Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.4, (2015), Issue 1, pp.35-43

DUALISTIC STRUCTURES ON TWISTED PRODUCT MANIFOLDS

ABDOUL SALAM DIALLO AND LEONARD TODJIHOUNDÉ

ABSTRACT. In this paper, we show that the projection of a dualistic structure defined on a twisted product manifold induces dualistic structures on the base and the fiber manifolds, and conversely. Then under some conditions on the Ricci curvature and the Weyl conformal tensor we characterize dually flat structures on twisted product manifolds.

2010 Mathematical Subject Classification: 53A15, 53B05, 53B15, 53B20.
Keywords and phrases: conjugate connections; dualistic structures; twisted product.

1. INTRODUCTION

Dualistic structures play an important role in of information geometry, specially in the investigation of the natural differential geometric structure possessed by families of probability distributions. Information geometry is a branch of the mathematics that applied the technique of differential geometry to the field of probability theory. This is done by taking probability distributions for a statistical model as the points of a Riemannian manifold, forming a statistical manifold. The fisher information metric provides the Riemannian matric (see [1], [2] for more details).

The information geometry is nowdays applied in a broad variety of different fields and contexts which include, for instance, information theory, stochastic processes, dynamical systems and times series, statistical physics, quantum systems and the mathematical theory of neural networks [3].

Dually flat manifolds constitutes fundamental objets of information geometry. However, due to the fact that the global theory of dually flat manifolds is still far from being complete, its range of application still suffers certain limitations since often only matters of mainly a local nature can be successfully pursued. Consequently, there is a strong need and desire for a further understanding of the global characteristics of dually flat manifolds (see [2], [3]).

In [8], the author obtained that a warped product manifold is dually flat if and only if the base manifold is dually flat and the fiber manifold is a constant sectional curvature. This result is closed related to the fact that the warping function is defined only on the base manifold and do not depend on the points on the fiber manifold. In the present

Work done during the first author stay at the Institut de Mathématiques et de Sciences Physiques of Porto-Novo, Benin, supported by the Germany Office of University Exchanges (DAAD).

paper, we investigate dually structures on twisted product manifolds and under some conditions we characterize dually flat twisted product manifolds.

2. Preliminaries

2.1. **Dualistic structures.** Let (M, g) be a Riemannian manifold and ∇ an affine connection on M. A connection ∇^* is called *conjugate connection* (or *dual connection*) of ∇ with respect to the metric g if

$$X \cdot g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z), \tag{2.1}$$

for arbitrary $X, Y, Z \in \Gamma(TM)$. The triple (g, ∇, ∇^*) satisfying (2.1) is called *dualistic structure* on M.

The geometry of conjugate connections is a natural generalization of geometry of Levi-Civita connections from Riemannian manifolds theory. Conjugate connections arise from affine differential geometry and from geometric theory of statistical inferences [2]. In [8], the author proved that the projection of a dualistic structure defined on a warped product space induces dualistic structures on the base and the fiber manifold. Recently in [4], the author extended the construction of doubly warped product for geometry of conjugate connections .

Proposition 2.1. *The torsion tensors T^∇ and T^{∇^*} of ∇ and ∇^*, respectively, satisfy:*

$$g(T^\nabla(X,Y), Z) = g(T^{\nabla^*}(X,Y), Z) + (\nabla^* g)(X, Y, Z) - (\nabla^* g)(Y, X, Z)$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. From the torsion tensor equation, we have:

$$\begin{aligned} g(T^\nabla(X,Y), Z) &:= g(\nabla_X Y, Z) - g(\nabla_Y X, Z) - g([X,Y], Z), \\ &= X \cdot g(Y, Z) - g(Y, \nabla^*_X Z) - Y \cdot g(X, Z) \\ &+ g(X, \nabla^*_Y Z) - g(\nabla^*_X Y - \nabla^*_Y X - T^{\nabla^*}(X,Y), Z) \\ &:= g(T^{\nabla^*}(X,Y), Z) + (\nabla^* g)(X, Y, Z) - (\nabla^* g)(Y, X, Z) \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. □

