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CONFORMAL RICCI SOLITON IN KENMOTSU MANIFOLD

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ABSTRACT. In this paper we introduce the conformal Ricci soliton equation which is generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation and study quasi conformal, conharmonic and projective curvature tensors in a Kenmotsu manifold admitting conformal Ricci soliton.

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1. INTRODUCTION

The concept of Ricci soliton [1] was introduced by Hamilton [2] in mid 80's. Ricci solitons are natural generalizations of Einstein metrices and also correspond to self-similar solutions of Hamilton's Ricci flow [3]. It often arises as limits of dilations of singularities in the Ricci flow. Ricci solitons are of interests to physicists as well and are called quasi-Einstein metrics in Physics literature. Ramesh Sharma [4] started the study of Ricci soliton in contact manifolds and after him M.M.Tripathi [5], Bejan[6], Crasmareanu [7] studied Ricci soliton in contact metric manifolds. The Ricci soliton equation is given by

$$\pounds_X g + 2S + 2\lambda g = 0 \tag{1.1}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ a real scalar.
The Ricci soliton is said to be shirking, steady and expanding according as λ is negative, zero and positive respectively. Nagaraja and Permalatha [8] studied the conditions for Ricci soliton in Kenmotsu manifold [9] to be shrinking, steady and expanding.

In 2005 A.E. Fischer [10] has introduced a new concept called conformal Ricci flow which a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The resulting equations are named the conformal Ricci flow equations because of the role that conformal geometry plays in constraining the scalar curvature and because these equations are the vector field sum of a conformal flow equation and a Ricci flow equation. These new equations are given by

$$\begin{aligned} \frac{\partial g}{\partial t} + 2(S + \frac{g}{n}) &= -pg \\ R(g) &= -1. \end{aligned} \tag{1.2}$$

Where $R(g)$ is the scalar curvature of the manifold and p is scalar non-dynamical field and n is the dimension of manifold.The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy, the time-dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint.

We introduce the conformal Ricci soliton equation as

$$\pounds_X g + 2S = [2\lambda - (p + \frac{2}{n})]g. \tag{1.3}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

2. Preliminaries

An $n(odd)-$dimensional smooth manifold M is said to be an almost contact metric manifold if it admits an almost contact metric structure (ϕ, ξ, η, g) consisting of a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η and a Riemannian metric g compatible with (ϕ, ξ, η) satisfying

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta o \phi = 0 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

An almost contact metric manifold is said to be a Kenmotsu manifold [8] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi(X) \tag{2.3}$$

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.4}$$

Where ∇ denotes the Riemannian connection of M.

In a Kenmotsu manifold the following relations hold

$$R(X, Y)Z = g(X, Z)Y - g(Y, Z)X \tag{2.5}$$

$$(\nabla_X \eta)Y = g(\phi X, \phi Y) \tag{2.6}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z) \tag{2.7}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X \tag{2.8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi \tag{2.9}$$

$$R(\xi, X)\xi = X - \eta(X)\xi. \tag{2.10}$$

Using (2.1) and (2.4) we have

$$(\pounds_{\xi} g)(X,Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.11}$$

Now applying conformal Ricci soliton(1.3) in (2.11), we get

$S(X,Y) = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 1]g(X,Y) + \frac{1}{2}\eta(X)\eta(Y).$

Let us take $A = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 1]$. So we can write the above equation as

$$S(X,Y) = Ag(X,Y) + \frac{1}{2}\eta(X)\eta(Y) \tag{2.12}$$

$$QX = AX + \frac{1}{2}\eta(X)\xi \tag{2.13}$$

$$S(X,\xi) = (A + \frac{1}{2})\eta(X), \tag{2.14}$$

which shows that the manifold is $\eta-$ Einstein.

If we put $X = Y = e_i$ in (2.14), where $\{e_i\}$ is an orthonormal basis, and summing over i, we get $R(g) = An + \frac{1}{2}$. But for conformal Ricci flow, $R(g) = -1$ which leads us to get $A = -\frac{3}{2n}$. As we have $A = \frac{1}{2}[2\lambda - (p + \frac{2}{n}) - 1]$, putting this in the previous result we get

$$\lambda = \frac{1}{2}(p + 1 - \frac{1}{n}). \tag{2.15}$$

Proposition 2.1: A Kenmotsu manifold admitting conformal Ricci soliton is $\eta-$Einstein and the scalar λ of the conformal Ricci soliton is equal to $\lambda = \frac{1}{2}(p + 1 - \frac{1}{n})$.

3. Kenmotsu manifold admitting conformal Ricci soliton and $R(\xi, W) \cdot \tilde{C} = 0$

Let M be an $n-$dimensional Kenmotsu manifold admitting a conformal Ricci soliton (g, V, λ). Quasi conformal curvature tensor $\tilde{C}$ in M is defined by

$$\begin{aligned} \tilde{C}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY] + \frac{1}{n}(\frac{a}{n-1} + 2b)[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{3.1}$$

where a, b are constants.

