ON C$^3$-LIKE FINSLER METRICS OF SCALAR FLAG CURVATURE

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Abstract. In this paper, we consider the class of C$^3$-like Finsler metrics. This class of Finsler metrics contains the class of $(\alpha, \beta)$-metrics as a special case. We study C$^3$-like metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature and find a condition under which these metrics reduce to C-reducible metrics.

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1. Introduction

For a Finsler manifold $(M, F)$, the flag curvature is a function $K(P, y)$ of tangent planes $P \subset T_x M$ and directions $y \in P$. If $F$ is Riemannian, $K = K(P)$ is independent of $y \in P \setminus \{0\}$, $K$ being called the sectional curvature in Riemannian geometry. $F$ is said to be of scalar flag curvature if the flag curvature $K(P, y) = K(x, y)$ is independent of flags $P$ associated with any fixed flagpole $y$. For example, every two-dimensional Finsler metric is of scalar flag curvature. One of the important problems in Finsler geometry is to characterize Finsler manifolds of scalar flag curvature. Let $(M, F)$ be a Finsler manifold. The second derivatives of $\frac{1}{2} F^2$ at $y \in T_x M_0$ is an inner product $g_y$ on $T_x M$. The third order derivatives of $\frac{1}{2} F^2$ at $y \in T_x M_0$ is a symmetric trilinear form $C_y$ on $T_x M$. We call $g_y$ and $C_y$ the fundamental form and the Cartan torsion, respectively. The rate of change of Cartan torsion $C$ along geodesics is called the Landsberg curvature $L$. A Finsler metric $F$ is said to be isotropic Landsberg metric if $L + c(x) F C = 0$, where $c = c(x)$ is a scalar function on $M$. Taking a trace of $C$ and $L$ give us mean Cartan torsion $I$ and mean Landsberg curvature $J$, respectively. A Finsler metric $F$ is said to be isotropic mean Landsberg metric if $J + c(x) F I = 0$, where $c = c(x)$ is a scalar function on $M$.

In [10], Prasad-Singh introduced a class of Finsler spaces named by C3-like spaces. A Finsler metric $F$ is called C3-like if its Cartan tensor is given by

$$C_{ijk} = \left\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \right\} + \left\{ b_i I_j I_k + I_i b_j I_k + I_i I_j b_k \right\},$$

where $a_i = a_i(x, y)$ and $b_i = b_i(x, y)$ are homogeneous scalar functions on $TM$ of degree -1 and 1, respectively and $h = h_{ij} dx^i \otimes dx^j$ is the angular metric. If we put

$$a_i = \frac{p}{n+1} l_i \quad \text{and} \quad b_i = \frac{q}{3C^2} l_i,$$
where $p = p(x, y)$ and $q = q(x, y)$ are scalar functions on $TM$, then $F$ is a semi-C-reducible metric. In [5], Matsumoto-Shibata introduced the notion of semi-C-reducibility and proved that every non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$ is semi-C-reducible. If $b_i = 0$, then $F$ is called C-reducible metric [3]. In [4], Matsumoto-Højø proved that a Finsler metric $F$ is C-reducible if and only if it is a Randers metric $F = \alpha + \beta$ or Kropina metric $F = \alpha^2 / \beta$. The Randers metrics and Kropina metrics are special $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is a Finsler metric on $M$ defined by $F := \alpha \phi(s), s = \beta / \alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$ [12]. Therefore the study of the class of C3-like Finsler spaces will enhance our understanding of the geometric meaning of $(\alpha, \beta)$-metrics.

Recently, Cheng-Shen have classified Randers metrics of scalar flag curvature with isotropic S-curvature [2]. This class of Randers metrics contains all projectively flat Randers metrics with isotropic S-curvature and Randers metrics of constant flag curvature. In [11], Cheng-Mo-Shen characterize Finsler metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature $J + c(x)FI = 0$, where $c = c(x)$ is a scalar function on $M$. Hence, it is natural problem to study the class of C3-like metrics of scalar curvature with relatively isotropic mean Landsberg curvature.

In this paper, we study C3-like metrics of scalar curvature with relatively isotropic mean Landsberg curvature and find a condition under which this metrics reduce to C-reducible metric. More precisely, we prove the following.

**Theorem 1.** Let $(M, F)$ be a C3-like Finsler manifold such that $b_i = b_i(x, y)$ is constant along Finslerian geodesics, i.e., $b'_i := b_{ij}y^j = 0$ and $a_{ij}y^j = 0$. Suppose that $J/I$ is isotropic,

$$J + c(x)FI = 0, \quad (1.2)$$

where $c = c(x)$ is a scalar function on $M$. Assume that $F$ is of scalar flag curvature $K$ such that

$$2c' + 4c^2 + K \neq 0,$$

where $c' := c_{ij}y^i$. Then $F$ reduces to a C-reducible metric with relatively isotropic Landsberg curvature $L + c(x)FC = 0$.

