A CURVATURE PROPERTY OF GENERALIZED RANDERS CHANGE OF M-TH ROOT METRICS

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ABSTRACT. In this paper, we consider generalized Randers change of m-th root Finsler metrics and find necessary and sufficient condition under which a generalized Randers change of an m-th root metric be locally dually flat.

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1. INTRODUCTION

Let \((M, F)\) be a Finsler manifold. A change of Finsler metric \(F \rightarrow \bar{F}\) is called a generalized Randers change of \(F\), if

\[
\bar{F}(x, y) = F(x, y) + \beta(x, y),
\]

(1)

where \(\beta(x, y) = b_i(x, y)y^i\) is a 1-form on a smooth manifold \(TM\). It is easy to see that, if \(\sup_{F(x, y)=1}|b_i(x, y)y^i| < 1\), then \(\bar{F}\) is again a Finsler metric. It is showed that if \(\beta\) is closed, then \(\bar{F}\) is pointwise projective to \(F\). The notion of a Randers change has been proposed by Matsumoto and studied in detail by Shibata [6, 10]. If \(F\) reduces to a Riemannian metric then \(\bar{F}\) reduces to a Randers metric. Due to this reason the transformation (1) has been called the generalized Randers change of Finsler metric. For other Finslerian transformations see [10, 21].

In [2], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [9]. A Finsler metric \(F\) on a manifold \(M\) is said to be locally dually flat if at any point there is a coordinate system \((x^i)\) in which the spray coefficients are in the following form

\[
G^i = -\frac{1}{2}g^{ij}H_{y^j},
\]

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where $H = H(x, y)$ is a $C^\infty$ homogeneous scalar function on $TM_0$. Such a coordinate system is called an adapted coordinate system [13,18,19]. Indeed, a Finsler metric $F$ on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies

$$\frac{\partial^2 F^2}{\partial x^i \partial y^j} y^k = 2 \frac{\partial F^2}{\partial x^i}.$$

Let $(M, F)$ be a Finsler manifold of dimension $n$, $TM$ its tangent bundle and $(x^i, y^j)$ the coordinates in a local chart on $TM$. Let $F$ be the following function on $M$, by $F = \sqrt[1/m]{A}$, where $A$ is given by $A := a_{i_1,\ldots,i_m}(x)^{i_1} y^{i_2} \cdots y^{i_m}$ with $a_{i_1,\ldots,i_m}$ symmetric in all its indices (see [4,7,11,13–17]). Then $F$ is called an $m$-th root Finsler metric. Recently studies show that the theory of $m$-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory [7].

For quartic metrics, a study of the geodesics and of the related geometrical objects is made by S. Lebedev [5]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and N. Brinzei in two papers [3]. Tensorial connections for such spaces have been recently studied by L. Tamassy [12]. In [13], Tayebi-Najafi characterize locally dually flat and Antonelli $m$-th root Finsler metrics. They show that every $m$-th root Finsler metric of isotropic mean Berwald curvature reduces to a weakly Berwald metric. In [14], they prove that every $m$-th root Finsler metric of isotropic Landsberg metric reduces to a Landsberg metric. Then, they show that every $m$-th root Finsler metric with almost vanishing $H$-curvature satisfies $H = 0$. Recently, Tayebi-Nankali-Peyghan define some non-Riemannian curvature properties for Cartan spaces and consider Cartan space with the $m$-th root metric [15]. They prove that every $m$-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively.

Suppose that $A_{ij}$ define a positive definite tensor and $A^{ij}$ denotes its inverse. For an $m$-th root metric $F$, put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i.$$

In this paper, we consider generalized Randers change of an $m$-th root Finsler metric and find necessary and sufficient condition under which a generalized Randers change of an $m$-th root metric be locally dually flat. More precisely, we prove the following.