Corollary 2.1. *If $\nabla^* g = 0$, then $T^\nabla = T^{\nabla^*}$.*

Let $C(X, Y, Z) = \nabla_X g(Y, Z)$ the cubic form of (∇, g) and $C^*(X, Y, Z) = \nabla^*_X g(Y, Z)$ the cubic form of (∇^*, g). We have the following property:

Proposition 2.2. *The cubic form of (∇, g) is symmetric if and only the cubic form of (∇^*, g) is symmetric.*

Proof. From definition, we have:

$$\begin{aligned} (\nabla^* g)(X, Y, Z) &:= X \cdot g(Y, Z) - g(\nabla^*_X Y, Z) - g(Y, \nabla^*_X Z) \\ &= X \cdot g(Y, Z) - X \cdot g(Y, Z) + g(Y, \nabla_X Z) \\ &- X \cdot g(Y, Z) + g(Z, \nabla_X Y) \\ &:= -(\nabla g)(X, Y, Z). \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. □

Corollary 2.2. *If $\nabla^* g$ is symmetric and ∇^* is torsion free, then ∇g is symmetric and ∇ is torsion free too.*

The triple (M, ∇, g) is called *statistical manifold* if ∇ is a torsion free affine connection and its cubic form is symmetric. If ∇^* is conjugate connection with respect to g on M, then (M, ∇^*, g) is also statistical manifold called the *dual statistical manifold* of (M, ∇, g). The statistical manifold was introduced by S. Amari [1], it connects information geometry, affine differential geometry and Hessian geometry.

Let R and R^* the curvature tensors of ∇ and ∇^* respectively. We have also the following:

Proposition 2.3. *The curvature tensors R and R^* of ∇ and ∇^* are related by*

$$g(R(X,Y)Z,W) = -g(R^*(X,Y)W,Z)$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Corollary 2.3. $R = 0$ *if and only if* $R^* = 0$.

The manifold M endowed with a dualistic structure (g, ∇, ∇^*) is called a *dually flat space* if both dual connections ∇ and ∇^* are torsion free and flat; that is the curvature tensors with respect to ∇ and ∇^* respectively vanishe identically. This does not imply that the manifold is Euclidean, because the Riemannian curvature due to the Levi-Civita connections does not necessarily vanish. Moreover the existence of a dually flat structure on a manifold points out some topological and geometrical properties of the manifold. For example if a manifold M admits a dually flat structure (g, ∇, ∇^*) and if one of the dual connection, say ∇, is complete, then only the first homotopy group of M is non trivial, and any two points in M can be joined by a $\nabla-$geodesic [3].

2.2. **Twisted product manifolds.** Let (B, g_B) and (F, g_F) be Riemannian manifolds of dimensions r and s respectively, and let $\pi : B \times F \to B$ and $\sigma : B \times F \to F$ be the canonical projections. Also let $b : B \times F \to (0, \infty)$ be positive smooth function. Then the *twisted product* of Riemannian manifolds (B, g_B) and (F, g_F) with twisting function b is the product manifold $B \times F$ with metric tensor

$$g = g_B \oplus b^2 g_F$$

given by

$$g = \pi^* g_B + (b \circ \pi)^2 \sigma^* g_F.$$

We denote this Riemannian manifold (M, g) by $B \times_b F$. In particular, if b is constant on F, then $B \times_b F$ is called the *warped product* of (B, g_B) and (F, g_F) with warping function b. Moreover if $b = 1$, then we obtain a *direct product*. If b is not constant, then we have a *proper twisted product*.

Let $\mathcal{L}(B)$ (respectivement $\mathcal{L}(F)$) be the set of all vector fields on $B \times F$ which is the horizontal lift (respectivement the vertical lift) of a vector field on B (respectivement on F). Thus a vector field on $B \times F$ can be written as

$$A = X + U, \quad \text{with} \quad X \in \mathcal{L}(B) \quad \text{and} \quad U \in \mathcal{L}(F).$$

Obviously

$$\pi_*(\mathcal{L}(B)) = \Gamma(TB) \quad \text{and} \quad \sigma_*(\mathcal{L}(F)) = \Gamma(TF).$$

For any vector field $X \in \mathcal{L}(B)$, we denote $\pi_*(X)$ by $\bar{X}$ and for any vector field $U \in \mathcal{L}(F)$, we denote $\sigma_*(U)$ by $\bar{U}$.