Finsler metrics with $J = 0$ are said to be weakly Landsbergian. Every Landsberg metric is a weakly Landsbergian. Using Theorem 1 we study weakly Landsberg C3-like metrics of scalar flag curvature.

**Corollary 1.** Let $(M, F)$ be a weakly Landsberg C3-like Finsler manifold such that $b_i = b_i(x, y)$ is constant along Finslerian geodesics and $a_{ij}y^j = 0$. Suppose that $F$ is of non-zero scalar flag curvature. Then $F$ is Riemannian metrics of constant sectional curvature.

It is remarkable that through of this paper, we use the Berwald connection and the $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by “|” and “,” respectively [11].

2. Preliminaries

Let $M$ be a n-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M}T_xM$ the tangent bundle of $M$. A Finsler metric on $M$ is a function
where $L$ is a Randers metric satisfies that $M$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, (iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,
\[
g_y(u,v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)] \big|_{s,t=0}, \quad u,v \in T_x M.
\]

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of $F_x$, define $C_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by
\[
C_y(u,v,w) := \frac{1}{2} \frac{d}{dt} [g_y + tw(u,v)] \big|_{t=0}, \quad u,v,w \in T_x M.
\]

The family $C := \{C_y\}_{y \in TM}$ is called the Cartan torsion. It is well known that $C=0$ if and only if $F$ is Riemannian.

For $y \in T_x M_0$, define mean Cartan torsion $I_y$ by $I_y(u) := I_i(y) u^i$, where $I_i := g^{jk} C_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}$. By Diecke Theorem, $F$ is Riemannian if and only if $I_y = 0$.

For $y \in T_x M_0$, define the Matsumoto torsion $M_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ by $M_y(u,v,w) := M_{ijk}(y) u^i v^j w^k$ where
\[
M_{ijk} := C_{ijk} - \frac{1}{n+1} \{I_j h_{ik} + I_k h_{ij} + I_k h_{ij}\}, \quad (2.1)
\]

and $h_{ij} := F F\{y\} = g_{ij} - \frac{1}{n+1} g^{kl} g_{ij} y^k y^l$ is the angular metric. A Finsler metric $F$ is said to be C-reducible if $M_y = 0$. This quantity is introduced by Matsumoto [6]. Matsumoto proves that every Randers metric satisfies that $M_y = 0$. It is remarkable that, a Randers metric $F = \alpha + \beta$ on a manifold $M$ is just a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ perturbed by a one form $\beta = b_i(x) y^i$ on $M$ such that $\|\beta\|_{\alpha} = \sqrt{a^{im} b_{m} b_{l}} < 1$. Later on, Matsumoto-Hôjô proves that the converse is true too.

**Lemma 1.** (4) A Finsler metric $F$ on a manifold of dimension $n \geq 3$ is a Randers metric if and only if $M_y = 0, \forall y \in TM_0$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by
\[
C_{ijk} = \frac{p}{1+n} \left\{ h_{ij} l_k + h_{jk} l_i + h_{ki} l_j \right\} + \frac{q}{C^2} I_i l_j l_k,
\]
where $p = p(x,y)$ and $q = q(x,y)$ are scalar function on $TM$ and $C^2 = I^i l_i$. Multiplying the definition of semi-C-reducibility with $g^{jk}$ shows that $p$ and $q$ must satisfy $p + q = 1$. If $p = 0$, then $F$ is called C2-like metric [7].

**Theorem 2.** (5,7) Let $F = \phi(\frac{\beta}{\alpha}) \alpha$ be a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ of dimension $n \geq 3$. Then $F$ is semi-C-reducible.

