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SPACETIMES OF CLASS ONE WITH PERFECT FLUID

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ABSTRACT. We consider spacetimes with perfect fluid embedded into E_5, and it is showed that the Newman-Penrose formalism [1-3] gives simple proofs for results obtained by Krishna Rao [4] and Barnes [5].

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1. INTRODUCTION

We here shall employ the notation and quantities of [3, 5-11]. The curvature of the spacetime is generated by a perfect fluid via the Einstein equations:

$$G_{ab} + \lambda g_{ab} = -(\rho + p)u_a u_b - p g_{ab},\ u^r u_r = -1 \tag{1}$$

where λ is the cosmological constant and u_j, ρ, p are the 4-velocity, density and presion of the fluid, respectively. The present work is dedicated to metrics which are solutions of (1) and class one (that is, they accept embedding into E_5 [3, 6]). Thus, it is convenient a brief resume of interesting results in the theme under study:

(a) Szekeres [12] and Greenberg [13] proved that when $\lambda = p = 0$, then (1) not accepts geometries with Petrov type N.

(b) Wainwright [14] realized a complete analysis of algebraically special metrics satisfying (1), and such that their degenerate Debever-Penrose vector [3] defines a null congruence geodesic without shear and expansion. In particular, he deduced that:

"The Gödel metric [15] is the unique solution of (1) with Petrov type $\neq O, I$ and $p = 0$, whose repeated null congruence is geodesic with zero expansion and shear", (2)

and it is possible to show that the Gödel metric not admits embedding into E_5 [8,16-18].

(c) Wainwright [19] and Carminati-Wainwright [20] investigated solutions of (1) with $p = 0$ and Petrov type D.

(d) The only known metric Petrov type III verifying (1), for $\lambda = 0$, was constructed by Allnut [21] and its corresponding degenerate null congruence is geodesic, has shear and expansion but without rotation.

(e) Bonnor-Davidson [22] obtained a solution of (1) with $\lambda = 0$ and Petrov type II, and its repeated principal congruence is geodesic, non-zero expansion but without shear and rotation.

(f) Stephani [23] used the embedding process to determine all solutions of (1) with Petrov type D, class one and zero acceleration of the matter. Besides, Stephani suggested that any R4 with perfect fluid and Petrov type O has class one, which is correct only when $\rho \neq \lambda$ [6].

(g) Barnes [5] employed the Gauss equation to study the compatibility between the Petrov and Churchill-Plebañski [24-28] types for the Weyl, Ricci and second fundamental tensors, thus he showed that:

"All perfect fluids of class one have Petrov types D or O and must be [11(11)] or [1(111)], respectively", (3)

which implies that the mentioned metrics of Allnut [21] and Bonnor-Davidson [22] not accept embedding into E_5.

Besides, from (3) it is immediate the following theorem of Pokhariyal [29]:

"Any spacetime of class one whose elementary divisors of b_{ij} are real (but not simples), not admits a perfect fluid distribution". (4)

(h) Krishna Rao [4] studied (1) when $\lambda = 0$ and he proved the results:

"All perfect fluids with Petrov type O and spherical symmetry, have class one", (5)

and

"All perfect fluids with state equation $\rho = 3p$, and spherical symmetry, have Petrov type O and class one". (6)

The interior solution of Schwarzschild can be embedded into E_5 [30] due to (5). Tikekar [31] showed a metric satisfying (6).

(i) Szekeres [16] obtained two interesting theorems:

"If R_4 with perfect fluid has class one, then the fluid must be without rotation", (7)

and

"If R_4 embedded into E_5 has a perfect fluid with $p = 0$, then these spacetime is of Friedman (Petrov type O)," (8)

both theorems (7) and (8) imply that the Gödel metric [15] (perfect fluid with rotation and Petrov type D) has not class one [8,17,18,30].

Goldman-Rosen [32] have used perfect fluids embedded into E_5 to construct cosmological models in general relativity.

In the next Section we shall employ the Newman-Penrose (NP) formalism [1-3] for the analysis of metrics verifying (1) with class one, resulting thus one relation of Krishna Rao [4] and a part of the theorem (3) of Barnes [5].

2. Equations of Gauss and Codazzi

A spacetime accepts isometric and local embedding into E_5 if and only if there is the second fundamental form $b_{ac} = b_{ca}$ satisfying the equations [3, 6]:

$$R_{ijkc} = \epsilon \left(b_{ik}b_{jc} - b_{ic}b_{jk}\right) \quad Gauss \tag{9}$$

$$b_{ij;k} = b_{ik;j} \qquad Codazzi \tag{10}$$

where R_{acrj} represents the curvature tensor, $\epsilon = \pm 1$ and ;r means the covariant derivative.

