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ON $C3$-LIKE FINSLER METRICS OF SCALAR FLAG CURVATURE

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ABSTRACT. In this paper, we consider the class of $C3$-like Finsler metrics. This class of Finsler metrics contains the class of (α, β)-metrics as a special case. We study $C3$-like metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature and find a condition under which these metrics reduce to C-reducible metrics.
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1. INTRODUCTION

For a Finsler manifold (M, F), the flag curvature is a function $\mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. If F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P \setminus \{0\}$, $\mathbf{K}$ being called the sectional curvature in Riemannian geometry. F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(P, y) = \mathbf{K}(x, y)$ is independent of flags P associated with any fixed flagpole y. For example, every two-dimensional Finsler metric is of scalar flag curvature. One of the important problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature.

Let (M, F) be a Finsler manifold. The second derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is an inner product $\mathbf{g}_y$ on T_xM. The third order derivatives of $\frac{1}{2}F_x^2$ at $y \in T_xM_0$ is a symmetric trilinear forms $\mathbf{C}_y$ on T_xM. We call $\mathbf{g}_y$ and $\mathbf{C}_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of Cartan torsion $\mathbf{C}$ along geodesics is called the Landsberg curvature $\mathbf{L}$. A Finsler metric F is said to be isotropic Landsberg metric if $\mathbf{L} + c(x)F\mathbf{C} = 0$, where $c = c(x)$ is a scalar function on M. Taking a trace of $\mathbf{C}$ and $\mathbf{L}$ give us mean Cartan torsion $\mathbf{I}$ and mean Landsberg curvature $\mathbf{J}$, respectively. A Finsler metric F is said to be isotropic mean Landsberg metric if $\mathbf{J} + c(x)F\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M.

In [10], Prasad-Singh introduced a class of Finsler spaces named by $C3$-like spaces. A Finsler metric F is called $C3$-like if its Cartan tensor is given by

$$C_{ijk} = \Big\{a_ih_{jk} + a_jh_{ki} + a_kh_{ij}\Big\} + \Big\{b_iI_jI_k + I_ib_jI_k + I_iI_jb_k\Big\}, \tag{1.1}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on TM of degree -1 and 1, respectively and $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric. If we put

$$a_i = \frac{p}{n+1}I_i \quad \text{and} \quad b_i = \frac{q}{3C^2}I_i,$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM, then F is a semi-C-reducible metric. In [5], Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible. If $b_i = 0$, then F is called C-reducible metric [3]. In [4], Matsumoto-Hōjō proved that a Finsler metric F is C-reducible if and only if it is a Randers metric $F = \alpha + \beta$ or Kropina metric $F = \alpha^2/\beta$. The Randers metrics and Kropina metrics are special (α,β)-metrics. An (α,β)-metric is a Finsler metric on M defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M [12]. Therefore the study of the class of $C3$-like Finsler spaces will enhance our understanding of the geometric meaning of (α,β)-metrics.
Recently, Cheng-Shen have classified Randers metrics of scalar flag curvature with isotropic S-curvature [2]. This class of Randers metrics contains all projectively flat Randers metrics with isotropic S-curvature and Randers metrics of constant flag curvature. In [1], Cheng-Mo-Shen characterize Finsler metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature $\mathbf{J} + c(x)F\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on M. Hence, it is natural problem to study the class of $C3$-like metrics of scalar curvature with relatively isotropic mean Landsberg curvature.
In this paper, we study $C3$-like metrics of scalar curvature with relatively isotropic mean Landsberg curvature and find a condition under which this metrics reduce to C-reducible metric. More precisely, we prove the following.

Theorem 1. *Let (M,F) be a C3-like Finsler manifold such that $b_i = b_i(x,y)$ is constant along Finslerian geodesics, i.e., $b'_i := b_{i|s}y^s = 0$ and $a_iy^i = 0$. Suppose that $\mathbf{J}/\mathbf{I}$ is isotropic,*

$$\mathbf{J} + c(x)F\mathbf{I} = 0, \tag{1.2}$$

where $c = c(x)$ is a scalar function on M. Assume that F is of scalar flag curvature $\mathbf{K}$ such that

$$2c' + 4c^2 + \mathbf{K} \neq 0,$$

where $c' := c_{|s}y^s$. Then F reduces to a C-reducible metric with relatively isotropic Landsberg curvature $\mathbf{L} + c(x)F\mathbf{C} = 0$.