Putting $Z = \xi$ in (3.1) we have by using (2.8), (2.13) and(2.14),

$\tilde{C}(X,Y)\xi = [-a + b(A + \frac{1}{2}) + A + \frac{1}{n}(\frac{a}{n-1} + 2b)](\eta(Y)X - \eta(X)Y).$

Let $B = [-a + b(A + \frac{1}{2}) + A + \frac{1}{n}(\frac{a}{n-1} + 2b)]$, so the above equation can be written as

$$\tilde{C}(X,Y)\xi = B\,[\,\eta(Y)\,X\;-\;\eta(X)\,Y]. \tag{3.2}$$

Similarly we obtain

$$\eta(\tilde{C}(X,Y)Z) = B[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.3}$$

Now we assume that $R(\xi,W)\cdot\tilde{C} = 0$ holds in M. Which implies

$$\begin{aligned} R(\xi,X)(\tilde{C}(Y,Z)W) - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)W)W \\ -\tilde{C}(Y,Z)R(\xi,X)W = 0. \end{aligned} \tag{3.4}$$

Using (2.9), we get

$$\begin{aligned} \eta(\tilde{C}(Y,Z)W)X - \grave{\tilde{C}}(X,Y,Z,W)\xi - \eta(Y)\tilde{C}(X,Z)W + g(X,Y)\tilde{C}(\xi,Z)W \\ -\eta(Z)\tilde{C}(Y,X)W + g(X,Z)\tilde{C}(Y,\xi)W - \eta(W)\tilde{C}(Y,Z)X + g(X,W)\tilde{C}(Y,Z)\xi = 0 \end{aligned} \tag{3.5}$$

where $\grave{\tilde{C}}(X,Y,Z,W) = g(X,\tilde{C}(Y,Z)W)$.

Operating η on (3.5) and using (2.1), (3.2) we obtain

$$B\,[-g(X,Y)\,g(Z,W) + g(X,Z)\,g(Y,W)] - \grave{\tilde{C}}(X,Y,Z,W) = 0. \tag{3.6}$$

Now using (3.1) in (3.6), we get

$$\begin{aligned} B[-g(X,Y)g(Z,W) + g(X,Z)g(Y,W)] - a\grave{R}(X,Y,Z,W) - b[S(Z,W)g(X,Y) - S(Y,W)g(X,Z) \\ +g(Z,W)g(X,QY) - g(Y,W)g(X,QZ)] - \frac{1}{n}(\frac{a}{n-1} + 2b)[g(W,Z)g(X,Y) - g(Y,W)g(X,Z)] = 0. \end{aligned} \tag{3.7}$$

Taking $X = Y = e_i$ in (3.7) and summing over i, we get

$$[B(1-n) - \frac{1}{n}(\frac{a}{n-1} + 2b)(n-1) + b]g(Z,W) - [a + b(n-1) - 1]S(Z,W) = 0. \tag{3.8}$$

Which shows that the manifold becomes Einstein.Thus we can state the following theorem.

Theorem 3.1: A Kenmotsu manifold admitting conformal Ricci soliton and $R(\xi,W)\cdot\tilde{C} = 0$ is an Einstein manifold, where $\tilde{C}$ is quasi conformal curvature tensor and $R(\xi,W)$ is derivation of tensor algebra of the tangent space of the manifold.

4. Kenmotsu manifold admitting conformal Ricci soliton and $H(\xi,X).S = 0$

Let M be an n-dimensional Kenmotsu manifold admitting a Ricci Soliton (g,V,λ). The conharmonic curvature tensor on M is defined by

$$H(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{4.1}$$

So we can write

$H(\xi, X)Y = R(\xi, X)Y - \frac{1}{n-2}[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX].$

Now using (2.9), (2.12) and (2.13) in the above equation we get

$$H(\xi, X)Y = C[\eta(Y)X - g(X, Y)\xi] \tag{4.2}$$

where $C = \frac{4A+1}{2(n-2)} + 1$.

Also we can write

$$\eta(H(\xi, X)Z) = C[\eta(Z)\eta(X) - g(X, Z)]. \tag{4.3}$$

We assume that $H(\xi, X) \cdot S = 0$ holds. Then we have

$$S(H(\xi, X)Y, Z) + S(Y, H(\xi, X)Z) = 0. \tag{4.4}$$

Using (2.12) in (4.4), we have

$$Ag(H(\xi, X)Y, Z) + \frac{1}{2}\eta(H(\xi, X)Y)\eta(Z) + Ag(H(\xi, X)Z, Y) + \frac{1}{2}\eta(H(\xi, X)Z)\eta(Y) = 0. \tag{4.5}$$

Using (4.2) and (4.3) in (4.5), we have

$C\eta(Z)\eta(X)\eta(Y) - Cg(X, Z)\eta(Y) = 0.$

As C and $\eta(Y)$ are not equals to zero so we have $g(X, Z) = \eta(X)\eta(Z)$.

Theorem4.1: If a Kenmotsu manifold admits conformal Ricci soliton equation and $H(\xi, X) \cdot S = 0$ then $g(X, Z) = \eta(X)\eta(Z)$ holds for all tangent vectors $X, Z \in \chi(M)$, where H is conharmonic curvature tensor and S is Ricci tensor.

