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ERDÖS - MORDELL INEQUALITY AND COMPLEX NUMBERS

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ABSTRACT. We present a proof of the Erdös-Mordell inequality using dot and cross products of complex numbers.

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1. INTRODUCTION

With their natural two dimensional vector representations, complex numbers are often used to obtain results in plane geometry [1–3]. Plane transformations as homotheties, translations, symmetries, and similarities can also be described using complex numbers [2–6]. In some complex analysis textbooks two other operations well defined for vectors in two and three dimensions, the dot and cross products, are extended to complex numbers [7, 8]. To our knowledge, these two operations on complex numbers are presented as a curiosity and are not used to obtain results in plane geometry. We will use these two operations here to establish a new proof of Erdös-Mordell inequality. Another proof of Erdös-Mordell inequality recently published in [9] is based on vector analysis and has motivated the present paper. For the history of Erdös-Mordell inequality and references about other proofs see [10].

2. THE SETTING

Let ΔABC be a triangle and O a point in it. Consider a coordinate system with O as it origin and A is on the positive direction of the X-axis, B is in the first or second quadrant, and C is in the third or fourth quadrant. We associate to the vertices A, B, and C their corresponding complex numbers

$$\begin{array}{lll} A = |A|e^{i\theta_A} & , & \theta_A = 0, \\ B = |B|e^{i\theta_B} & , & \theta_B \in (0,\pi), \\ C = |C|e^{i\theta_C} & , & \theta_C \in [\pi, 2\pi), \quad \text{and} \quad \theta_C - \theta_B \in (0,\pi). \end{array} \tag{1}$$

The conditions (1) on the angles imply that O is in the triangle. We also associate to the foot of the perpendiculars from O to the sides their corresponding complex numbers : P for the side joining the vertices B and C, Q for C and A, and R for A and B. With this notation, the Erdös-Mordell inequality is

$$2(|P| + |Q| + |R|) \leq |A| + |B| + |C|. \tag{2}$$

3. Dot and cross products of complex numbers

The dot and cross products of complex numbers are two real numbers defined by the dot and cross products of their corresponding vectors in the plane, or in the space with a null third component. For $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, they are respectively defined by

$$z_1 \odot z_2 = x_1x_2 + y_1y_2 = \mathrm{Re}\{\bar{z}_1 z_2\} = \frac{\bar{z}_1 z_2 + z_1 \bar{z}_2}{2},$$

and

$$z_1 \otimes z_2 = x_1y_2 - x_2y_1 = \mathrm{Im}\{\bar{z}_1 z_2\} = \frac{\bar{z}_1 z_2 - z_1 \bar{z}_2}{2i},$$

where $\mathrm{Re}\{z\} = x$ and $\mathrm{Im}\{z\} = y$ stands respectively for the real part and the imaginary part of $z = x + iy$. So we have

$$\bar{z}_1 z_2 = z_1 \odot z_2 + i\, z_1 \otimes z_2. \tag{3}$$

and $z_1 \odot z_2 = z_2 \odot z_1$, $z_1 \otimes z_2 = -z_2 \otimes z_1$, $\bar{z}z = z \odot z = |z|^2$, $z \otimes z = 0$, $\bar{z}(iz) = i\ [z \otimes (iz)] = i\,|z|^2$, and $z \odot (iz) = 0$.
Considering (3) and $(\bar{z}_1 z_2)^n = \overline{z_1^n} z_2^n$, we obtain

$$z_1^n \odot z_2^n = \sum_{l=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^l \binom{n}{2l} (z_1 \odot z_2)^{n-2l} (z_1 \otimes z_2)^{2l}$$

and

$$z_1^n \otimes z_2^n = \sum_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^l \binom{n}{2l+1} (z_1 \odot z_2)^{n-2l-1} (z_1 \otimes z_2)^{2l+1}$$

which is another form of De Moivre's Theorem. For the special case $n = 2$ we obtain

$$z_1^2 \odot z_2^2 = (z_1 \odot z_2)^2 - (z_1 \otimes z_2)^2, \tag{4}$$

and

$$z_1^2 \otimes z_2^2 = 2\,(z_1 \odot z_2)\,(z_1 \otimes z_2). \tag{5}$$

A useful relation for our purpose is obtained from the identity

$$\begin{aligned} \overline{(z_1 - z_2)}(z_1 + z_2) &= (z_1 - z_2) \odot (z_1 + z_2) + i\,(z_1 - z_2) \otimes (z_1 + z_2) \\ &= \left(|z_1|^2 - |z_2|^2\right) + 2i\, z_1 \otimes z_2, \end{aligned}$$

combined to the fact that $|\overline{(z_1 - z_2)}(z_1 + z_2)| = |(z_1 - z_2)(z_1 + z_2)| = |z_1^2 - z_2^2|$, leads to

$$|z_1^2 - z_2^2|^2 = \left(|z_1|^2 - |z_2|^2\right)^2 + (2\, z_1 \otimes z_2)^2. \tag{6}$$

It follows that

$$|z_1^2 - z_2^2| \geq 2\,|z_1 \otimes z_2| \tag{7}$$

with equality if and only if $|z_1| = |z_2|$.