Finsler metrics with $\mathbf{J} = 0$ are said to be weakly Landsbergian. Every Landsberg metric is a weakly Landsbergian. Using Theorem 1, we study weakly Landsberg C3-like metrics of scalar flag curvature.

Corollary 1. *Let (M,F) be a weakly Landsberg C3-like Finsler manifold such that $b_i = b_i(x,y)$ is constant along Finslerian geodesics and $a_iy^i = 0$. Suppose that F is of non-zero scalar flag curvature. Then F is Riemannian metrics of constant sectional curvature.*

It is remarkable that through of this paper, we use the Berwald connection and the h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively [11].

2. Preliminaries

Let M be a n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM = \cup_{x\in M}T_xM$ the tangent bundle of M. A Finsler metric on M is a function

$F : TM \to [0, \infty)$ which has the following properties: (i) F is C^∞ on $TM_0 := TM \setminus \{0\}$; (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM, (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on T_xM is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\left[F^2(y + su + tv)\right]|_{s,t=0}, \ \ u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]|_{t=0}, \ \ u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion. It is well known that **C=0** if and only if F is Riemannian.
For $y \in T_xM_0$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$. By Diecke Theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

For $y \in T_xM_0$, define the Matsumoto torsion $\mathbf{M}_y : T_xM \otimes T_xM \otimes T_xM \to R$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^iv^jw^k$ where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\}, \tag{2.1}$$

and $h_{ij} := FF_{y^iy^j} = g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible if $\mathbf{M}_y = 0$. This quantity is introduced by Matsumoto [6]. Matsumoto proves that every Randers metric satisfies that $\mathbf{M}_y = 0$. It is remarkable that, a Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ perturbated by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha = \sqrt{a^{im}b_mb_i} < 1$. Later on, Matsumoto-Hōjō proves that the converse is true too.

Lemma 1. ([4]) A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, $\forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n}\left\{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\right\} + \frac{q}{C^2}I_iI_jI_k,$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM and $C^2 = I^iI_i$. Multiplying the definition of semi-C-reducibility with g^{jk} shows that p and q must satisfy $p + q = 1$. If $p = 0$, then F is called $C2$-like metric [7].

Theorem 2. ([5][7]) Let $F = \phi(\frac{\beta}{\alpha})\alpha$ be a non-Riemannian (α, β)-metric on a manifold M of dimension $n \geq 3$. Then F is semi-C-reducible.

The horizontal covariant derivatives of **C** along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to R$ defined by

$$\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^iv^jw^k,$$

where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$ and $w = w^i\frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L} = 0$. The quantity $\mathbf{L}/\mathbf{C}$ is regarded as the relative rate of change of $\mathbf{C}$ along geodesics. Then F is said to be relatively isotropic Landsberg metric if

$$\mathbf{L} = cF\mathbf{C}$$

for some scalar function $c = c(x)$ on M.
The horizontal covariant derivatives of $\mathbf{I}$ along geodesics give rise to the mean Landsberg curvature

$$\mathbf{J}_y(u) := J_i(y)u^i,$$

where $J_i := g^{jk}L_{ijk}$. A Finsler metric is called a weakly Landsberg metric if $\mathbf{J} = 0$. The quantity $\mathbf{J}/\mathbf{I}$ is regarded as the relative rate of change of $\mathbf{I}$ along geodesics. Then F is said to be relatively isotropic mean Landsberg metric if

$$\mathbf{J} = cF\mathbf{I},$$

for some scalar function $c = c(x)$ on M.