Lemma 2.1. [7] *Let $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(TB)$ and $X, Y, Z \in \mathcal{L}(B)$ be their corresponding horizontal lifts. We have:*

$$\bar{X} \cdot g(\bar{Y}, \bar{Z}) \circ \pi = X \cdot g(Y, Z). \tag{2.2}$$

Also, let $\bar{U}, \bar{V}, \bar{W} \in \Gamma(TF)$ and $U, V, W \in \mathcal{L}(F)$ be their corresponding vertical lifts. Then

$$\bar{U} \cdot g(\bar{V}, \bar{W}) \circ \sigma = U \cdot g(V, W). \tag{2.3}$$

Let (B, g_B) and (F, g_F) be Riemannian manifolds with Levi-Civita connection ${}^B\nabla$ and ${}^F\nabla$, respectively, and let ∇ denote the Levi-Civita connection and the gradient of the twisted product manifold $(B \times_b F)$ of (B, g_B) and (F, g_F) with twisting function b. We have the following proposition.

Proposition 2.4 ([5]). *Let $M = B \times_b F$ be a twisted product manifold with the $g = g_B \oplus b^2 g_F$ and let $X, Y \in \mathcal{L}(B)$ and $U, V \in \mathcal{L}(F)$. Then we have*

$$\begin{aligned}
\nabla_X Y &= {}^B\nabla_X Y; \\
\nabla_X U &= \nabla_U X = X(k)U; \\
\nabla_U V &= {}^F\nabla_U V + U(k)V + V(k)U - g_F(U, V)\nabla k
\end{aligned}$$

where $k = \log b$.

Let M be an m-dimensional manifold with the metric tensor g. If $(E_1, \cdots, E_m)$ is a orthonormal base of M, then we define the curvature tensor, Ricci curvature and scalar curvature, respectively, as follows:

$$\begin{aligned}
R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z; \\
Ricc(X, Y) &= \sum_{i=1}^{m} g(R(E_i, X)Y, E_i) \\
S &= \sum_i Ric(E_i, E_i).
\end{aligned}$$

The Weyl conformal curvature tensor field of M is the tensor field C of type $(1, 3)$ defined by

$$\begin{aligned}
C(X, Y)Z &= R(X, Y)Z \\
&+ \frac{1}{m-2}[Ric(X, Z)Y - R(Y, Z)X + g(X, Z)QY - g(Y, Z)QX] \\
&- \frac{S}{(m-1)(m-2)}[g(X, Z)Y - g(Y, Z)X]
\end{aligned}$$

for any vector fields X, Y and Z on M, where S is scalar curvature.

Define $h^k_B(X,Y) = XY(k) - {}^B\nabla_X Y(k)$ for $X,Y \in \mathcal{L}(B)$. If $X,Y \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, then $XV(k) = VX(k)$ and the Hessian form h^k of k on $B \times_b F$ satisfies

$$\begin{aligned} h^k(X,V) &= XV(k) - X(k)V(k), \\ h^k(X,Y) &= h^k_B(X,Y). \end{aligned}$$

Let R^B and R^F be the curvature tensors of (B, g_B) and (F, g_F), respectively, and let R be the curvature tensor of $B \times_b F$. Then we have the following proposition:

Proposition 2.5 ([5]). *Let $M = B \times_b F$ be a twisted product manifold. If $X,Y,Z \in \mathcal{L}(B)$ and $U,V,W \in \mathcal{L}(F)$, then we have*

$$\begin{aligned} R(X,Y)Z &= R^B(X,Y)Z; \\ R(X,Y)U &= 0; \\ R(X,U)Y &= \frac{h^b_B(X,Y)}{b}U; \\ R(U,V)X &= UX(k)V - VX(k)U; \\ R(X,U)V &= [X(k)V(k) + h^k(X,V)]U - g(U,V)[X(k)\nabla k + H^k(X)]; \\ R(U,V)W &= R^F(U,V)W + g(U,W)grad_B(V(\log b)) - g(V,U)grad_B(U(\log b)) \\ &- \frac{|grad_B b|^2}{b^2}[g(V,W)U - g(U,W)V]. \end{aligned}$$

Proposition 2.6 ([5]). *Let $M = B \times_b F$ be a twisted product manifold. If $X,Y \in \mathcal{L}(B)$ and $U,V,\in \mathcal{L}(F)$, then we have*

$$\begin{aligned} Ric(X,Y) &= Ric^B(X,Y) - s[h^k_B(X,Y) + X(k)Y(k)]; \\ Ric(X,V) &= (s-1)XV(k). \end{aligned}$$

A twisted product manifold $B \times_b F$ is called *mixed Ricci-flat* if $Ric(X,U) = 0$ for all $X\mathcal{L}(B)$ and $U \in \mathcal{L}(F)$.

Proposition 2.7 ([6]). *Let $M = B \times_b F$ be a twisted product manifold with a twisting function f. Then for $X,Y \in \mathcal{L}(B)$ and $U,V \in \mathcal{L}(F)$, we have*

$$\begin{aligned} C(X,Y)V &= \Big(\frac{1-s}{m-2}\Big)[XV(k)Y - YV(k)X] \\ C(V,W)X &= \Big(\frac{r-1}{m-2}\Big)[XV(k)W - XW(k)V]. \end{aligned}$$

We say that $B \times_b F$ is *mixed Weyl conformal flat* if $C(X,V) = 0$ for all $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$. Moreover, F is *Weyl conformal-flat* along B if $C(X,Y) = 0$, and B is *Weyl conformal-flat* along F if $C(U,V) = 0$ for all $X,Y\mathcal{L}(B)$ and $U,V \in \mathcal{L}(F)$.

In [5], Fernandez-Lopez, Garcia-Rio, Kupeli and Unal gave characterization of a twisted product manifold to be a warped product manifold using the Ricci tensor of the manifold. Similar characterization were given by Kazan and Sahin using the Weyl conformal curvature tensor and the Weyl projective curvature tensor in [6].

3. Dualistic structures on twisted product manifolds

Let $\bar{X}, \bar{Y}, \bar{Z} \in \Gamma(TB)$ and $X, Y, Z \in \mathcal{L}(B)$ be their corresponding horizontal lifts respectively, we put:

$$\pi_*(D_X Y) = {}^B\nabla_{\bar{X}}\bar{Y} \quad \text{and} \quad \pi_*(D^*_X Y) = {}^B\nabla^*_{\bar{X}}\bar{Y}.$$

Since D are D^* are affine connections on $B \times_b F$ and π is a projection of $B \times F$ on B then ${}^B\nabla$ and ${}^B\nabla^*$ are affine connections on B.

Let $\bar{U}, \bar{V}, \bar{W} \in \Gamma(TF)$ and $U, V, W \in \mathcal{L}_V(F)$ their corresponding vertical lifts, we put:

$$\sigma_*(D_U V) = {}^F\nabla_{\bar{U}}\bar{V} \quad \text{and} \quad \sigma_*(D^*_U V) = {}^F\nabla^*_{\bar{U}}\bar{V}.$$

Since D are D^* are affine connections on $B \times F$ and σ is a projection of $B \times F$ on F then ${}^F\nabla$ and ${}^F\nabla^*$ are affine connections on F.