**Theorem 1.1.** Let $F = \sqrt[1/m]{A}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $F = F + \beta$ be generalized Randers change of $F$ where $\beta = b_i(x, y) y^i$. Then $F$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_i(x) y^i$ on $U$ such that the following hold

1. $\beta_{00} \beta + \beta_{i0} \beta_0 = 2 \beta_{x^i} \beta_0$, \hspace{1cm} (2)
2. $A_{x^i} = \frac{1}{3m} \left[m A \theta_0 + 2 \theta A_i\right]$, \hspace{1cm} (3)
3. $2 \psi \left[ \psi_{00} \beta + \psi_{0i} \beta_i + \psi_{i0} \beta_0 - 2 \psi_{x^i} \beta \right] - 4 \psi^2 \left[ 2 \beta_{x^i} - \beta_{00} \right] = \psi_{0} \psi_{0} \beta$, \hspace{1cm} (4)
where

\[\psi := A^{\frac{2}{m}},\]
\[\psi_p := \frac{2}{m} A^{\frac{2}{m} - 1} A_p,\]
\[\psi_{0p} := \frac{2}{m} A^{\frac{2}{m} - 2} \left[ \left( \frac{2}{m} - 1 \right) A_p A_0 + A A_{0p} \right],\]
\[\beta_l := \beta_{y^l}, \quad \beta_{0l} := \beta_{x^l} y^k, \quad \beta_{xl} := (b_l)_{y^l} y^k, \quad \beta_{0} := \beta_x y^l, \quad \beta_{0l} = (b_l)_0 + (b_l)_{x^l} y^l y^k.\]

2. **Proof of the Theorem 1.1**

Recently, Li-Shen defined the notion of generalized \(m\)-th root Finsler metrics and studied locally projectively flat generalized fourth root metrics. The Finsler metrics defined by \(F = \sqrt[m]{A^2 + B}\) are called generalized \(m\)-th root metrics, where \(B := b_{ij}(x) y^i y^j\). In [16], Tayebi-Nankali-Peyghan find necessary and sufficient condition under which conformal \(\beta\)-change of an \(m\)-th root metric be locally dually flat. In [21], Tayebi-Tabatabaeifar-Peyghan consider Kropina change of \(m\)-th root Finsler metrics and find necessary and sufficient condition under which the Kropina change of an \(m\)-th root Finsler metric be locally dually flat. For the Kropina metric and its properties, see [20]. Then in [17], Tayebi-Peyghan-Shahbazi consider these metrics and characterize locally dually flat generalized \(m\)-th root Finsler metrics. They find a condition under which a generalized \(m\)-th root metric is projectively related to a \(m\)-th root metric. This motivates us to consider the other curvature properties of generalized \(m\)-th root metrics.

In this section, we will prove a generalized version of Theorem 1.1. Indeed, we find necessary and sufficient condition under which a generalized Randers change of an generalized \(m\)-th root metric be locally dually flat. Let \(F\) be a scalar function on \(TM\) defined by following

\[F = \sqrt[m]{A^2 + B},\]

where \(A\) and \(B\) are given by

\[A := a_{i_1 \ldots i_m}(x) y^{i_1} \ldots y^{i_m}, \quad B := b_{ij}(x) y^i y^j.\]  \hspace{1cm} (5)

Then \(F\) is called generalized \(m\)-th root Finsler metric.
Suppose that the matrix \((A_{ij})\) defines a positive definite tensor and \((A^{ij})\) denotes its inverse. Then the following hold

\[
g_{ij} = \frac{A_{m}^{2}}{m^{2}} [mA_{ij} + (2 - m)A_{i}A_{j}] + b_{ij},
\]

\[
A_{i} := \frac{\partial A}{\partial y^{i}}, \quad A_{ij} := \frac{\partial^{2} A}{\partial y^{i} \partial y^{j}},
\]

\[
B_{i} := \frac{\partial B}{\partial y^{i}}, \quad B_{ij} := \frac{\partial^{2} B}{\partial y^{i} \partial y^{j}},
\]

\[
A_{x^{i}} := \frac{\partial A}{\partial x^{i}}, \quad A_{0} := A_{x^{i}y^{i}},
\]

\[
B_{x^{i}} := \frac{\partial B}{\partial x^{i}}, \quad B_{0} := B_{x^{i}y^{i}},
\]

\[
A_{0l} := A_{x^{i}y^{i}y^{k}} = \frac{\partial^{2} A}{\partial x^{i} \partial y^{i} \partial y^{k}} y^{k},
\]

\[
B_{0l} := B_{x^{i}y^{i}y^{k}} = \frac{\partial^{2} B}{\partial x^{i} \partial y^{i} \partial y^{k}} y^{k}.
\]

Now, we are going to prove the following.

