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A SYNTHETIC PROOF OF DAO'S GENERALIZATION OF GOORMAGHTIGH'S THEOREM

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ABSTRACT. Using the concept of cross ratio, we give a synthetic proof of Dao's generalization of Goormaghtigh's theorem.

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1. INTRODUCTION

In 1930, René Goormaghtigh, French engineer and geometrician, expanded Droz-Farny theorem [1, 2, 3] with a nice theorem as follow.

Theorem 1.1 (Goormaghtigh [4])**.** *Given triangle ABC and point P distinct from A, B, C. A line Δ passes through P. A_1, B_1, C_1 belong to BC, CA, AB respectively such that PA_1, PB_1, PC_1 are the images of PA, PB, PC respetively by reflection R_Δ. Then, A_1, B_1, C_1 are collinear.*

Notation R_Δ refers to reflection against Δ.
Theorem 1.1 is called Goormaghtigh's theorem [4].
When P is the orthocenter of triangle ABC, theorem 1.1 actually becomes Droz-Farny theorem.
Proof of theorem 1.1 can be found in [5, 6].
In 2014, O.T.Dao expanded theorem 1.1 with two theorems [7].
In this article, we are first going to expand O.T.Dao's second theorem with theorem 1.2 and more beautifully restate O.T.Dao's first theorem with theorem 1.3. Then, we are going to prove theorems 1.2 and 1.3. Please note that, in terms of ideas, the way we prove theorems 1.2 and 1.3 is completely different from the way that theorem 1.1 is proved in [1] and [2].

Theorem 1.2. *Given triangle ABC and point P distinct from A, B, C. A line Δ passes through P. α is any real number. Let A_1, B_1, C_1 belong to BC, CA, AB respectively such that PA_1, PB_1, PC_1 are the images of PA, PB, PC respectively by transformation $R_P^\alpha \circ R_\Delta$. Then, A_1, B_1, C_1 are collinear.*

Notation R_P^α refers rotation around P with angle of rotation α
When $\alpha = 0$, theorem 1.2 becomes theorem 1.1.
When $\alpha = \frac{\pi}{2}$, theorem 1.2 becomes O.T.Dao's second theorem.

Theorem 1.3 (Dao [7])**.** *Given triangle ABC and point P distinct from A, B, C. Lines Δ and Δ' cut at P. Points A_1, B_1, C_1 belong to BC, CA, AB respectively such that $(PA,\ PA_1,\ \Delta,\ \Delta') = (PB,\ PB_1,\ \Delta,\ \Delta') = (PC,\ PC_1,\ \Delta,\ \Delta') = -1$. Then, A_1, B_1, C_1 are collinear.*

When $\Delta \perp \Delta'$, theorem 1.3 becomes theorem 1.1.
Theorem 1.3 is a different, more interesting reiteration of O.T.Dao's first theorem.
Before we prove theorems 1.2 and 1.3, note that notation $\overline{AB}$ refers to the signed length from point A to point B.

2. PROOF OF THEOREM 1.2

There are two cases to consider.
Case 1. $\alpha \equiv 0 \pmod{2\pi}$. Then, $R_P^\alpha \circ R_\Delta = R_\Delta$.
Ignore platitudinous situations: Δ passes through a vertex of triangle ABC; Δ passes through two vertices of triangle of ABC.
Let A_2, B_2 be the intersections of PC_1 and BC, CA respectively (see f.1).

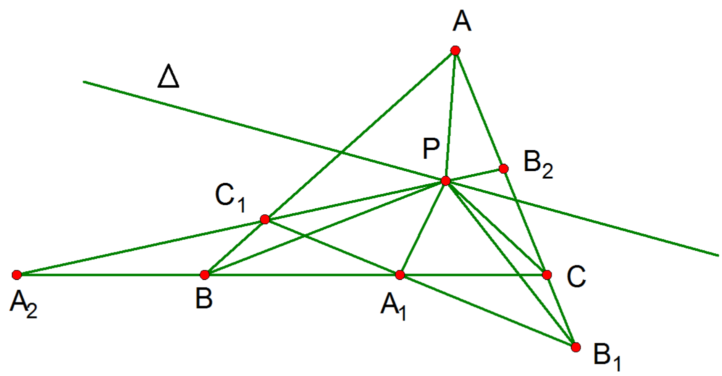

Figure 1

Since reflection preserves cross ratio,

$$\frac{\overline{A_1B}}{\overline{A_1C}} : \frac{\overline{A_2B}}{\overline{A_2C}} = (BCA_1A_2) = P(BCA_1A_2) = P(BCA_1C_1) = P(B_1C_1AC)$$
$$= P(B_1B_2AC) = P(ACB_1B_2) = (ACB_1B_2) = \frac{\overline{B_1A}}{\overline{B_1C}} : \frac{\overline{B_2A}}{\overline{B_2C}}.$$

