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AN INTERESTING APPLICATION OF THE BRITISH FLAG THEOREM

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ABSTRACT. We will use the British flag theorem to prove an elegant theorem for two similarly oriented regular polygons-2n.

MSC 2010: 51M04, 51M25.
Keywords: signed distances, centroid, circumcenter.

1. INTRODUCTION

The British flag theorem is one of the simplest theorems in plane geometry.

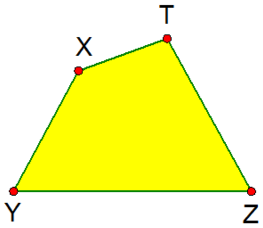
Theorem 1.1 (British flag)**.** *If $ABCD$ be a rectangle and P be any point on the plane, then*

$$PA^2 + PC^2 = PB^2 + PD^2 \tag{1}$$

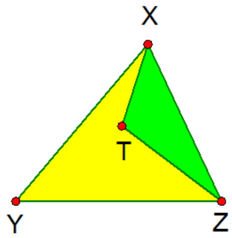
Theorem 1.1 could easily be given as an assignment for secondary school students after they have learnt the Pythagoras theorem. Theorem 1.1 can be found in [1,p.87]. It is impossible to list all the applications of theorem 1.1. In this article, by proving a new theorem, an elegant theorem for two similarly oriented regular polygons-2n, we will be introducing another interesting application of theorem 1.1.
The new theorem is stated using the concept of signed area of a quadrilateral.

Definition 1.1. *The signed area of a quadrangle $XYZT$ is a number, denoted as $S\left[XYZT\right]$, and defined as $S\left[XYZT\right] = \frac{1}{2}\boldsymbol{XZ} \wedge \boldsymbol{YT}$, where notation $\boldsymbol{a} \wedge \boldsymbol{b}$ refers to the cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. $\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}\left|\boldsymbol{a}\right|\left|\boldsymbol{b}\right| sin(\boldsymbol{a}, \boldsymbol{b})$, where $(\boldsymbol{a}, \boldsymbol{b})$ is the directional angle between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.*

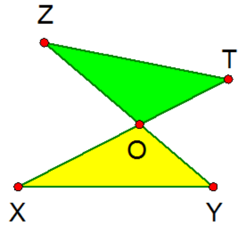
Apparently, $S\left[XYZT\right] = S\left[YZTX\right] = S\left[ZTXY\right] = S\left[TXYZ\right]$.
Denote the area of a polygon as $S(.)$.
• $S\left[XYZT\right] = S\left(XYZT\right)$ if quadrangle XYZT is convex and positively orientated (f.1a);
• $S\left[XYZT\right] = S\left(XYZ\right) - S\left(XTZ\right)$ if quadrangle XYZT is concave at T and triangle XYZ is positively orientated (f.1b);
• $S\left[XYZT\right] = S\left(XYO\right) - S\left(ZTO\right)$ if quadrangle $XYZT$ cuts itself at $O = XT \cap YZ$ and triangle XYO is positively orientated (f.1c);
• $S\left[XYZT\right] = S\left(ZTO\right) - S\left(XYO\right)$ if quadrangle XYZT cuts itself at $O = XT \cap YZ$ and triangle XYO is negatively orientated (f.1.d).
The yellow triangles on figures 1 are positively orientated (1.a, 1.b, 1.c, 1.d) and the green ones are negatively orientated (1.b, 1,c, 1.d). Definition 1.1 can be found in [2, pp. 178-184].



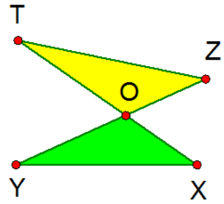
(f.1.a)



(f.1.b)



(f.1.c)



(f.1.d)

Theorem 1.2. *If $A_1A_2...A_{2n}$ and $B_1B_2...B_{2n}$ are two similarly oriented regular polygons, then $S\left[A_iA_{i+1}B_{i+1}B_i\right] + S\left[A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}\right]$ is constant for any $i \in \{1; 2; ...; 2n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.*

Due to the concept of signed area in theorem 1.2, regular polygon $B_1B_2...B_{2n}$ does not have to lie inside regular polygon $A_1A_2...A_{2n}$; quadrangles $A_iA_{i+1}B_{i+1}B_i$ and $A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}$ can cut themselves for any $i \in \{1; 2; ...; 2n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.

2. Proof of the theorem 1.2

First, we need one lemma.