Given a Finsler manifold (M, F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i(y)$ are local functions on TM given by

$$G^i := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l} \right\}, \ \ y \in T_xM. \tag{2.2}$$

$\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of $\mathbf{G}$ is called a geodesic in M. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

The Riemann curvature $\mathbf{K}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ is a family of linear maps on tangent spaces, defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

For a flag $P = \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{K}_y(u))}{\mathbf{g}_y(y, y)\mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2},$$

When F is Riemannian, $\mathbf{K} = \mathbf{K}(P)$ is independent of $y \in P$, which is just the sectional curvature of P in Riemannian geometry. We say that a Finsler metric F is of scalar curvature if for any $y \in T_xM$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0. If $\mathbf{K} = constant$, then the Finsler metric F is said to be of constant flag curvature.

3. PROOF OF THEOREM 1

In this section, we are going to prove the Theorem 1. To prove it, we need the following.

Lemma 2. *Let (M, F) be a C3-like Finsler manifold and $b_i = b_i(x, y)$ is constant along Finslerian geodesics, i.e., $b'_i := b_{i|s}y^s = 0$ and $a_i y^i = 0$. Suppose that F has relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a non-zero scalar function on M. Then the Matsumoto torsion satisfies*

$$M'_{ijk} = 2cFM_{ijk}, \tag{3.1}$$

where $M'_{ijk} = M_{ijk|s}y^s$.

Proof. F is $C3$-like metric

$$C_{ijk} = \Big\{a_i h_{jk} + a_j h_{ki} + a_k h_{ij}\Big\} + \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}, \tag{3.2}$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are scalar functions on TM. By multiplying (3.2) with g^{ij} and considering $a_i y^i = 0$, we have

$$a_i = \frac{1}{n+1}\Big\{(1 - 2I^m b_m)I_i - C^2 b_i\Big\}, \tag{3.3}$$

where $C^2 = I^m I_m$. By plugging (3.3) in (3.2), we get

$$\begin{aligned} C_{ijk} = &\ \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{2I^m b_m}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ - &\ \frac{C^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} + \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}, \end{aligned} \tag{3.4}$$

or equivalently

$$\begin{aligned} M_{ijk} = -\frac{2I^m b_m}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - &\ \frac{C^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ + &\ \Big\{b_i I_j I_k + I_i b_j I_k + I_i I_j b_k\Big\}. \end{aligned} \tag{3.5}$$

By taking a horizontal derivation of (3.5), we have

$$\begin{aligned} M'_{ijk} = - &\ \frac{2}{n+1}\Big(J^m b_m + I^m b'_m\Big)\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ - &\ \frac{2I^m b_m}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} - \frac{C^2}{n+1}\Big\{b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij}\Big\} \\ - &\ \frac{1}{n+1}\Big(J^m I_m + I^m J_m\Big)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ + &\ \Big\{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\Big\} \\ + &\ \Big\{b'_i I_j I_k + b'_j I_i I_k + b'_k I_i I_j\Big\}, \end{aligned} \tag{3.6}$$

where $b'_i = b_{i|s}y^s$ and

$$M'_{ijk} = L_{ijk} - \frac{1}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\}. \tag{3.7}$$

Since b_i is constant along geodesics, i.e., $b'_i = 0$, then (3.6) reduces to following

$$\begin{aligned} M'_{ijk} =& -\frac{2J^m b_m}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} - \frac{2I^m b_m}{n+1}\Big\{J_i h_{jk} + J_j h_{ki} + J_k h_{ij}\Big\} \\ & - \frac{1}{n+1}\Big(J^m I_m + I^m J_m\Big)\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\} \\ & + \Big\{b_i J_j I_k + b_i I_j J_k + b_j J_i I_k + b_j I_i J_k + b_k J_i I_j + b_k I_i J_j\Big\}. \end{aligned} \tag{3.8}$$

Let F is of relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$. Then from (3.8), we get

$$\begin{aligned} M'_{ijk} = 2cF\Big\{b_i I_j I_k + b_j I_i I_k + b_k I_i I_j\Big\} &- \frac{4cFI^m b_m}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ &- \frac{2cFC^2}{n+1}\Big\{b_i h_{jk} + b_j h_{ki} + b_k h_{ij}\Big\}. \end{aligned} \tag{3.9}$$

