{b^2}{S_A + S_B + S_C}\,(c^2a^2\alpha + S_BS_C\beta - c^2S_C\gamma),$$

$$z_T = -c^2(a^2\alpha + S_B\gamma - a^2) = -c^2\left(a^2\alpha + S_B\gamma - \frac{a^2\alpha + b^2\beta + c^2\gamma}{S_A + S_B + S_C}\,a^2\right)$$
$$= \frac{c^2}{S_A + S_B + S_C}\,[-(a^2\alpha + S_B\gamma)(c^2 + S_C) + (a^2\alpha + b^2\beta + c^2\gamma)a^2]$$
$$= \frac{c^2}{S_A + S_B + S_C}\,(-a^2S_A\alpha + a^2b^2\beta + S_CS_A\gamma).$$

Consequently, the absolute barycentric coordinates of the T point are

$$T = \frac{1}{(S_A + S_B + S_C)S^2}\,[a^2(S_AS_B\alpha - b^2S_B\beta + b^2c^2\gamma),$$
$$b^2(c^2a^2\alpha + S_BS_C\beta - c^2S_C\gamma), c^2(-a^2S_A\alpha + a^2b^2\beta + S_CS_A\gamma)]$$

Let A', B', C' be the midpoints of the side BC, CA, AB and the pairs of points (T_a, T_a'), (T_b, T_b'), (T_c, T_c') to be symmetric with respect to the A', B', C' midpoints (Figure 3).

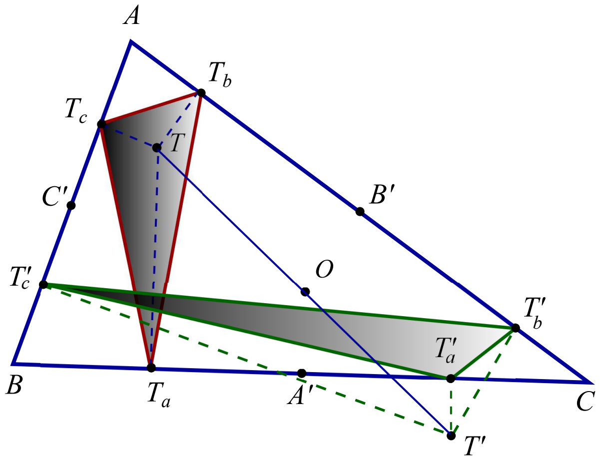

Figure 3

The absolute barycentric coordinates of the T'_a, T'_b, T'_c points are $T'_a = (0, 1-\alpha, \alpha)$, $T'_b = (\beta, 0, 1-\beta)$, $T'_c = (1-\gamma, \gamma, 0)$. Whether the triangle $T'_aT'_bT'_c$ is a pedal triangle as the triangle $T_aT_bT_c$?

Theorem 3. *The symmetries of the vertex in a pedal triangle with respect to the midpoints of sides of ABC reference triangle are the vertex of a pedal triangle.*

Proof. The triangle $T_aT_bT_c$ is a pedal triangle, so

$$E(\alpha, \beta, \gamma) = a^2(2\alpha - 1) + b^2(2\beta - 1) + c^2(2\gamma - 1) = 0.$$

The triangle $T'_aT'_bT'_c$ is pedal triangle, if and only if $E(1-\alpha, 1-\beta, 1-\gamma) = 0$, which is true, since $E(1-\alpha, 1-\beta, 1-\gamma) = -E(\alpha, \beta, \gamma) = 0$.

Definition 2. If the pairs of (T_a, T'_a), $T_b, T'_b)$, (T_c, T'_c) points are symmetric with respect to the A', B', C' midpoints of the sides BC, CA, AB then the two $T_aT_bT_c$ and $T'_aT'_bT'_c$ triangles we call *symmetric pedal triangles*.

Let $T'_aT'_bT'_c$ be the pedal triangle of the point T'. The absolute barycentric coordinates of T' are:

$$\begin{aligned} T' = \frac{1}{(S_A+S_B+S_C)S^2}\{ & a^2[S_AS_B(1-\alpha) - b^2S_B(1-\beta) + b^2c^2(1-\gamma)], \\ & b^2[c^2a^2(1-\alpha) + S_BS_C(1-\beta) - c^2S_C(1-\gamma)], \\ & c^2[-a^2S_A(1-\alpha) + a^2b^2(1-\beta) + S_CS_A(1-\gamma)]\}. \end{aligned}$$

Theorem 4. *The area of symmetric pedal triangles $T_aT_bT_c$ and $T'_aT'_bT'_c$ are equal and the T and T' points are symmetric with respect to O, the circumcenter of the ABC triangle.*

Proof. We calculate the area of $T_aT_bT_c$ and $T'_aT'_bT'_c$ triangles:
$\sigma[T_aT_bT_c] = |\Delta_1|\sigma$, where

$$\Delta_1 = \begin{vmatrix} 0 & \alpha & 1-\alpha \\ 1-\beta & 0 & \beta \\ \gamma & 1-\gamma & 0 \end{vmatrix} = \alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma),$$

$\sigma[T'_aT'_bT'_c] = |\Delta_2|\sigma$, where

$$\Delta_2 = \begin{vmatrix} 0 & 1-\alpha & \alpha \\ \beta & 0 & 1-\beta \\ 1-\gamma & \gamma & 0 \end{vmatrix} = \alpha\beta\gamma + (1-\alpha)(1-\beta)(1-\gamma).$$

We will demonstrate that $x_T + x_{T'} = 2x_0$, where

$$O = (x_0, y_0, z_0) = \left(\frac{a^2S_A}{2S^2}, \frac{b^2S_B}{2S^2}, \frac{c^2S_C}{2S^2}\right):$$

$$\begin{aligned} x_T + x_{T'} = 2x_0 \Leftrightarrow x_T + x_{T'} = \frac{a^2S_A}{S^2} \Leftrightarrow \\ \Leftrightarrow a^2(S_AS_B - b^2S_B + b^2c^2) = a^2S_A(S_A+S_B+S_C) \Leftrightarrow \\ \Leftrightarrow -b^2S_B + b^2c^2 = S_A(S_A+S_C) \Leftrightarrow \\ \Leftrightarrow -b^2S_B + b^2c^2 = S_Ab^2 \Leftrightarrow S_A + S_B = c^2, \end{aligned}$$

which is true (Figure 3). Similarly it is possible to demonstrate that $y_T + y_{T'} = 2y_0$ and $z_T + z_{T'} = 2z_0$.