Proposition 3.1. *Let (g, D, D^*) be a dualistic structure on a twisted product manifold $B \times_b F$. Then the projections induces dualistic structures on the base and the fiber manifolds.*

Proof. From (2.1) and (2.2), we have:

$$\begin{aligned}
\bar{X} \cdot g_B(\bar{Y}, \bar{Z}) \circ \pi &= X \cdot g(Y, Z) \\
&= \Big[g(D_X Y, Z) + g(Y, D^*_X Z)\Big] \\
&= \Big[g_B(\pi_*(D_X Y), \pi_*(Z)) \circ \pi \\
&+ g_B(\pi_*(Y), \pi_*(D^*_X Z)) \circ \pi\Big] \\
&= \Big[g_B({}^B\nabla_{\bar{X}}\bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}}\bar{Z})\Big] \circ \pi.
\end{aligned}$$

Thus

$$\bar{X} \cdot g_B(\bar{Y}, \bar{Z}) = g_B({}^B\nabla_{\bar{X}}\bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}}\bar{Z}).$$

Hence ${}^B\nabla$ and ${}^B\nabla^*$ are conjugate with respect to g_B.

From (2.1) and (2.3), we have:

$$\begin{aligned}
\bar{U} \cdot g_F(\bar{V}, \bar{W}) \circ \sigma &= b^{-2} U \cdot g(V, W) \\
&= b^{-2}\Big[g(D_U V, W) + g(V, D^*_U W)\Big] \\
&= b^{-2}\Big[b^2 g_F(\sigma_*(D_U V), \sigma_*(W)) \circ \sigma \\
&+ b^2 g_F(\sigma_*(V), \sigma_*(D^*_U W)) \circ \sigma\Big] \\
&= \Big[g_F({}^F\nabla_{\bar{U}}\bar{V}, \bar{W}) + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}}\bar{W})\Big] \circ \sigma.
\end{aligned}$$

Hence ${}^F\nabla$ and ${}^F\nabla^*$ are conjugate with respect to g_F. □

Now, we construct a dualistic structure on the twisted product space from those on its base and fiber manifolds.

Proposition 3.2. *Let $(g_B, {}^B\nabla, {}^B\nabla^*)$ and $(g_F, {}^F\nabla, {}^F\nabla^*)$ be dualistic structures on B and F. Then the triple (g, D, D^*) is a dualistic structure on $B \times_b F$.*

Proof. Let $X, Y, Z \in \mathcal{L}(B)$. We have:

$$\begin{aligned}
X \cdot g(Y, Z) &= \bar{X} \cdot g_B(\bar{Y}, \bar{Z}) \circ \pi \\
&= \Big[g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \Big] \circ \pi \\
&= \Big[g_B({}^B\nabla_{\bar{X}} \bar{Y}, \bar{Z}) \circ \pi + g_B(\bar{Y}, {}^B\nabla^*_{\bar{X}} \bar{Z}) \circ \pi \Big] \\
&= g_B(\pi_*(D_X Y), \pi_*(Z)) \circ \pi + g_B(\pi_*(Y), \pi_*(D^*_X Z)) \circ \pi \\
&= g(D_X Y, Z) + g(Y, D^*_X Z).
\end{aligned}$$

Let $U, V, W \in \mathcal{L}(F)$. We have:

$$\begin{aligned}
U \cdot g(V, W) &= b^2 \bar{U} \cdot g_F(\bar{V}, \bar{W}) \circ \sigma \\
&= b^2 \Big[g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \Big] \circ \sigma \\
&= b^2 \Big[g_F({}^F\nabla_{\bar{U}} \bar{V}, \bar{W}) \circ \sigma + g_F(\bar{V}, {}^F\nabla^*_{\bar{U}} \bar{W}) \circ \sigma \Big] \\
&= b^2 g_F(\sigma_*(D_U V), \sigma_*(W)) \circ \sigma + b^2 g_F(\sigma_*(V), \sigma_*(D^*_U W)) \circ \sigma \\
&= g(D_U V, W) + g(V, D^*_U W).
\end{aligned}$$

□

We call (g, D, D^*) the dualistic strucure on $B \times_b F$ induced from $(g_B, {}^B\nabla, {}^B\nabla^*)$ on B and $(g_F, {}^F\nabla, {}^F\nabla^*)$. We have the following result:

Proposition 3.3. *Let (g, D, D^*) the dualistic strucure on $B \times_b F$ induced from $(g_B, {}^B\nabla, {}^B\nabla^*)$ on B and $(g_F, {}^F\nabla, {}^F\nabla^*)$. If the connections ${}^B\nabla, {}^B\nabla^*, {}^F\nabla$ and ${}^F\nabla^*$ are symmetric and torsion free, then the induced connections D and D^* are also symmetric and torsion free.*

4. Dually flat twisted product manifolds

Let (g, D, D^*) the dualistic strucure on $B \times_b F$ induced from the dualistic structures $(g_B, {}^B\nabla, {}^B\nabla^*)$ and $(g_F, {}^F\nabla, {}^F\nabla^*)$ on B and F respectively. By Proposition 2.5, we have:

Lemma 4.1. *Let $R, {}^BR$ and ${}^FR$ be the Riemannian curvature operators with respect to $\nabla, {}^B\nabla$ and ${}^F\nabla$ respectively. It holds*

$$\begin{aligned}
R(X, Y)Z &= {}^BR(X, Y)Z; \\
R(X, Y)U &= 0; \\
R(X, U)Y &= \frac{h^b_B(X, Y)}{b} U; \\
R(U, V)X &= UX(\log b)V - VX(\log b)U; \\
R(X, U)V &= [VX(\log b)]U - g(U, V)\Big[\frac{\nabla^B_X(grad_B b)}{b} + grad_F(X(\log b))\Big]; \\
R(U, V)W &= {}^FR(U, V)W + g(U, W) grad_B(V(\log b)) - g(V, U) grad_B(U(\log b)) \\
&\quad - \frac{|grad_B b|^2}{b^2}[g(V, W)U - g(U, W)V].
\end{aligned}$$

and let $R^*, {}^BR^*$ and ${}^FR^*$ be the Riemannian curvature operators with respect to $\nabla^*, {}^B\nabla^*$ and ${}^F\nabla^*$ respectively.

$$\begin{aligned}
R^*(X,Y)Z &= {}^BR^*(X,Y)Z;\\
R^*(X,Y)U &= 0;\\
R^*(X,U)Y &= \frac{h^b_B(X,Y)}{b}U;\\
R^*(U,V)X &= UX(\log b)V - VX(\log b)U;\\
R^*(X,U)V &= [VX(\log b)]U - g(U,V)\Big[\frac{\nabla^B_X(grad_B b)}{b} + grad_F(X(\log b))\Big];\\
R^*(U,V)W &= {}^FR^*(U,V)W + g(U,W)grad_B(V(\log b)) - g(V,U)grad_B(U(\log b))\\
&- \frac{|grad_B b|^2}{b^2}[g(V,W)U - g(U,W)V].
\end{aligned}$$

Remark 4.1. [8] *Assume M is a warped product. Then $(B\times_b F, g, D, D^*)$ is a dually flat space if and only if $(B, g_B, {}^B\nabla, {}^B\nabla^*)$ is also dually and $(F, g_F, {}^F\nabla, {}^F\nabla^*)$ is a Riemannian manifold of constant sectional curvature.*

Now, we can give our main theorem:

Theorem 4.1. *Let $B\times_b F$ be a twisted product of (B, g_B) and (F, g_F) with twisting function b and $\dim F > 1$. Assume $Ric(X,V) = 0$ for all $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, then $(B\times_b F, g, D, D^*)$ is a dually flat space if and only if $(B, g_B, {}^B\nabla, {}^B\nabla^*)$ is dually flat and $(F, g_F, {}^F\nabla, {}^F\nabla^*)$ is of constant sectional curvature.*