5. Kenmotsu manifold admitting conformal Ricci soliton and $P(\xi, X) \cdot \tilde{C} = 0$

The projective curvature tensor is given as

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{5.1}$$

We know

$\tilde{C}(\xi, X)Y = aR(\xi, X)Y + b[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX]$

$+ \frac{1}{n}(\frac{a}{n-1} + 2b)[g(Y, X)\xi - g(\xi, Y)X].$

Using (2.13) and (2.14), we get

$$\tilde{C}(\xi,X)Y = aR(\xi,X)Y + b[S(X,Y)\xi - (A+\frac{1}{2})\eta(Y)X + g(X,Y)\xi(A+\frac{1}{2}) \\ -A\eta(Y)X - \frac{1}{2}\eta(Y)\eta(X)\xi] + \frac{1}{n}(\frac{a}{n-1} + 2b)[g(X,Y)\xi - \eta(Y)X]. \tag{5.2}$$

Using (2.9) in the above equation, we get

$\tilde{C}(\xi,X)Y = a[\eta(Y)X - g(X,Y)\xi] + b[S(X,Y)\xi - (A+\frac{1}{2})\eta(Y)X$

$+g(X,Y)\xi(A+\frac{1}{2}) - A\eta(Y)X - \frac{1}{2}\eta(Y)\eta(X)\xi] + \frac{1}{n}(\frac{a}{n-1} + 2b)[g(X,Y)\xi - \eta(Y)X].$

Operating with η we have

$$\eta(\tilde{C}(\xi,X)Y) = [a - \frac{1}{n}(\frac{a}{n-1} + 2b)](\eta(Y)\eta(X) - g(X,Y)) \\ +b[S(X,Y) + (A+\frac{1}{2})g(X,Y) - (2A+1)\eta(X)\eta(Y)] \tag{5.3}$$

$$\eta(\tilde{C}(X,Y)\xi) = 0 \tag{5.4}$$

Now from (5.1) we have

$P(\xi,X)Y = R(\xi,X)Y - \frac{1}{n-1}[S(X,Y)\xi - S(\xi,Y)X].$

Using (2.12) and (2.14) in the above equation, we obtain

$$P(\xi,X)Y = (\frac{1}{2} - A)\eta(Y)X - (1 + \frac{A}{n-1})g(X,Y)\xi - \frac{1}{2(n-1)}\eta(X)\eta(Y)\xi. \tag{5.5}$$

Assume that in M, $P(\xi,X)\cdot\tilde{C} = 0$ holds.
So we can write

$$P(\xi,X)\tilde{C}(Y,Z)W - \tilde{C}(P(\xi,X)Y,Z)W - \tilde{C}(Y,P(\xi,X)Z)W - \tilde{C}(Y,Z)P(\xi,X)W = 0 \tag{5.6}$$

for all vector fields X, Y, Z, W on M.

Using (5.5) in(5.6) and then operating η we get,

$$(\frac{1}{2}-A)\eta(\tilde{C}(Y,Z)W)\eta(X)-(1+\frac{A}{n-1})g(X,\tilde{C}(Y,Z)W)-\frac{1}{2(n-1)}\eta(X)\eta(\tilde{C}(Y,Z)W)$$
$$-(\frac{1}{2}-A)\eta(Y)\eta(\tilde{C}(X,Z)W)-(1+\frac{A}{n-1})g(X,Z)\eta(\tilde{C}(Y,\xi)W)-\frac{1}{2(n-1)}\eta(X)\eta(Z)\eta(\tilde{C}(Y,\xi)W)$$
$$-(\frac{1}{2}-A)\eta(W)\eta(\tilde{C}(Y,Z)X)-(1+\frac{A}{n-1})g(X,W)\eta(\tilde{C}(Y,Z)\xi)-\frac{1}{2(n-1)}\eta(X)\eta(W)\eta(\tilde{C}(Y,Z)\xi)=0. \quad (5.7)$$

.
Using (3.2) and (3.3) in (5.7) and then putting $X=Y=Z=W=e_i$ and summing over i,we get
$B[n+A-\frac{1}{2}]=0$

which implies either $B=0$ or $A=\frac{1}{2}-n$. If we consider $B=0$ and putting the value of B, we have, $a+(b+1)[\lambda-\frac{1}{2}(p+\frac{2}{n})]+\frac{1}{n}(\frac{a}{n-1}+2b)-\frac{1}{2}=0$.

Similarly if we take $A=\frac{1}{2}-n$ and put the values of A, we obtain $\lambda-\frac{1}{2}(p+\frac{1}{n})=1-n$. Thus we can state the following theorem.

Theorem: If M is a n dimensional Kenmotsu manifold satisfying conformal Ricci soliton and if in M, $P(\xi,X)\cdot\tilde{C}=0$ holds, where P is the Projective curvature tensor, then either $\lambda=\frac{\frac{-1}{n}(\frac{a}{n-1}+2b)+\frac{1}{2}-a}{b+1}+\frac{1}{2}(p+\frac{2}{n})$ or $\lambda=\frac{1}{2}(p+\frac{1}{n})+1-n$.

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