The horizontal covariant derivatives of $C$ along geodesics give rise to the Landsberg curvature $L_y : T_x M \otimes T_x M \otimes T_x M \to \mathbb{R}$ defined by
\[
L_y(u,v,w) := L_{ijk}(y) u^i v^j w^k,
\]
where $L_{ijk} := C_{ijk} y^i$, $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^j \frac{\partial}{\partial x^i}$. The family $L := \{L_y\}_{y \in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg...
metric if \( L = 0 \). The quantity \( L/C \) is regarded as the relative rate of change of \( C \) along geodesics. Then \( F \) is said to be relatively isotropic Landsberg metric if
\[
L = cFC
\]
for some scalar function \( c = c(x) \) on \( M \).
The horizontal covariant derivatives of \( I \) along geodesics give rise to the mean Landsberg curvature
\[
J_y(u) := J_i(y)u^i,
\]
where \( J_i := g^{jk}L_{ijk} \). A Finsler metric is called a weakly Landsberg metric if \( J = 0 \). The quantity \( J/I \) is regarded as the relative rate of change of \( I \) along geodesics. Then \( F \) is said to be relatively isotropic mean Landsberg metric if
\[
J = cFI,
\]
for some scalar function \( c = c(x) \) on \( M \).

Given a Finsler manifold \((M, F)\), then a global vector field \( G \) is induced by \( F \) on \( TM_0 \), which in a standard coordinate \((x^i, y^i)\) for \( TM_0 \) is given by
\[
G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},
\]
where \( G^i(y) \) are local functions on \( TM \) given by
\[
G^i := \frac{1}{4}g^{ij}\left\{ \frac{\partial^2[F^2]}{\partial x^j\partial y^k}y^k - \frac{\partial[F^2]}{\partial x^j}\right\}, \quad y \in T_xM.
\]

\( G \) is called the associated spray to \((M, F)\). The projection of an integral curve of \( G \) is called a geodesic in \( M \). In local coordinates, a curve \( c(t) \) is a geodesic if and only if its coordinates \((c^i(t))\) satisfy \( \ddot{c}^i + 2G^i(\dot{c}) = 0 \).

The Riemann curvature \( K_y = R^i_d x^k \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM \) is a family of linear maps on tangent spaces, defined by
\[
R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j\partial y^k} + 2G^i \frac{\partial^2 G^i}{\partial y^j\partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]
For a flag \( P = \text{span}\{y, u\} \subset T_xM \) with flagpole \( y \), the flag curvature \( K = K(P, y) \) is defined by
\[
K(P, y) := \frac{g_y(u, K_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2}.
\]
When \( F \) is Riemannian, \( K = K(P) \) is independent of \( y \in P \), which is just the sectional curvature of \( P \) in Riemannian geometry. We say that a Finsler metric \( F \) is of scalar curvature if for any \( y \in T_xM \), the flag curvature \( K = K(x, y) \) is a scalar function on the slit tangent bundle \( TM_0 \). If \( K = \text{constant} \), then the Finsler metric \( F \) is said to be of constant flag curvature.
3. Proof of Theorem

In this section, we are going to prove the Theorem. To prove it, we need the following.

Lemma 2. Let \((M, F)\) be a C3-like Finsler manifold and \(b_i = b_i(x, y)\) is constant along Finslerian geodesics, i.e., \(b_i := b_i|_{|y|} = 0\) and \(a_i y^i = 0\). Suppose that \(F\) has relatively isotropic mean Landsberg curvature \(J = c F I\), where \(c = c(x)\) is a non-zero scalar function on \(M\). Then the Matsumoto torsion satisfies

\[
M'_{ijk} = 2c F M_{ijk}, \tag{3.1}
\]

where \(M'_{ijk} = M_{ijk|y^i}\).

Proof. \(F\) is C3-like metric

\[
C_{ijk} = \left\{ a_i h_{jk} + a_j h_{ki} + a_k h_{ij} \right\} + \left\{ b_i I_j I_k + I_k b_j I_k + I_j b_i b_k \right\}, \tag{3.2}
\]

where \(a_i = a_i(x, y)\) and \(b_i = b_i(x, y)\) are scalar functions on \(TM\). By multiplying (3.2) with \(g^{ij}\) and considering \(a_i y^i = 0\), we have

\[
a_i = \frac{1}{n+1} \left\{ (1 - 2I^m b_m) I_i - C^2 b_i \right\}, \tag{3.3}
\]

where \(C^2 = I^m I_m\). By plugging (3.3) in (3.2), we get

\[
C_{ijk} = \frac{1}{n+1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{2I^m b_m}{n+1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\}
- \frac{C^2}{n+1} \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\} + \left\{ b_i I_j I_k + I_k b_j I_k + I_j b_i b_k \right\}, \tag{3.4}
\]

or equivalently

\[
M_{ijk} = -\frac{2I^m b_m}{n+1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{C^2}{n+1} \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\}
+ \left\{ b_i I_j I_k + I_k b_j I_k + I_j b_i b_k \right\}. \tag{3.5}
\]