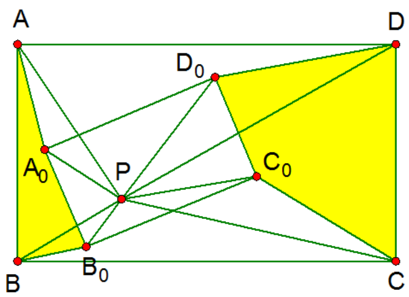
Lemma 2.1. *If $ABCD$ and $A_0B_0C_0D_0$ are two similar and similarly oriented rectangles, then*

$$S\left[ABB_0A_0\right] + S\left[CDD_0C_0\right] = \frac{1}{2}\left(\boldsymbol{AB} \wedge \boldsymbol{AC} - \boldsymbol{A_0B_0} \wedge \boldsymbol{A_0C_0}\right).$$

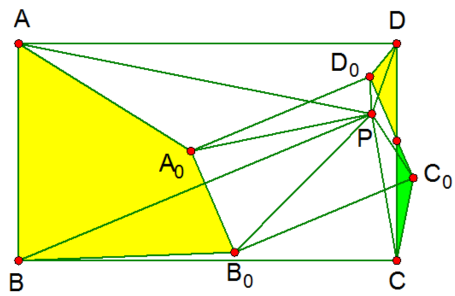
Proof of lemma 2.1. Because $ABCD$ and $A_0B_0C_0D_0$ are similar and similarly oriented, there exist a point P, which is the centre of spiral similarity transforming $ABCD$ into $A_0B_0C_0D_0$ and real numbers k and α such that (f.2).

$$\frac{PA_0}{PA} = \frac{PB_0}{PB} = \frac{PC_0}{PC} = \frac{PD_0}{PD} = k;$$
$$(\mathbf{PA},\, \mathbf{PA_0}) \equiv (\mathbf{PB},\, \mathbf{PB_0}) \equiv (\mathbf{PC},\, \mathbf{PC_0}) \equiv (\mathbf{PD},\, \mathbf{PD_0}) \equiv \alpha \,(\mathrm{mod} 2\pi).$$

Thus, by theorem 1.1, noting that $\mathbf{CD} = -\mathbf{AB}$; $\mathbf{C_0D_0} = -\mathbf{A_0B_0}$, we have



(f.2.a)



(f.2.b)

$$
\begin{aligned}
&2\left(S\left[ABB_0A_0\right] + S\left[CDD_0C_0\right]\right)\\
&= \mathbf{AB_0} \wedge \mathbf{BA_0} + \mathbf{CD_0} \wedge \mathbf{DC_0}\\
&= (\mathbf{PB_0} - \mathbf{PA}) \wedge (\mathbf{PA_0} - \mathbf{PB}) + (\mathbf{PD_0} - \mathbf{PC}) \wedge (\mathbf{PC_0} - \mathbf{PD})\\
&= -\mathbf{PB_0} \wedge \mathbf{PB} - \mathbf{PA} \wedge \mathbf{PA_0} + \mathbf{PA} \wedge \mathbf{PB} + \mathbf{PB_0} \wedge \mathbf{PA_0}\\
&\quad -\mathbf{PD_0} \wedge \mathbf{PD} - \mathbf{PC} \wedge \mathbf{PC_0} + \mathbf{PC} \wedge \mathbf{PD} + \mathbf{PD_0} \wedge \mathbf{PC_0}\\
&= PB_0 \cdot PB \sin\alpha - PA \cdot PA_0 \sin\alpha + PD_0 \cdot PD \sin\alpha - PC \cdot PC_0 \sin\alpha\\
&\quad +\mathbf{PA} \wedge (\mathbf{PA} + \mathbf{AB}) + \mathbf{PC} \wedge (\mathbf{PC} + \mathbf{CD}) + (\mathbf{PA_0} + \mathbf{A_0B_0}) \wedge \mathbf{PA_0} + (\mathbf{PC_0} + \mathbf{C_0D_0}) \wedge \mathbf{PC_0}\\
&= k \sin\alpha \left(PB^2 + PD^2 - PA^2 - PC^2\right) + \mathbf{PA} \wedge \mathbf{AB} + \mathbf{PC} \wedge \mathbf{CD} + \mathbf{A_0B_0} \wedge \mathbf{PA_0} + \mathbf{C_0D_0} \wedge \mathbf{PC_0}\\
&= -\mathbf{AB} \wedge \mathbf{PA} + \mathbf{AB} \wedge \mathbf{PC} + \mathbf{A_0B_0} \wedge \mathbf{PA_0} - \mathbf{A_0B_0} \wedge \mathbf{PC_0}\\
&= \mathbf{AB} \wedge (\mathbf{PC} - \mathbf{PA}) - \mathbf{A_0B_0} \wedge (\mathbf{PC_0} - \mathbf{PA_0})\\
&= (\mathbf{AB} \wedge \mathbf{AC} - \mathbf{A_0B_0} \wedge \mathbf{A_0C_0}).
\end{aligned}
$$