Collinson [17, 33] used (9) to demonstrate the identity:

$$^*R^{jm}_{qt} R_{jmpa} = -\frac{k_2}{12}\eta_{qtpa} \tag{11}$$

for any R_4 of class one, where $^*R^{ijrc}$ is the simple dual of Riemann tensor, η_{abcd} is the Levi-Civita tensor and $k_2 =^* R^*_{ijrc}R^{ijrc}$ is an invariant of Lanczos [34,35] in terms of the double dual $^*R^*_{jmpa}$. The projection of (11) onto a null tetrad of NP leads to a set of 14 equations (which are explicitly in [36]) for the quantities ψ_a and ϕ_{ab} (NP components of the Weyl and Ricci, tensors, respectively), and their complex conjugates $\overline{\psi}_a$ and $\overline{\phi}_{ab}$.
We now shall employ the NP formalism to study perfect fluids of class one. Thus, we construct an orthonormal real tetrad such that $e_{(4)b} = u_b$, that is, the unitary temporal vector coincides with the 4-velocity of the fluid. We define the NP tetrad in the usual manner ($i = \sqrt{-1}$):

$$m^r = \frac{1}{\sqrt{2}}\left(e^r_1 - ie^r_2\right), \ \overline{m}^r = \frac{1}{\sqrt{2}}\left(e^r_1 + ie^r_2\right),$$
$$l^r = \frac{1}{\sqrt{2}}\left(e^r_4 - e^r_3\right), \ n^r = \frac{1}{\sqrt{2}}\left(e^r_4 + e^r_3\right). \tag{12}$$

If we project (1) onto (12), then:

$$R = -\rho + 3p + 4\lambda, \ \phi_{00} = \phi_{22} = 2\phi_{11} = -\frac{1}{4}(\rho + p) \neq 0 \tag{13}$$

and the another ϕ_{ar} are zero; we suppose that $(\rho + p) \neq 0$ because if $(\rho + p) = 0$ then R_4 would be an Einstein type spacetime, but it is known [16] that these space must be the DeSitter model. When we use (13) in the 14 NP equations [36] (which are equivalent to (11)) result the restrictions:

$$\begin{gathered} \psi_0 = \overline{\psi}_4, \ \ \psi_1 = -\overline{\psi}_3, \ \ \psi_2 = \overline{\psi}_2, \\ \psi_0\overline{\psi}_1 + \psi_1\left(\psi_2 + \frac{R}{6}\right) = 0, \\ \psi_0\left(-\psi_2 + \frac{R}{6}\right) + \psi_1^2 = 0 \\ 3\psi_2\left(\psi_2 + \frac{R}{6}\right) - \psi_0\overline{\psi}_0 + 2\psi_1\overline{\psi}_1 = 0 \end{gathered} \tag{14}$$

which imply that:

$$\psi_1 = \psi_3 = 0 \tag{15}$$

then the system (14) adopts the form:

$$\psi_0 \left(-\psi_2 + \frac{R}{12} \right) = 0, \;\; 3\psi_2 \left(\psi_2 + \frac{R}{6} \right) - \psi_0 \overline{\psi}_0 = 0 \qquad (16)$$

and thus we have two possibilities:

(A) $\psi_0 = 0$ The equations (14) and (16) lead to $\psi_4 = 0$ such that:

$$\psi_2 \left(\psi_2 + \frac{R}{6} \right) = 0, \qquad (17)$$

relation obtained by Krishna Rao [4] for the case of spherical symmetry, we here not need some symmetry. The condition (17) permits the options $\psi_2 = 0$ (Petrov type O) and $\psi_2 = -\frac{R}{6} \neq 0$ (Petrov type D).

(B) $\psi_0 \neq 0$

Then (14) and (16) imply the relations:

$$\psi_4 = \overline{\psi_0}, \quad \psi_0 \overline{\psi}_0 = \frac{R^2}{16} \neq 0, \quad \psi_2 = \frac{R}{12} \neq 0 \qquad (18)$$

which jointly with algorithms [3,37-39] to determine the Petrov type gives the type D.

The expressions (17) and (18) are the only alternatives for a perfect fluid, thus we have obtained a part of the theorem:

”All perfect fluids (with any cosmological constant) of class one are type D or conformally flat”,

(19)

showed by Barnes [5] via the Churchill-Plebañski algebraic classification [24-28], here we not need such classification.

References

[1] E. Newman, R. Penrose, *An approach to gravitational radiation by a method of spin coefficients*, J. Math. Phys. 3, (1962), 566-578.

[2] S. J. Campbell, J. Wainwright, *Algebraic computing and the Newman-Penrose formalism in general relativity*, Gen. Rel. Grav. 8, (1977), 987-1001.

[3] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact solutions of Einsteins field equations*, Cambridge University Press, (1980).

[4] J. Krishna Rao, *On spherically symmetric perfect fluid distributions and class one property*, Gen. Rel. Grav. 2 (1971) 385-386.

[5] A. Barnes, *Spacetimes of embedding class one in general relativity*, Gen. Rel. Grav. 5, (1974), 147-162.

[6] H. F. Goenner, *General relativity and gravitation*, Vol. I, Ed. A. Held, Plenum NY (1980), Chap.14.

[7] R. Fuentes, J. López-Bonilla, T. Matos, G. Ovando, *Spacetime of class one*, Gen. Rel. Grav. 21, (1989), 777-784.

[8] G. González, J. López-Bonilla, M. Rosales, An identity for R_n embedded into E_{n+1}, Pramana J. Phys. 42, (1994), 85-88.