**Theorem 2.1.** Let \(F = \sqrt{A^{2/m} + B}\) be an generalized \(m\)-th root Finsler metric on an open subset \(U \subset \mathbb{R}^{n}\), where \(A\) is irreducible. Suppose that \(\bar{F} = F + \beta\) be generalized Randers change of \(F\) where \(\beta = b_{i}(x, y)y^{i}\). Then \(\bar{F}\) is locally dually flat if and only if there exists a 1-form \(\theta = \theta_{i}(x)y^{i}\) on \(U\) such that the following holds

\[
2 \left[ \beta_{0l}\beta + \beta_{l}\beta_{0} - 2\beta_{x^{i}}\beta \right] = 2B_{x^{i}} - B_{0l}, \quad (6)
\]

\[
A_{x^{i}} = \frac{1}{3m} [mA_{i} + 2\theta A_{i}], \quad (7)
\]

\[
2Y \left[ Y_{0l}\beta + Y_{0}\beta_{l} + Y_{l}\beta_{0} - 2Y_{x^{i}}\beta \right] - 4Y^{2} \left[ 2\beta_{x^{i}} - \beta_{0l} \right] = Y_{i}Y_{0}\beta, \quad (8)
\]

where

\[
Y := A^{2/m} + B,
\]

\[
Y_{p} := \frac{2}{m}A^{m-1}A_{p} + B_{p},
\]

\[
Y_{0p} := \frac{2}{m}A^{m-2} \left[ \left( \frac{2}{m} - 1 \right)A_{p}A_{0} + AA_{0p} \right] + B_{0p},
\]

\[
\beta_{0l} = \beta_{x^{i}y^{k}}, \quad \beta_{x^{i}} = (b_{i})_{x^{i}}y^{i}, \quad \beta_{0} = (b_{i})_{0}y^{i}, \quad \beta_{0l} = (b_{i})_{0} + (b_{i})_{x^{i}y^{i}y^{k}}.
\]

To prove Theorem 2.1, we need the following.
Lemma 2.2. Suppose that the following equation holds
\[ \Phi A^\frac{2}{m} - 2 + \psi A^\frac{1}{m} - 1 + \Theta = 0, \]
where \( \Phi, \psi, \Theta \) are polynomials in \( y \) and \( m > 2 \). Then \( \Phi = \psi = \Theta = 0 \).

Now, we are going to prove the Theorem 2.1.

Proof of the Theorem 2.1: Let \( \bar{F} \) be a locally dually flat metric. The following hold
\[ \bar{F}^2 = A^\frac{2}{m} + B + 2\beta \left[ A^\frac{2}{m} + B \right]^\frac{1}{2} + \beta^2, \]
\[ (\bar{F}^2)_{x} = \frac{2}{m} A^\frac{2}{m} - 1 A_{x} + B_{x} + \beta \left[ \frac{2}{m} A^\frac{2}{m} - 1 A_{x} + B_{x} \right] \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} \]
\[ + 2 \left[ A^\frac{2}{m} + B \right]^\frac{1}{2} \beta_{x} + 2\beta_{x} \beta. \]

Then
\[ [\bar{F}^2]_{x} y = \frac{2}{m} A^\frac{2}{m} - 2 \left[ \left( \frac{2}{m} - 1 \right) A_{1} A_{0} + A A_{0l} \right] + 2 \left[ \beta_{0l} + \beta_{l} \beta_{0} \right] \]
\[ + B_{0l} - \frac{1}{2} \beta_{0l} \left[ A_{0} \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} + Y_{0l} \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} \right] \]
\[ + \beta Y_{0l} \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} + \beta_{0l} Y_{l} \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} \]
\[ + 2\beta_{0l} \left[ A^\frac{2}{m} + B \right]^\frac{1}{2}. \]