By taking a horizontal derivation of (3.5), we have

\[
M'_{ijk} = -\frac{2}{n+1} \left\{ I^m b_m + I^m b'_m \right\} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\}
- \frac{2I^m b_m}{n+1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{C^2}{n+1} \left\{ b'_i h_{jk} + b'_j h_{ki} + b'_k h_{ij} \right\}
- \frac{1}{n+1} \left\{ I^m I_m + I^m I'_m \right\} \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\}
+ \left\{ b'_i I_j I_k + b'_j I_i I_k + b'_j I_i I_j + b_k I_i I_j \right\}
+ \left\{ b'_i I_j I_k + b'_j I_i I_k + b'_j I_i I_j \right\}, \tag{3.6}
\]

where \(b'_i = b_i|_{|y|}\) and

\[
M'_{ijk} = L_{ijk} - \frac{1}{n+1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\}. \tag{3.7}
\]
Let \( b_i \) is constant along geodesics, i.e., \( b_i' = 0 \), then (3.6) reduces to following
\[
M'_{ijk} = - \frac{2F^m b_m}{n + 1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} - \frac{2F^m b_m}{n + 1} \left\{ J_i h_{jk} + J_j h_{ki} + J_k h_{ij} \right\}
- \frac{1}{n + 1} \left( F^m I_m + 1 \right) \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\}
+ \left\{ b_i I_j I_k + b_i I_j I_k + b_j I_i I_k + b_k I_i I_j + b_k I_i I_j \right\}.
\]

(3.8)

Let \( F \) is of relatively isotropic mean Landsberg curvature \( J = cF^I \). Then from (3.8), we get
\[
M'_{ijk} = 2cF \left\{ b_i I_j I_k + b_j I_i I_k + b_k I_i I_j \right\} - \frac{4cF^m b_m}{n + 1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\}
- \frac{2cF^2}{n + 1} \left\{ b_i h_{jk} + b_j h_{ki} + b_k h_{ij} \right\}.
\]

(3.9)

By considering (3.4) and (3.9), we have
\[
M'_{ijk} = 2cF I_{ijk} - \frac{2cF}{n + 1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\}
= 2cF \left[ C_{ijk} - \frac{1}{n + 1} \left\{ I_i h_{jk} + I_j h_{ki} + I_k h_{ij} \right\} \right]
\]

(3.10)

By (2.1) and (3.10), we get (3.1).

On the other hand, we need to following.

**Lemma 3.** (8) Landsberg curvature and Riemann Curvature are related by the following equation
\[
L_{ijk} m^m + C_{ijm} R_{k}^m = - \frac{1}{6} g_{im} \left[ 2R_{m}^i j + R_{m}^j k \right] - \frac{1}{6} g_{jm} \left[ 2R_{m}^i k + R_{m}^i j \right].
\]

(3.11)

Contracting (3.11) with \( g^{ij} \) gives
\[
J_{k} m^m + I_{m} R_{k}^m = - \frac{1}{3} \left[ 2R_{k}^m_{k} + R_{m}^m_{k} \right].
\]

(3.12)

**Proof of Theorem** We will first prove that the Matsumoto torsion vanishes. To prove this, let
\[
M_y(u, u, u) = M_{ijk}(x, y) u^i u^j u^k \neq 0
\]
for some \( y, u \in T_x M_0 \) with \( F(x, y) = 1 \). Let \( \sigma(t) \) be the unit speed geodesic with \( \sigma(0) = x \) and \( \dot{\sigma}(0) = y \). Let \( U(t) \) denote the linear parallel vector field along \( \sigma \), that is, \( D_{\sigma} U(t) = 0 \). From the above equation, we see that a linearly parallel vector field \( U(t) \) along \( \sigma \) linearly depends on its initial value \( U(t_0) \) at a point \( \sigma(t_0) \).

Let
\[
\Xi(t) := M_{\sigma(t)} \left( U(t), \dot{U}(t), U(t) \right)
= M_{ijk} \left( \sigma(t), \dot{\sigma}(t) \right) U_i(t) U_i(t) U_i(t).
\]

(6)
We have
\[ \Xi''(t) = M_{ijk|p|q} \dot{\delta}^p(t) \dot{\delta}^q(t) \left( \sigma(t), \dot{\sigma}(t) \right) U^i(t) U^j U^k(t). \]

Now we assume that \( F \) is of scalar curvature with flag curvature \( K = K(x, y) \). This is equivalent to the following identity:
\[ R^i_k = K F^2 h^i_k, \quad (3.13) \]
where \( h^i_k := g^{ij} h_{jk} \). Differentiating (3.13) yields
\[ R^i_{k'l} = K L^2 h^i_k + K \left\{ 2 g_{lp} y^p \delta^i_k - g_{kp} y^p \delta^i_l - g_{kl} y^i \right\}. \quad (3.14) \]

