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[1]

A CURVATURE PROPERTY OF GENERALIZED RANDERS CHANGE OF M-TH ROOT METRICS

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ABSTRACT. In this paper, we consider generalized Randers change of m-th root Finsler metrics and find necessary and sufficient condition under which a generalized Randers change of an m-th root metric be locally dually flat.

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Keywords: Locally dually flat metric, projectively flat metric, m-th root metric.

1. INTRODUCTION

Let (M, F) be a Finsler manifold. A change of Finsler metric $F \to \bar{F}$ is called a generalized Randers change of F, if

$$\bar{F}(x, y) = F(x, y) + \beta(x, y), \tag{1}$$

where $\beta(x, y) = b_i(x, y)y^i$ is a 1-form on a smooth manifold TM. It is easy to see that, if $\sup_{F(x,y)=1}|b_i(x, y)y^i| < 1$, then $\bar{F}$ is again a Finsler metric. It is showed that if β is closed, then $\bar{F}$ is pointwise projective to F. The notion of a Randers change has been proposed by Matsumoto and studied in detail by Shibata [6, 10]. If F reduces to a Riemannian metric then $\bar{F}$ reduces to a Randers metric. Due to this reason the transformation (1) has been called the generalized Randers change of Finsler metric. For other Finslerian transformations see [10, 21].

In [2], Amari-Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory and multi-terminal information theory [1]. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [9]. A Finsler metric F on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

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where $H = H(x, y)$ is a C^∞ homogeneous scalar function on TM_0. Such a coordinate system is called an adapted coordinate system [13, 18, 19]. Indeed, a Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called dually flat if it satisfies

$$\frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k = 2\frac{\partial F^2}{\partial x^l}.$$

Let (M, F) be a Finsler manifold of dimension n, TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM. Let F be the following function on M, by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ with $a_{i_1 \dots i_m}$ symmetric in all its indices (see [4, 7, 11, 13–17]). Then F is called an m-th root Finsler metric. Recently studies show that the theory of m-th root Finsler metrics plays a very important role in physics, theory of space-time structure, gravitation, general relativity and seismic ray theory [7]. For quartic metrics, a study of the geodesics and of the related geometrical objects is made by S. Lebedev [5]. Also, Einstein equations for some relativistic models relying on such metrics are studied by V. Balan and N. Brinzei in two papers [3]. Tensorial connections for such spaces have been recently studied by L. Tamassy [12]. In [13], Tayebi-Najafi characterize locally dually flat and Antonelli m-th root Finsler metrics. They show that every m-th root Finsler metric of isotropic mean Berwald curvature reduces to a weakly Berwald metric. In [14], they prove that every m-th root Finsler metric of isotropic Landsberg metric reduces to a Landsberg metric. Then, they show that every m-th root Finsler metric with almost vanishing H-curvature satisfies $\mathbf{H} = 0$. Recently, Tayebi-Nankali-Peyghan define some non-Riemannian curvature properties for Cartan spaces and consider Cartan space with the m-th root metric [15]. They prove that every m-th root Cartan space of isotropic Landsberg curvature, or isotropic mean Landsberg curvature, or isotropic mean Berwald curvature reduces to a Landsberg, weakly Landsberg and weakly Berwald space, respectively.

Suppose that A_{ij} define a positive definite tensor and A^{ij} denotes its inverse. For an m-th root metric F, put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^j \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i.$$

In this paper, we consider generalized Randers change of an m-th root Finsler metric and find necessary and sufficient condition under which a generalized Randers change of an m-th root metric be locally dually flat. More precisely, we prove the following.