Thus, we get
\[ + \left[ A^\frac{2}{m} + B \right]^{-\frac{1}{2}} \left[ - \frac{1}{2} Y_{0l} \beta + \left( A^\frac{2}{m} + B \right) \left( Y_{0l} \beta + Y_{0l} \beta_{l} + Y_{l} \beta_{0} - 2Y_{x} \beta \right) \right] \]
\[ + 2 \left[ A^\frac{2}{m} + B \right]^2 \left[ \beta_{0l} - 2\beta_{x} \beta \right] + 2 \left[ \beta_{0l} + \beta_{l} \beta_{0} - 2\beta_{x} \beta \right] \]
\[ + \frac{1}{m} A^\frac{2}{m} - 2 \left[ \left( \frac{2}{m} - 1 \right) A_{1} A_{0} + A A_{0l} - 2 A A_{x} \right] + B_{0l} - 2B_{x} = 0. \]

By Lemma 2.2, we have
\[ \left[ \frac{2}{m} - 1 \right] A_{1} A_{0} + A A_{0l} = 2 A A_{x}, \]
\[ - \frac{1}{2} Y_{0l} \beta + Y \left[ Y_{0l} \beta + Y_{0l} \beta_{l} + Y_{l} \beta_{0} - 2Y_{x} \beta \right] = 2Y^{2} \left[ 2\beta_{x} \beta - \beta_{0l} \right], \]
\[ 2 \left[ \beta_{0l} + \beta_{l} \beta_{0} - 2\beta_{x} \beta \right] = 2B_{x} - B_{0l}. \]
One can rewrite (9) as follows
\[ A \left[ 2A_{x^l} - A_{0l} \right] = \left[ \frac{2}{m} - 1 \right] A_l A_0. \]  
(11)

Irreducibility of \( A \) and \( \deg(A_l) = m - 1 \) imply that there exists a 1-form \( \theta = \theta_l y^l \) on \( U \) such that
\[ A_0 = \theta A. \]  
(12)

Plugging (12) into (11), we get
\[ A_{0l} = A \theta_l + \theta A_l - A_{x^l}. \]  
(13)

Substituting (12) and (13) into (11) yields
\[ A_{x^l} = \frac{1}{3m} \left[ m A \theta_l + 2 \theta A_l \right]. \]

The converse is a direct computation. This completes the proof. □

**Proof of the Theorem 1.1:** In the Theorem 2.1, let us put \( B = 0 \). Then, we get the proof of the Theorem 1.1.

Now, suppose that \( \beta \) is a one-form on a manifold \( M \). In this case, \( \beta = b_i(x) y^i \) and then we have
\[ (b_i)_l = 0, \quad \beta_{0l} = (b_i)_0. \]

Then by the Theorem 2.1, we get the following.

**Corollary 2.1.** Let \( F = \sqrt{A^{2/m} + B} \) be an generalized \( m \)-th root Finsler metric on an open subset \( U \subset \mathbb{R}^n \), where \( A \) is irreducible. Suppose that \( \bar{F} = F + \beta \) be Randers change of \( F \) where \( \beta = b_i(x) y^i \). Then \( \bar{F} \) is locally dually flat if and only if there exists a 1-form \( \theta = \theta_l y^l \) on \( U \) such that the following holds
\[ 2 \left[ \beta_{0l} \beta + \beta_l \beta_0 - 2 \beta_{x^l} \beta \right] = 2B_{x^l} - B_{0l}, \]  
(14)
\[ A_{x^l} = \frac{1}{3m} \left[ m A \theta_l + 2 \theta A_l \right], \]  
(15)
\[ -\frac{1}{2} \mathbf{Y}_l \mathbf{Y}_0 \beta + \mathbf{Y}_0 \beta + \mathbf{Y}_0 \beta_l + \mathbf{Y}_l \beta_0 - 2 \mathbf{Y}_{x^l} \beta = 2 \mathbf{Y}^2 \left[ 2 \beta_{x^l} - \beta_{0l} \right], \]  
(16)