By considering (3.4) and (3.9), we have

$$\begin{aligned} M'_{ijk} &= 2cFC_{ijk} - \frac{2cF}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\} \\ &= 2cF\Bigg[C_{ijk} - \frac{1}{n+1}\Big\{I_i h_{jk} + I_j h_{ki} + I_k h_{ij}\Big\}\Bigg] \end{aligned} \tag{3.10}$$

By (2.1) and (3.10), we get (3.1).

On the other hand, we need to following.

Lemma 3. ([8]) Landsberg curvature and Riemann Curvature are related by the following equation

$$L_{ijk|m}y^m + C_{ijm}R^m{}_k = -\frac{1}{6}g_{im}\Big[2R^m{}_{k\cdot j} + R^m{}_{j\cdot k}\Big] - \frac{1}{6}g_{jm}\Big[2R^m{}_{k\cdot i} + R^m{}_{i\cdot k}\Big]. \tag{3.11}$$

Contracting (3.11) with g^{ij} gives

$$J_{k|m}y^m + I_m R^m{}_k = -\frac{1}{3}\Big[2R^m{}_{k\cdot m} + R^m{}_{m\cdot k}\Big]. \tag{3.12}$$

Proof of Theorem 1: We will first prove that the Matsumoto torsion vanishes. To prove this, let

$$\mathbf{M}_y(u,u,u) = M_{ijk}(x,y)u^i u^j u^k \neq 0$$

for some $y, u \in T_x M_0$ with $F(x,y) = 1$. Let $\sigma(t)$ be the unit speed geodesic with $\sigma(0) = x$ and $\dot\sigma(0) = y$. Let $U(t)$ denote the linear parallel vector field along σ, that is, $D_{\dot\sigma}U(t) = 0$. From the above equation, we see that a linearly parallel vector field $U(t)$ along σ linearly depends on its initial value $U(t_o)$ at a point $\sigma(t_o)$.
Let

$$\begin{aligned} \Xi(t) :=& \ \mathbf{M}_{\dot\sigma(t)}\Big(U(t), U(t), U(t)\Big) \\ =& \ M_{ijk}\Big(\sigma(t), \dot\sigma(t)\Big)U^i(t)U^j(t)U^k(t). \end{aligned}$$

We have

$$\Xi''(t) = M_{ijk|p|q}\dot{\sigma}^p(t)\dot{\sigma}^q(t)\Big(\sigma(t),\dot{\sigma}(t)\Big)U^i(t)U^j(t)U^k(t).$$

Now we assume that F is of scalar curvature with flag curvature $\mathbf{K} = \mathbf{K}(x,y)$. This is equivalent to the following identity:

$$R^i{}_k = \mathbf{K}F^2\, h^i_k, \tag{3.13}$$

where $h^i_k := g^{ij}h_{jk}$. Differentiating (3.13) yields

$$R^i{}_{k\cdot l} = \mathbf{K}_{.l}F^2\, h^i_k + \mathbf{K}\Big\{2g_{lp}y^p\delta^i_k - g_{kp}y^p\delta^i_l - g_{kl}y^i\Big\}. \tag{3.14}$$

By (3.11), (3.12) and (3.14), we obtain

$$L_{ijk|m}y^m = -\frac{1}{3}F^2\Big\{\mathbf{K}_{.i}h_{jk} + \mathbf{K}_{.j}h_{ik} + \mathbf{K}_{.k}h_{ij} + 3\mathbf{K}C_{ijk}\Big\} \tag{3.15}$$

and

$$J_{k|m}y^m = -\frac{1}{3}F^2\Big\{(n+1)\mathbf{K}_{.k} + 3\mathbf{K}I_k\Big\}. \tag{3.16}$$