Theorem 1.1. *Let $F = \sqrt[m]{A}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of F where $\beta = b_i(x, y) y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x) y^l$ on U such that the following hold*

$$\beta_{0l}\beta + \beta_l \beta_0 = 2\beta_{x^l}\beta, \tag{2}$$

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big], \tag{3}$$

$$2\psi\Big[\psi_{0l}\beta + \psi_0\beta_l + \psi_l\beta_0 - 2\psi_{x^l}\beta\Big] - 4\psi^2\Big[2\beta_{x^l} - \beta_{0l}\Big] = \psi_l\psi_0\beta, \tag{4}$$

where

$$\psi := A^{\frac{2}{m}},$$
$$\psi_p := \frac{2}{m} A^{\frac{2}{m}-1} A_p,$$
$$\psi_{0p} := \frac{2}{m} A^{\frac{2}{m}-2} \left[\left(\frac{2}{m} - 1 \right) A_p A_0 + A A_{0p} \right],$$
$$\beta_l := \beta_{y^i} = b_l, \quad \beta_{0l} := \beta_{x^k y^l} y^k, \quad \beta_{x^l} := (b_i)_{x^l} y^i, \quad \beta_0 := \beta_{x^l} y^i, \quad \beta_{0l} = (b_l)_0 + (b_i)_{x^k y^l} y^i y^k.$$

2. Proof of the Theorem 1.1

Recently, Li-Shen defined the notion of generalized m-th root Finsler metrics and studied locally projectively flat generalized fourth root metrics. The Finsler metrics defined by $F = \sqrt{\sqrt[m]{A^2} + B}$ are called generalized m-th root metrics, where $B := b_{ij}(x) y^i y^j$. In [16], Tayebi-Nankali-Peyghan find necessary and sufficient condition under which conformal β-change of an m-th root metric be locally dually flat. In [21], Tayebi-Tabatabaeifar-Peyghan consider Kropina change of m-th root Finsler metrics and find necessary and sufficient condition under which the Kropina change of an m-th root Finsler metric be locally dually flat. For the Kropina metric and its properties, see [20]. Then in [17], Tayebi-Peyghan-Shahbazi consider these metrics and characterize locally dually flat generalized m-th root Finsler metrics. They find a condition under which a generalized m-th root metric is projectively related to a m-th root metric. This motivates us to consider the other curvature properties of generalized m-th root metrics.
In this section, we will prove a generalized version of Theorem 1.1. Indeed, we find necessary and sufficient condition under which a generalized Randers change of an generalized m-th root metric be locally dually flat. Let F be a scalar function on TM defined by following

$$F = \sqrt{A^{2/m} + B},$$

where A and B are given by

$$A := a_{i_1 \ldots i_m}(x) y^{i_1} \ldots y^{i_m}, \quad B := b_{ij}(x) y^i y^j. \tag{5}$$

Then F is called generalized m-th root Finsler metric.

Suppose that the matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Then the following hold

$$
\begin{aligned}
g_{ij} &= \frac{A^{\frac{2}{m}-2}}{m^2}[mAA_{ij} + (2-m)A_iA_j] + b_{ij},\\
A_i &:= \frac{\partial A}{\partial y^i}, \qquad A_{ij} := \frac{\partial^2 A}{\partial y^j \partial y^j},\\
B_i &:= \frac{\partial B}{\partial y^i}, \qquad B_{ij} := \frac{\partial^2 B}{\partial y^j \partial y^j},\\
A_{x^i} &:= \frac{\partial A}{\partial x^i}, \qquad A_0 := A_{x^i}y^i,\\
B_{x^i} &:= \frac{\partial B}{\partial x^i}, \qquad B_0 := B_{x^i}y^i,\\
A_{0l} &:= A_{x^k y^l}y^k = \frac{\partial^2 A}{\partial x^i \partial y^l}y^k,\\
B_{0l} &:= B_{x^k y^l}y^k = \frac{\partial^2 B}{\partial x^i \partial y^l}y^k.
\end{aligned}
$$

Now, we are going to prove the following.