4. The area of a triangle

Let ΔOFG be a triangle with vertices O (the origin of the axes), and two other vertices noted F and G. Let H be the foot of the perpendicular from O to the side joining F and G. Considering the area of the triangle ΔOFG and the complex numbers associated to F, G, and H, we have

$$|H||F-G| = |F \otimes G| = 2\,\text{Area}(\Delta OFG),$$

from which we get

$$|H| = \frac{|F \otimes G|}{|F-G|} = \frac{|\text{Im}\{\bar{F}G\}|}{|F-G|} = \frac{|\bar{F}G - F\bar{G}|}{2|F-G|}. \tag{8}$$

If $F = f^2$ and $G = g^2$, using (5) and (7), (8) becomes

$$|H| = \frac{|f^2 \otimes g^2|}{|f^2 - g^2|} = \frac{2|f \odot g||f \otimes g|}{|f^2 - g^2|} \leq \frac{2|f \odot g||f \otimes g|}{2|f \otimes g|} = |f \odot g| \tag{9}$$

with equality holding iff $|f| = |g|$.

5. Proof of the inequality

We consider that $A = a^2$, $B = b^2$, and $C = c^2$, where

$$\begin{array}{lcll} a = |a|e^{i\theta_a} & , & \theta_a = 0, & \text{and} \quad 2\theta_a = \theta_A, \\ b = |b|e^{i\theta_b} & , & \theta_b \in (0, \pi/2), & \text{and} \quad 2\theta_b = \theta_B, \\ c = |c|e^{i\theta_c} & , & \theta_c \in [\pi/2, \pi), & \text{and} \quad 2\theta_c = \theta_C. \end{array} \tag{10}$$

It follows from (9) and (10) that

$$|A| + |B| + |C| - 2|P| - 2|Q| - 2|R| \geq |a|^2 + |b|^2 + |c|^2 - 2|b \odot c| - 2|c \odot a| - 2|a \odot b|$$

with equality iff $|a| = |b| = |c|$. Because $a \odot b \geq 0$, $b \odot c \geq 0$, and $c \odot a \leq 0$, we obtain

$$\begin{aligned} &|a|^2 + |b|^2 + |c|^2 - 2|b \odot c| - 2|c \odot a| - 2|a \odot b| \\ &\quad = a \odot a + b \odot b + c \odot c - 2b \odot c + 2c \odot a - 2a \odot b \\ &\quad = (a - b + c) \odot (a - b + c) \\ &\quad = |a - b + c|^2 \\ &\quad \geq 0, \end{aligned}$$

with equality iff $a - b + c = 0$. Hence (2) holds.

Equality holds if and only if $|a| = |b| = |c|$ and $a - b + c = 0$. In this case, using the dot product successively with a, b, and c we obtain the following system of equations

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \begin{bmatrix} |a|^2 \\ |b|^2 \\ |c|^2 \end{bmatrix},$$

and its solution is

$$\begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} |a|^2 \\ |b|^2 \\ |c|^2 \end{bmatrix}.$$

If $|a| = |b| = |c| = d$, we get

$$\begin{bmatrix} a \odot b \\ b \odot c \\ c \odot a \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} d^2,$$

from which we conclude that $\theta_b = \pi/3$ and $\theta_c = 2\pi/3$. Then, in case of equality, ΔABC is an equilateral triangle.

6. Other forms of the Erdös-Mordell inequality

Using (8), we can rewrite (2) as

$$\frac{|\bar{A}B - A\bar{B}|}{|A-B|} + \frac{|\bar{B}C - B\bar{C}|}{|B-C|} + \frac{|\bar{C}A - C\bar{A}|}{|C-A|} \leq |A| + |B| + |C|. \tag{11}$$

If we use inversion, which is the application $z \mapsto \frac{1}{z} = \frac{\bar{z}}{|z|^2}$, the vertices become $A' = \frac{\bar{A}}{|A|^2}$, $B' = \frac{\bar{B}}{|B|^2}$, and $C' = \frac{\bar{C}}{|C|^2}$, and the distances from O to the sides are now $|P'| = \frac{|P|}{|B||C|}$, $|Q'| = \frac{|Q|}{|C||A|}$, and $|R'| = \frac{|R|}{|A||B|}$, and we obtain

$$2\,\frac{|A||P| + |B||Q| + |C||R|}{|A||B||C|} \leq \frac{1}{|A|} + \frac{1}{|B|} + \frac{1}{|C|}. \tag{12}$$

Finally using the polar reciprocity with respect to the unit circle centered at O, which is the application $z \mapsto \frac{z}{|z|^2}$, the foot P, Q, and R are mapped to the vertices of the polar triangle in $A'' = \frac{P}{|P|^2}$, $B'' = \frac{Q}{|Q|^2}$ and $C'' = \frac{R}{|R|^2}$, and the vertices A, B, and C are mapped to the foot of the perpendiculars $P'' = \frac{A}{|A|^2}$, $Q'' = \frac{Q}{|Q|^2}$, and $R'' = \frac{R}{|R|^2}$. Then we obtain

$$2\left[\frac{1}{|A|} + \frac{1}{|B|} + \frac{1}{|C|}\right] \leq \frac{1}{|P|} + \frac{1}{|Q|} + \frac{1}{|R|}. \tag{13}$$

Inequalities (11), (12), and (13) also appeared in [11].

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