Proof. Let $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$, then from Proposition 2.6, we have:

$$Ric(X,V) = (1-s)XV(k).$$

If $Ric(X,V) = 0$, then it follows that $XV(k) = 0$ and $VX(k) = 0$ for all $X \in \mathcal{L}(B)$ and $V \in \mathcal{L}(F)$. Now, $XV(k) = 0$ implies that $V(k)$ only depends on the points of F, and likewise, $VX(k) = 0$ implies that $X(k) = 0$ only depends on the points of B. Thus k can be expressed as a sum of two functions α and β which are defined on B on F, respectively, that is, $k(p,q) = \alpha(p) + \beta(q)$ for any $(p,q) \in B\times F$. Hence $b = \exp(\alpha)\exp(\beta)$, that is, $b(p,q) = \delta(p)\gamma(q)$, where $\delta = \exp(\alpha)$ and $\beta = \exp(\beta)$ for any $(p,q) \in B\times F$. Thus we can write $g = g_B \oplus \delta^2 g_{\mathcal{F}}$ where $g_{\mathcal{F}} = \gamma^2 g_F$, that is, the twisted product manifold $B\times_b F$ can be expressed as a warped product $B\times_\delta F$, where the metric tensor of F is $g_{\mathcal{F}}$ is given above. Thus Theorem is obvious from Remark 4.1. ☐

Theorem 4.2. *Let $B\times_b F$ be a twisted product of (B, g_B) and (F, g_F) with twisting function b. Assume either B is Weyl conformal flat along F or F is Weyl conformal flat along B. Then $(B\times_b F, g, D, D^*)$ is a dually flat space if and only if $(B, g_B, {}^B\nabla, {}^B\nabla^*)$ is dually flat and $(F, g_F, {}^F\nabla, {}^F\nabla^*)$ is of constant sectional curvature.*

Proof. From Proposition 2.7, it follows that $VX(k) = 0$ and $XV(k) = 0$. The rest of the proof is similar to the previous theorem. ☐

Theorem 4.3. *Let $B\times_b F$ be a twisted product manifold. Assume*

(1) *either the conformal Weyl tensor is parallel and $H^k(X) \neq -X(k)\nabla k$ with $\dim B \neq 1$;*
(2) *or $H^k(X) = -X(k)\nabla k$.*

Then $(B \times_b F, g, D, D^*)$ *is a dually flat space if and only if* $(B, g_B, {}^B\nabla, {}^B\nabla^*)$ *is dually flat and* $(F, g_F, {}^F\nabla, {}^F\nabla^*)$ *is of constant sectional curvature.*

Proof. From Theorem 3.6 and Theorem 3.7 of [6] and Corollary 2.5. □

REFERENCES

[1] S. Amari, *Differential-Geometrical Methods in Statistics*, Lecture Notes in Statistics, 28, Springer-Verlag, 1985.

[2] S. Amari and H. Nagaoka, *Methods of Information geometry*, AMS, Oxford University Press, vol. 191, 2000.

[3] N. Ay and W. Tuschmann, *Dually flat manifolds and global information geometry*, Open Syst. and Information Dyn. 9 (2002), 195-200.

[4] A. S. Diallo, *Dualistic structures on doubly warped product manifolds*, Int. Electron. J. Geom., 6, (1), (2013), 41-45.

[5] M. Fernandez-Lopez, E. Garcia-Río, D. N. Kupeli and B. Unal, *A curvature condition for a twisted product to be a warped product*, Manuscripta Math. 106 (2001), 213-217.

[6] S. Kazan and B. Sahin, *Characterization of Twisted Product manifolds to be warped product manifold*, Acta Math. Univ. Comenianae, Vol. LXXXII, 2 (2013), 253-263.

[7] B. O'Neill, *Semi-Riemannian geometry*, Academic Press, New-York, 1983.

[8] L. Todjihounde *Dualistic structures on warped product manifolds*, Diff. Geom.-Dyn. Syst. 8, (2006), 278-284.

UNIVERSITÉ ALIOUNE DIOP DE BAMBEY,
UFR SATIC, DÉPARTEMENT DE MATHÉMATIQUES,
B.P. 30, BAMBEY, SÉNÉGAL,
E-mail address: abdoulsalam.diallo@uadb.edu.sn

INSTITUT DE MATHÉMATIQUES ET DE SCIENCES PHYSIQUES,
B.P. 630, PORTO-NOVO, BÉNIN
E-mail address: leonardt@imsp-uac.org