By (3.11), (3.12) and (3.14), we obtain
\[ L_{ijk|m} y^m = -\frac{1}{3} F^2 \left\{ K_i h_{jk} + K_j h_{ik} + K_k h_{ij} + 3 K C_{ijk} \right\}. \quad (3.15) \]
and
\[ J_{k|m} y^m = -\frac{1}{3} F^2 \left\{ (n + 1) K_k + 3 K I_k \right\}. \quad (3.16) \]
See [9]. Thus the L-curvature \( L = L_{ijk} \omega^i \otimes \omega^j \otimes \omega^k \) and the J-curvature \( J = J_i \omega^i \) are given by
\[ L_{ijk} = C_{ijk|m} y^m, \quad J_i = I_{i|m} y^m. \quad (3.17) \]

By (3.17), we have
\[ C_{ijk|p|q} y^p y^q = L_{ijk|m} y^m, \quad I_{k|p} y^p y^q = J_{k|m} y^m. \]
\[ M_{ijk|p|q} = L_{ijk|m} y^m - \frac{1}{n + 1} \left\{ I_{i|m} y^m h_{jk} + I_{j|m} y^m h_{ik} + I_{k|m} y^m h_{ij} \right\}. \quad (3.18) \]

Plugging (3.15) and (3.16) into (3.18) yields
\[ M_{ijk|p|q} y^p y^q + K F^2 M_{ijk} = 0. \quad (3.19) \]
See [9]. It follows from (3.19) that
\[ \Xi''(t) + K(t) \Xi(t) = 0. \quad (3.20) \]
On the other hand, (3.1) is equal to following
\[ \Xi'(t) = 2 c(t) \Xi(t). \quad (3.21) \]
By (3.21) we have
\[ \Xi''(t) = 2 c'(t) \Xi(t) + 2 c(t) \Xi'(t) = \left( 2 c' + 4 c^2 \right) \Xi. \quad (3.22) \]

By (3.22) and (3.20) we get
\[ (2 c' + 4 c^2 + K) \Xi = 0. \quad (3.23) \]
By assumption \( 2 c' + 4 c^2 + K \neq 0 \), then \( \Xi = 0 \). Then \( F \) is a C-reducible metric:
\[ C_{ijk} = \frac{1}{n + 1} \left\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \right\}. \quad (3.24) \]
By (3.24), we get
\[ L_{ijk} = \frac{1}{n + 1} \left\{ I_i h_{jk} + I_j h_{ik} + I_k h_{ij} \right\}. \quad (3.25) \]
Since \( J = cFI \), then by (3.24) and (3.25) it follows that \( L = cFC \). This completes the proof. \( \square \)

**Proof of Corollary 1** Since \( J = 0 \), then by (3.8) we get

\[ M'_{ijk} = 0. \]

Thus (3.7), implies that \( F \) is a Landsberg metric. From (3.19), it follows that

\[ KM = 0. \]

Since \( K \neq 0 \), then \( M = 0 \) and \( F \) is a \( C \)-reducible metric. By Numata Theorem, \( F \) reduces to a Riemannian metric of constant sectional curvature. \( \square \)

Now, we are going to consider complete C3-like Finsler manifold with vanishing flag curvature. We prove the following

**Theorem 3.** Let \((M, F)\) be a complete C3-like Finsler manifold of dimension \( n \geq 3 \) which is not a C-reducible. Suppose that \( F \) is of vanishing scalar flag curvature \( K = 0 \) with relatively isotropic mean Landsberg curvature \( J = cFI \), where \( c = c(x) \) is a scalar function on \( M \). If \( b_i = b_i(x, y) \) is constant along Finslerian geodesics and \( a_iy^i = 0 \), then \( F \) reduces to a Landsberg metric.

**Proof.** Since \( F \) is not C-reducible, then \( \Xi \neq 0 \). Thus by (3.23), we get

\[ 2c' + 4c^2 = 0. \] (3.26)

Considering (3.26) on the indicatrix, implies the following

\[ c(t) = \frac{c(0)}{1 - 2tc(0)}. \] (3.27)

Since \((M, F)\) is a complete manifold, then by letting \( t \to \pm \infty \), we conclude that \( c = 0 \). Thus by assumption, we have

\[ J = 0. \]

Then (3.8) implies that

\[ M'_{ijk} = 0. \] (3.28)

By plugging \( J = 0 \) in (3.7) and considering (3.28), it follows that \( F \) is a Landsberg metric.

**References**


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