Theorem 2.1. *Let $F = \sqrt{A^{2/m} + B}$ be an generalized m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of F where $\beta = b_i(x,y)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following holds*

$$2\Big[\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta\Big] = 2B_{x^l} - B_{0l}, \tag{6}$$

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big], \tag{7}$$

$$2\mathbf{Y}\Big[\mathbf{Y}_{0l}\beta + \mathbf{Y}_0\beta_l + \mathbf{Y}_l\beta_0 - 2\mathbf{Y}_{x^l}\beta\Big] - 4\mathbf{Y}^2\Big[2\beta_{x^l} - \beta_{0l}\Big] = \mathbf{Y}_l\mathbf{Y}_0\beta, \tag{8}$$

where

$$
\begin{aligned}
&\mathbf{Y} := A^{\frac{2}{m}} + B,\\
&\mathbf{Y}_p := \frac{2}{m}A^{\frac{2}{m}-1}A_p + B_p,\\
&\mathbf{Y}_{0p} := \frac{2}{m}A^{\frac{2}{m}-2}\left[\left(\frac{2}{m}-1\right)A_pA_0 + AA_{0p}\right] + B_{0p},\\
&\beta_{0l} = \beta_{x^k y^l}y^k, \quad \beta_{x^l} = (b_i)_{x^l}y^i, \quad \beta_0 = (b_i)_0 y^i, \quad \beta_{0l} = (b_l)_0 + (b_i)_{x^k y^l}y^i y^k.
\end{aligned}
$$

To prove Theorem 2.1, we need the following.

Lemma 2.2. *Suppose that the following equation holds*

$$\mathbf{\Phi} A^{\frac{2}{m}-2} + \psi A^{\frac{1}{m}-1} + \mathbf{\Theta} = 0,$$

where $\mathbf{\Phi}, \psi, \mathbf{\Theta}$ *are polynomials in* y *and* $m > 2$. *Then* $\mathbf{\Phi} = \psi = \mathbf{\Theta} = 0$.

Now, we are going to prove the Theorem 2.1.

Proof of the Theorem 2.1: Let $\bar{F}$ be a locally dually flat metric. The following hold

$$\begin{aligned}
\bar{F}^2 &= A^{\frac{2}{m}} + B + 2\beta\Big[A^{\frac{2}{m}} + B\Big]^{\frac{1}{2}} + \beta^2,\\
(\bar{F}^2)_{x^k} &= \frac{2}{m}A^{\frac{2}{m}-1}A_{x^k} + B_{x^k} + \beta\left[\frac{2}{m}A^{\frac{2}{m}-1}A_{x^k} + B_{x^k}\right]\Big[A^{\frac{2}{m}} + B\Big]^{-\frac{1}{2}}\\
&+ \;\; 2\Big[A^{\frac{2}{m}} + B\Big]^{\frac{1}{2}}\beta_{x^k} + 2\beta_{x^k}\beta.
\end{aligned}$$

Then

$$\begin{aligned}
[\bar{F}^2]_{x^k y^l}y^k = \; & \frac{2}{m}A^{\frac{2}{m}-2}\left[(\frac{2}{m} - 1)A_l A_0 + AA_{0l}\right] + 2\Big[\beta_{0l}\beta + \beta_l\beta_0\Big]\\
& + B_{0l} - \frac{1}{2}\beta\mathbf{Y}_l\mathbf{Y}_0\Big[A^{\frac{2}{m}} + B\Big]^{-\frac{3}{2}} + \mathbf{Y}_0\beta_l\Big[A^{\frac{2}{m}} + B\Big]^{-\frac{1}{2}}\\
& + \beta\mathbf{Y}_{0l}\Big[A^{\frac{2}{m}} + B\Big]^{-\frac{1}{2}} + \beta_0\mathbf{Y}_l\Big[A^{\frac{2}{m}} + B\Big]^{-\frac{1}{2}}\\
& + 2\beta_{0l}\Big[A^{\frac{2}{m}} + B\Big]^{\frac{1}{2}}.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
& + \Big[A^{\frac{2}{m}} + B\Big]^{-\frac{3}{2}}\Bigg[-\frac{1}{2}\mathbf{Y}_l\mathbf{Y}_0\beta + \Big(A^{\frac{2}{m}} + B\Big)\Big(\mathbf{Y}_{0l}\beta + \mathbf{Y}_0\beta_l + \mathbf{Y}_l\beta_0 - 2\mathbf{Y}_{x^l}\beta\Big)\\
& + 2\Big[A^{\frac{2}{m}} + B\Big]^2\Big[\beta_{0l} - 2\beta_{x^l}\Big]\Bigg] + 2\Big[\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta\Big]\\
& + \frac{1}{m}A^{\frac{2}{m}-2}\left[(\frac{2}{m} - 1)A_l A_0 + AA_{0l} - 2AA_{x^k}\right] + B_{0l} - 2B_{x^l} = 0.
\end{aligned}$$