where
\[ \beta_{0l} = \beta_{x^l} y^k, \quad \beta_{x^l} = (b_i)_x y^i, \quad \beta_0 = (b_i)_0 y^i, \quad \beta_{0l} = (b_i)_0, \]
\[ \mathbf{Y} := \dot{A}^\frac{2}{m} + B, \]
\[ \mathbf{Y}_p := \frac{2}{m} A^{-\frac{2}{m} - 1} A_p + B_p, \]
\[ \mathbf{Y}_{0p} := \frac{2}{m} A^{-\frac{2}{m} - 2} \left( \frac{2}{m} - 1 \right) A_p A_0 + A A_{0p} + B_{0p}. \]
In the Theorem 1.1, let us put $B = 0$. Then the generalized $m$-th root metric $F = \sqrt[A]{B^{2/m} + B}$ reduces to a $m$-th root metric $F = A^{1/m}$. In this case, we have

$$B_p = 0, \quad B_{0p} = 0.$$  

Then by the Theorem 1.1, we obtain the following.

**Corollary 2.2.** Let $F = A^{1/m}$ be an $m$-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of $F$ where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta_l = \theta_l(x)y^l$ on $U$ such that the following holds

$$A_{x^l} = \frac{1}{3m} \left[ m A \theta_l + 2 \theta A_l \right],$$  

$$-\frac{1}{2} \psi_l \psi_0 \beta + \psi \left[ \psi_{0l} \beta + \psi_0 \beta_l + \psi_l \beta_0 - 2 \psi_{x^l} \beta \right] = 2 \psi^2 \left[ 2 \beta_{x^l} - \beta_{0l} \right],$$  

$$\beta_{0l} \beta + \beta_l \beta_0 = 2 \beta_{x^l} \beta,$$

where

$$\psi := A^{2/m},$$  

$$\psi_p := \frac{2}{m} A^{2/m-1} A_p,$$  

$$\psi_{0p} := \frac{2}{m} A^{2/m-2} \left[ \left( \frac{2}{m} - 1 \right) A_p A_0 + A A_{0p} \right],$$  

$$\beta_{0l} = \beta_{x^l} y^l, \quad \beta_{x^l} = (b_i)_{x^l} y^i, \quad \beta_0 = \beta_{x^l} y^l, \quad \beta_{0l} = (b_i)_0.$$

### 3. Berwald-Moór Metric

In 4-dimension, the special fourth root metric in the form

$$F = \sqrt[4]{y^1 y^2 y^3 y^4}$$

is called the Berwald-Moór metric. This metric is singular in $y$ and not positive definite. Recently, physicists are interested in fourth-root of metrics [8]. Thus it is important to study the geometric properties of fourth root metrics. For the Berwald-Moór metric $F = \sqrt[4]{y^1 y^2 y^3 y^4}$, we have the following:

$$A_{x^l} = 0, \quad A_0 = 0.$$  

Then, we get the following.

**Theorem 3.1.** Let $F = \sqrt[4]{y^1 y^2 y^3 y^4}$ be the Berwald-Moór Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of $F$ where $\beta = b_i(x,y)y^i$. Then $\bar{F}$ is locally dually flat if and only if $\beta = 0$. In this case, $\bar{F} = F$.  

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Proof. By the Theorem 1.1, $F = \sqrt[4]{y^1 y^2 y^3 y^4}$ is locally dually flat if and only if the following hold
\begin{align}
2A\theta_l + \theta A_l &= 0, \quad (20) \\
\beta A_l &= Ab_l, \quad (21) \\
\beta_0\beta + \beta_1\beta_0 &= 2\beta_{x^l} \beta. \quad (22)
\end{align}
By multiplying (20) with $y^l$, we get $\theta = 0$. Contracting (21) with $y^l$ implies that $\beta = 0$. □

Let us suppose that $\beta = b_i(x)y^i$ is a 1-form on $M$. In this case, we have $(b_i)_l = 0$. By Theorem 3.1, we conclude the following.

**Corollary 3.1.** Let $F = \sqrt[4]{y^1 y^2 y^3 y^4}$ be the Berwald-Moór Finsler metric on an open subset $U \subset \mathbb{R}^n$, where $A$ is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of $F$ where $\beta = b_i(x)y^i$ is a 1-form on $M$. Then $\bar{F}$ is locally dually flat if and only if $\beta = 0$. In this case, $\bar{F} = F$.

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