4. Relation between the area of pedal triangles of isogonal conjugates

The isogonal conjugate N of an M point in the plane of the ABC triangle is constructed by reflecting the lines AM, BM and CM of the angle bisectors at A, B and C. The three reflected lines concur at the isogonal conjugate [4], [5]. Consequently, the barycentric coordinates of N are
$N = \left(\frac{a^2}{u} : \frac{b^2}{v} : \frac{c^2}{w}\right) = (a^2vw : b^2wu : c^2uv)$. Note with the symbol μ_N the sum of the coordinates of N : $\mu_N = a^2vw + b^2wu + c^2uv$. Let $x_N = \frac{a^2vw}{\mu_N}$, $y_N = \frac{b^2wu}{\mu_N}$, $z_N = \frac{c^2uv}{\mu_N}$ be the absolute barycentric coordinates of N (Figure 4).

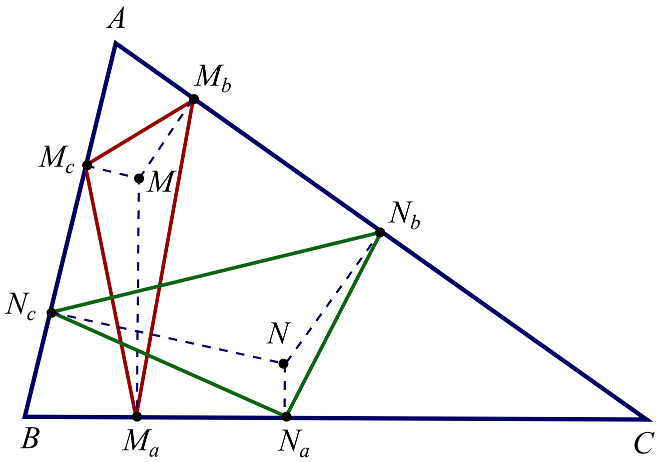

Figure 4

Theorem 5. *The following relation stands:*

$$|x_N y_N z_N| \cdot \sigma[M_a M_b M_c] = |x_M y_M z_M| \cdot \sigma[N_a N_b N_c]. \tag{4.1}$$

Proof. The area of the pedal triangle $M_a M_b M_c$ is:

$$\sigma[M_a M_b M_c] = \frac{4\sigma^3}{a^2b^2c^2|\mu_M^2|} |a^2vw + b^2wu + c^2uv| = \frac{4\sigma^3|\mu_N|}{a^2b^2c^2\mu_M^2} .$$

The area of the $N_aN_bN_c$ pedal triangle is:

$$\begin{aligned}\sigma[N_aN_bN_c] &= \frac{4\sigma^3}{a^2b^2c^2\mu_N^2}\,|a^2b^2wu\cdot c^2uv + b^2c^2uv\cdot a^2vw + c^2a^2vw\cdot b^2wu| \\ &= \frac{4\sigma^3}{a^2b^2c^2\mu_N^2}\cdot a^2b^2c^2|uvw|\,|\mu_M| \\ &= \frac{4\sigma^3|\mu_N|}{a^2b^2c^2\mu_M^2}\cdot\frac{|\mu_M^3|}{|\mu_N^3|}\,a^2b^2c^2|uvw| \\ &= \left|\frac{a^2vw}{\mu_N}\cdot\frac{b^2wu}{\mu_N}\cdot\frac{c^2uv}{\mu_N}\right|\cdot\frac{|\mu_M^3|}{|uvw|}\cdot\sigma[M_aM_bM_c].\end{aligned}$$

From this, the relation (4.1) follows.
With the formula (4.1) we can determine the area of $N_aN_bN_c$ pedal triangle, if we know the area of pedal triangle $M_aM_bM_c$ and vice versa.
For example: we calculate the area of pedal triangle of the symmedian point

$$K=\left(\frac{a^2}{a^2+b^2+c^2},\frac{b^2}{a^2+b^2+c^2},\frac{c^2}{a^2+b^2+c^2}\right),$$

which is the isogonal conjugate of the centroid $G=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.
So

$$\begin{aligned}&|x_Gy_Gz_G|\cdot\sigma[K_aK_bK_c] = |x_Ky_Kz_K|\cdot\sigma[G_aG_bG_c] \Leftrightarrow \\ &\Leftrightarrow \frac{1}{27}\cdot\sigma[K_aK_bK_c] = \frac{a^2b^2c^2}{(a^2+b^2+c^2)^3}\cdot\sigma[G_aG_bG_c] \Leftrightarrow \\ &\Leftrightarrow \sigma[K_aK_bK_c] = \frac{27a^2b^2c^2}{(a^2+b^2+c^2)^3}\cdot\frac{4\sigma^3}{a^2b^2c^2}\cdot\frac{1}{9}\,(a^2+b^2+c^2) \Leftrightarrow \\ &\Leftrightarrow \sigma[K_aK_bK_c] = \frac{12\sigma^3}{(a^2+b^2+c^2)^2}\,.\end{aligned}$$

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