By Lemma 2.2, we have

$$\Big[\frac{2}{m} - 1\Big]A_l A_0 + AA_{0l} = 2AA_{x^k}, \tag{9}$$

$$-\frac{1}{2}\mathbf{Y}_l\mathbf{Y}_0\beta + \mathbf{Y}\Big[\mathbf{Y}_{0l}\beta + \mathbf{Y}_0\beta_l + \mathbf{Y}_l\beta_0 - 2\mathbf{Y}_{x^l}\beta\Big] = 2\mathbf{Y}^2\Big[2\beta_{x^l} - \beta_{0l}\Big],$$

$$2\Big[\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta\Big] = 2B_{x^l} - B_{0l}. \tag{10}$$

One can rewrite (9) as follows

$$A\Big[2A_{x^l} - A_{0l}\Big] = \Big[\frac{2}{m} - 1\Big] A_l A_0. \tag{11}$$

Irreducibility of A and $deg(A_l) = m - 1$ imply that there exists a 1-form $\theta = \theta_l y^l$ on U such that

$$A_0 = \theta A. \tag{12}$$

Plugging (12) into (11), we get

$$A_{0l} = A\theta_l + \theta A_l - A_{x^l}. \tag{13}$$

Substituting (12) and (13) into (11) yields

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big].$$

The converse is a direct computation. This completes the proof. □

Proof of the the Theorem 1.1: In the Theorem 2.1, let us put $B = 0$. Then, we get the proof of the the Theorem 1.1.

Now, suppose that β is a one-form on a manifold M. In this case, $\beta = b_i(x)y^i$ and then we have

$$(b_i)_l = 0, \quad \beta_{0l} = (b_l)_0.$$

Then by the Theorem 2.1, we get the following.

Corollary 2.1. *Let $F = \sqrt{A^{2/m} + B}$ be an generalized m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following holds*

$$2\Bigg[\beta_{0l}\beta + \beta_l\beta_0 - 2\beta_{x^l}\beta\Bigg] = 2B_{x^l} - B_{0l}, \tag{14}$$

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big], \tag{15}$$

$$-\frac{1}{2}\mathbf{Y}_l\mathbf{Y}_0\beta + \mathbf{Y}\Bigg[\mathbf{Y}_{0l}\beta + \mathbf{Y}_0\beta_l + \mathbf{Y}_l\beta_0 - 2\mathbf{Y}_{x^l}\beta\Bigg] = 2\mathbf{Y}^2\Bigg[2\beta_{x^l} - \beta_{0l}\Bigg], \tag{16}$$

where

$$\beta_{0l} = \beta_{x^k y^l}y^k, \quad \beta_{x^l} = (b_i)_{x^l}y^i, \quad \beta_0 = (b_i)_0 y^i, \quad \beta_{0l} = (b_l)_0,$$

$$\mathbf{Y} := A^{\frac{2}{m}} + B,$$

$$\mathbf{Y}_p := \frac{2}{m}A^{\frac{2}{m}-1}A_p + B_p,$$

$$\mathbf{Y}_{0p} := \frac{2}{m}A^{\frac{2}{m}-2}\Bigg[\Big(\frac{2}{m} - 1\Big)A_pA_0 + AA_{0p}\Bigg] + B_{0p}.$$

In the Theorem 1.1, let us put $B = 0$. Then the generalized m-th root metric $F = \sqrt{A^{2/m} + B}$ reduces to a m-th root metric $F = A^{\frac{1}{m}}$. In this case, we have

$$B_p = 0, \quad B_{0p} = 0.$$

Then by the Theorem 1.1, we obtain the following.