[9] J. López-Bonilla, J. Rivera, H. Yee, Embedding of the Riemann's space, Braz. J. Phys. 25, (1995), 80-81.

[10] J. López-Bonilla, H. Núñez-Yépez, *An identity for spacetimes embedded into E_5*, Pramana J. Phys. 46, (1996), 219-221.

[11] J. López-Bonilla, J. Morales, G. Ovando, *Codazzi's equation in four dimensions*, Indian J. Phys. B74, (2000), 397-398.
[12] P. Szekeres, *On the propagation of gravitational fields in matter*, J. Math Phys. 7, (1966), 751-761.
[13] P. Greenberg, Stud. Appl. Math. 51, (1972), 415-420.
[14] J. Wainwright, *A class of algebraically special perfect fluid space-times*, Commun. Math. Phys. 17, (1970), 42-60.
[15] K. Gödel, *An example of a new type of cosmological solution of Einsteins field equation of gravitation*, Rev. Mod. Phys. 21, (1949), 447-450.
[16] P. Szekeres, *Embedding properties of general relativistic manifolds*, Nuovo Cim. A43, (1966), 1062-1070.
[17] J. López-Bonilla, J. Morales, M. Rosales, *A theorem on spacetimes of class one*, Braz. J. Phys. 24, (1994), 522-525.
[18] J. López-Bonilla, G. Ovando, J. Rivera, *Embedding of the Gödel metric*, Aligarh Bull. Math. 17, (1997-98), 63-66.
[19] J. Wainwright, *Classification of the type D perfect fluid solutions of the Einstein equations*, Gen. Rel. Grav. 8, (1977), 797-807.
[20] J. Carminati, J. Wainwright, *Perfect-fluid space-times with type-D Weyl tensor*, Gen. Rel. Grav. 17, (1985), 853-867.
[21] J. A. Allnut, *An approach to perfect-fluid space-times by a method of spin-coefficients*, Gen. Rel. Grav. 13, (1981) 1017-1020.
[22] W. B. Bonnor, W. Davidson, *Petrov type II perfect fluid spacetimes with vorticity*, Class. Quantum Grav. 2, (1985), 775-780.
[23] H. Stephani, *Einige Lösunggen der Einsteinschen Feldgleichungen mit idealer Flssigkeit, die sich in einen fnfdimensionalen flachen Raum einbetten lassen*, Commun. Math. Phys. 9, (1968), 53-54.
[24] R. V. Churchill, *Canonical forms for symmetric linear vector functions in pseudo-euclidean space*, Trans. Am. Math. Soc. 34, (1932), 784-794.
[25] J. Plebañski, *The algebraic structure of the tensor of matter*, Acta Phys. Polon. 26, (1964), 963-1020.
[26] J. Plebañski, J. Stachel, *Einstein tensor and spherical symmetry*, J. Math. Phys. 9, (1968), 269-283.
[27] H. F. Goenner, J. Stachel, *Einstein tensor and 3-parameter groups of isometries with 2-dimensional orbits*, J. Math. Phys. 11, (1970), 3358-3370.
[28] C. B. G. McIntosh, J. M. Foyster, A. W. C. Lun, *The classification of the Ricci and Plebañski tensors in general relativity using Newman-Penrose formalism*, J. Math. Phys. 22, (1981), 2620-2623.
[29] G. P. Pokhariyal, *Perfect fluid distribution in class one spacetime*, Gen. Rel. Grav. 3, (1972), 87-93.
[30] C. D. Collinson, *Embedding of the plane-fronted waves and other space-times*, J. Math Phys. 9, (1968), 403-410.
[31] R. S. Tikekar, *A suspended converse of a theorem regarded spherically symmetric space-times*, Curr. Sci. 39, (1970), 460-461.
[32] I. Goldman, N. Rosen, *A universe embedded in a five-dimensional flat space*, Gen. Rel. Grav. 2, (1971), 367-384.
[33] C. D. Collinson, *Einstein-Maxwell fields of embedding class one*, Commun. Math. Phys. 8, (1968), 1-11.
[34] C. Lanczos, *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, Ann. of Math. 39, (1938), 842-850.
[35] V. Gaftoi, J. López-Bonilla, D. Navarrete, G. Ovando, *Sobre unos invariantes de Lanczos*, Rev. Mex. Fs. 36, (1990), 503-509.
[36] D. Ladino, J. López-Bonilla, *Espacio-tiempo de clase uno*, Rev. Mex. Fs. 35, (1989), 623-637.
[37] R. A. D'Inverno, R. A. Russel-Clark, *Classification of the Harrison metrics*, J. Math. Phys. 12, (1971), 1258-1265.
[38] F. W. Letniowski, R. G. McLenaghan, *An improved algorithm for quartic equation classification and Petrov classification*, Gen. Rel. Grav. 20, (1988), 463-483.
[39] G. Ares de Parga, O. Chavoya, J. López-Bonilla, G. Ovando, *Clasificaciòn algebraica del tensor conformal*, Rev. Mex. Fs. 35, (1989), 201-207.

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