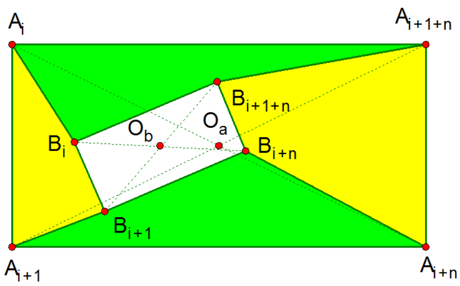
Therefore, $S[ABB_0A_0] + S\left[CDD_0C_0\right] = \frac{1}{2}\left(\mathbf{AB} \wedge \mathbf{AC} - \mathbf{A_0B_0} \wedge \mathbf{A_0C_0}\right).$ □

Note. A Spiral similarity with center P, rotation angle α and similarity coefficient k is the sum of a central similarity with center P and similarity coefficient k and a rotation about P through the angle α, taken in either order [3, p.36].

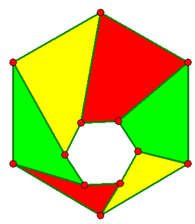
Next, we are going to prove theorem 1.2 (f.3.a, f.3.b).

Without the loss of generality, assume that $A_1A_2...A_{2n}$ and $B_1B_2...B_{2n}$ are positively oriented.

Let O_a and O_b are the centres of $A_1A_2...A_{2n}$ and $B_1B_2...B_{2n}$ respectively.



(f.3.a)



(f.3.b)

Because $A_1A_2...A_{2n}$ and $B_1B_2...B_{2n}$ are regular polygons that share a positive orientation, $A_iA_{i+1}A_{i+n}A_{i+1+n}$ and $B_iB_{i+1}B_{i+n}B_{i+1+n}$ are similar and positively oriented rectangles for any $i \in \{1; 2; ...; n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.

Hence, by the lemma 2.1, we have

$$
\begin{aligned}
&S\left[A_iA_{i+1}B_{i+1}B_i\right] + S\left[A_{i+n}A_{i+1+n}B_{i+1+n}B_{i+n}\right]\\
&= \tfrac{1}{2}\left(\mathbf{A_iA_{i+1}} \wedge \mathbf{A_iA_{i+n}} - \mathbf{B_iB_{i+1}} \wedge \mathbf{B_iB_{i+n}}\right)\\
&= \tfrac{1}{2}\left(A_iA_{i+1}.A_iA_{i+n}\sin(\mathbf{A_iA_{i+1}}, \mathbf{A_iA_{i+n}}) - B_iB_{i+1}.B_iB_{i+n}\sin(\mathbf{B_iB_{i+1}}, \mathbf{B_iB_{i+n}})\right)\\
&= \tfrac{1}{2}\left(A_iA_{i+1}.A_iA_{i+n}\sin\widehat{A_{i+1}A_iA_{i+n}} - B_iB_{i+1}.B_iB_{i+n}\sin\widehat{B_{i+1}B_iB_{i+n}}\right)\\
&= \tfrac{1}{2}\left(2S(A_iA_{i+1}A_{i+n}) - 2S(B_iB_{i+1}B_{i+n})\right)\\
&= \tfrac{1}{2}\left(4S(O_aA_iA_{i+1}) - 4S(O_bB_iB_{i+1})\right)\\
&= 2\left(S(O_aA_iA_{i+1}) - S(O_bB_iB_{i+1})\right)\\
&= 2\left(\tfrac{1}{2n}S\left(A_1A_2...A_{2n}\right) - \tfrac{1}{2n}S\left(B_1B_2...B_{2n}\right)\right)\\
&= \tfrac{1}{n}\left(S\left(A_1A_2...A_{2n}\right) - S\left(B_1B_2...B_{2n}\right)\right).
\end{aligned}
$$

This means that $S\left[A_iA_{i+1}B_{i+1}B_i\right] + S\left[A_{n+i}A_{n+i+1}B_{n+i+1}B_{n+i}\right]$ is constant for any $i \in \{1; 2; ...; 2n\}$, assuming that $A_{2n+1} = A_1$ and $B_{2n+1} = B_1$.

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