Corollary 2.2. *Let $F = A^{\frac{1}{m}}$ be an m-th root Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be Randers change of F where $\beta = b_i(x)y^i$. Then $\bar{F}$ is locally dually flat if and only if there exists a 1-form $\theta = \theta_l(x)y^l$ on U such that the following holds*

$$A_{x^l} = \frac{1}{3m}\Big[mA\theta_l + 2\theta A_l\Big], \tag{17}$$

$$-\frac{1}{2}\psi_l\psi_0\beta + \psi\Bigg[\psi_{0l}\beta + \psi_0\beta_l + \psi_l\beta_0 - 2\psi_{x^l}\beta\Bigg] = 2\psi^2\Bigg[2\beta_{x^l} - \beta_{0l}\Bigg], \tag{18}$$

$$\beta_{0l}\beta + \beta_l\beta_0 = 2\beta_{x^l}\beta, \tag{19}$$

where

$$\psi := A^{\frac{2}{m}},$$
$$\psi_p := \frac{2}{m}A^{\frac{2}{m}-1}A_p,$$
$$\psi_{0p} := \frac{2}{m}A^{\frac{2}{m}-2}\Bigg[\Big(\frac{2}{m} - 1\Big)A_pA_0 + AA_{0p}\Bigg],$$
$$\beta_{0l} = \beta_{x^ky^l}y^k, \quad \beta_{x^l} = (b_i)_{x^l}y^i, \quad \beta_0 = \beta_{x^l}y^i, \quad \beta_{0l} = (b_l)_0.$$

3. Berwald-Moór Metric

In 4-dimension, the special fourth root metric in the form

$$F = \sqrt[4]{y^1y^2y^3y^4}$$

is called the Berwald-Moór metric. This metric is singular in y and not positive definite. Recently, physicists are interested in fourth-root of metrics [8]. Thus it is important to study the geometric properties of fourth root metrics. For the Berwald-Moór metric $F = \sqrt[4]{y^1y^2y^3y^4}$, we have the following:

$$A_{x^l} = 0, \quad A_0 = 0.$$

Then, we get the following.

Theorem 3.1. *Let $F = \sqrt[4]{y^1y^2y^3y^4}$ be the Berwald-Moór Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of F where $\beta = b_i(x, y)y^i$. Then $\bar{F}$ is locally dually flat if and only if $\beta = 0$. In this case, $\bar{F} = F$.*

Proof. By the Theorem 1.1, $F = \sqrt[4]{y^1y^2y^3y^4}$ is locally dually flat if and only if the following hold

$$2A\theta_l + \theta A_l = 0, \tag{20}$$

$$\beta A_l = Ab_l, \tag{21}$$

$$\beta_{0l}\beta + \beta_l\beta_0 = 2\beta_{x^l}\beta. \tag{22}$$

By multiplying (20) with y^l, we get $\theta = 0$. Contracting (21) with y^l implies that $\beta = 0$. □

Let us suppose that $\beta = b_i(x)y^i$ is a 1-form on M. In this case, we have $(b_i)_l = 0$. By Theorem 3.1, we conclude the following.

Corollary 3.1. *Let $F = \sqrt[4]{y^1y^2y^3y^4}$ be the Berwald-Moór Finsler metric on an open subset $U \subset \mathbb{R}^n$, where A is irreducible. Suppose that $\bar{F} = F + \beta$ be generalized Randers change of F where $\beta = b_i(x)y^i$ is a 1-form on M. Then $\bar{F}$ is locally dually flat if and only if $\beta = 0$. In this case, $\bar{F} = F$.*

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