See [9]. Thus the L-curvature $\mathbf{L} = L_{ijk}\omega^i\otimes\omega^j\otimes\omega^k$ and the J-curvature $\mathbf{J} = J_i\omega^i$ are given by

$$L_{ijk} = C_{ijk|m}y^m, \qquad J_i = I_{i|m}y^m. \tag{3.17}$$

By (3.17), we have

$$C_{ijk|p|q}y^py^q = L_{ijk|m}y^m, \qquad I_{k|p|q}y^py^q = J_{k|m}y^m.$$

$$M_{ijk|p|q} = L_{ijk|m}y^m - \frac{1}{n+1}\Big\{J_{i|m}y^m h_{jk} + J_{j|m}y^m h_{ik} + J_{k|m}y^m h_{ij}\Big\}. \tag{3.18}$$

Plugging (3.15) and (3.16) into (3.18) yields

$$M_{ijk|p|q}y^py^q + \mathbf{K}F^2M_{ijk} = 0. \tag{3.19}$$

See [9]. It follows from (3.19) that

$$\Xi''(t) + \mathbf{K}(t)\Xi(t) = 0. \tag{3.20}$$

On the other hand, (3.1) is equal to following

$$\Xi'(t) = 2c(t)\Xi(t). \tag{3.21}$$

By (3.21) we have

$$\begin{aligned}\Xi''(t) &= 2c'(t)\Xi(t) + 2c(t)\Xi'(t)\\ &= (2c' + 4c^2)\Xi.\end{aligned} \tag{3.22}$$

By (3.22) and (3.20) we get

$$(2c' + 4c^2 + \mathbf{K})\Xi = 0. \tag{3.23}$$

By assumption $2c' + 4c^2 + \mathbf{K} \neq 0$, then $\Xi = 0$. Then F is a C-reducible metric:

$$C_{ijk} = \frac{1}{n+1}\Big\{I_ih_{jk} + I_jh_{ki} + I_kh_{ij}\Big\}. \tag{3.24}$$

By (3.24), we get

$$L_{ijk} = \frac{1}{n+1}\Big\{J_ih_{jk} + J_jh_{ki} + J_kh_{ij}\Big\}. \tag{3.25}$$

Since $\mathbf{J} = cF\mathbf{I}$, then by (3.24) and (3.25) it follows that $\mathbf{L} = cF\mathbf{C}$. This completes the proof. □

Proof of Corollary 1: Since $\mathbf{J} = 0$, then by (3.8) we get

$$M'_{ijk} = 0.$$

Thus (3.7), implies that F is a Landsberg metric. From (3.19), it follows that

$$\mathbf{KM} = 0.$$

Since $\mathbf{K} \neq 0$, then $\mathbf{M} = 0$ and F is a C-reducible metric. By Numata Theorem, F reduces to a Riemannian metric of constant sectional curvature. □

Now, we are going to consider complete $C3$-like Finsler manifold with vanishing flag curvature. We prove the following

Theorem 3. *Let (M, F) be a complete C3-like Finsler manifold of dimension $n \geq 3$ which is not a C-reducible. Suppose that F is of vanishing scalar flag curvature $\mathbf{K} = 0$ with relatively isotropic mean Landsberg curvature $\mathbf{J} = cF\mathbf{I}$, where $c = c(x)$ is a scalar function on M. If $b_i = b_i(x, y)$ is constant along Finslerian geodesics and $a_i y^i = 0$, then F reduces to a Landsberg metric.*

Proof. Since F is not C-reducible, then $\Xi \neq 0$. Thus by (3.23), we get

$$2c' + 4c^2 = 0. \tag{3.26}$$

Considering (3.26) on the indicatrix, implies the following

$$c(t) = \frac{c(0)}{1 - 2tc(0)}. \tag{3.27}$$

Since (M, F) is a complete manifold, then by letting $t \to \pm\infty$, we conclude that $c = 0$. Thus by assumption, we have

$$\mathbf{J} = 0.$$

Then (3.8) implies that

$$M'_{ijk} = 0. \tag{3.28}$$

By plugging $\mathbf{J} = 0$ in (3.7) and considering (3.28), it follows that F is a Landsberg metric.

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