From this, noting that A_2, B_2, C_1 are collinear, by Menelaus theorem, we have

$$\frac{\overline{A_1B}}{\overline{A_1C}} . \frac{\overline{B_1C}}{\overline{B_1A}} . \frac{\overline{C_1A}}{\overline{C_1C}} = \frac{\overline{A_2B}}{\overline{A_2C}} . \frac{\overline{B_2C}}{\overline{B_2A}} . \frac{\overline{C_1A}}{\overline{C_1C}} = 1.$$

Hence, by Menelaus theorem, A_1, B_1, C_1 are collinear.
Case 2. $\alpha \not\equiv 0 \pmod{2\pi}$.
Let line Δ' pass through P such that $\angle(\Delta, \Delta') \equiv \frac{\alpha}{2} \pmod{\pi}$.
Apparently, $R_P^\alpha \circ R_\Delta = (R_{\Delta'} \circ R_\Delta) \circ R_\Delta = R_{\Delta'} \circ (R_\Delta \circ R_\Delta) = R_{\Delta'} \circ id = R_{\Delta'}$
From this, noting that P belongs to Δ', according to case 1, we can deduce that A_1, B_1, C_1 are collinear.

3. PROOF OF THEOREM 1.3

We need two lemmas.

Lemma 3.1. *If BC, CA, AB are parallel to B_1C_1, C_1A_1, A_1B_1 respectively, then two triangles ABC and $A_1B_1C_1$ are similar in the same direction.*

Proof. We have $BC \,//\, B_1C_1$; $CA \,//\, C_1A_1$; $AB \,//\, A_1B_1$. Therefore, $\angle(BA,\, BC) \equiv \angle(B_1A_1,\, B_1C_1)$ $(\bmod\, \pi)$ and $\angle(CA,\, CB) \equiv \angle(C_1A_1,\, C_1B_1)\ (\bmod\, \pi)$.
Hence, triangles ABC and $A'B'C'$ are similar in the same direction.

Lemma 3.2. *Given two triangles ABC and $A_1B_1C_1$ which are similar in the same direction. A_2, B_2, C_2 are the midpoints of AA_1, BB_1, CC_1 respectively. Then, triangle $A_2B_2C_2$ are similar to triangles ABC and $A_1B_1C_1$ in the same direction.*

Proof. Let M, N be the midpoints of AB_1, AC_1 respectively (see f.2).
Because M, N, A_2 are the midpoints of AB_1, AC_1, AA_1 respectively, MN, NA_2, A_2M are parallel to B_1C_1, C_1A_1, A_1B_1 respectively.
Therefore, by lemma 3.1, triangles A_2MN and $A_1B_1C_1$ are similar in the same direction (1).
As A_2, B_2, C_2, M, N are the midpoints of $AA_1, BB_1, CC_1, AB_1, AC_1$ respectively,

$$\overrightarrow{MA_2} = \frac{1}{2}\overrightarrow{B_1A_1};\ \overrightarrow{MB_2} = \frac{1}{2}\overrightarrow{AB};\ \overrightarrow{NA_2} = \frac{1}{2}\overrightarrow{C_1A_1};\ \overrightarrow{NC_2} = \overrightarrow{AC}.$$

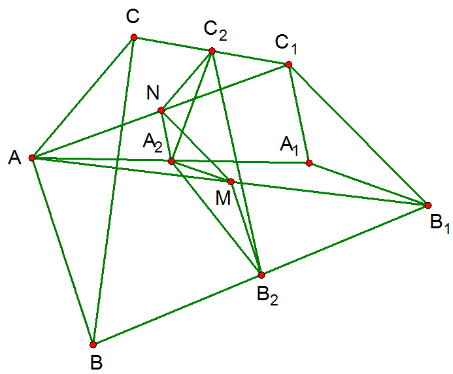

Figure 2

From this, noting that triangles ABC and $A_1B_1C_1$ are similar in the same direction,

$$\begin{aligned}\angle(\overrightarrow{MA_2}, \overrightarrow{MB_2}) &\equiv \angle(\overrightarrow{B_1A_1},\, \overrightarrow{AB}) \equiv \pi + \angle(\overrightarrow{B_1A_1},\, \overrightarrow{BA})\ (\bmod 2\pi)\\ &\equiv \pi + \angle(\overrightarrow{C_1A_1},\, \overrightarrow{CA}) \equiv \angle(\overrightarrow{C_1A_1},\, \overrightarrow{AC}) \equiv \angle(\overrightarrow{NA_2}, \overrightarrow{NC_2})(\bmod 2\pi).\end{aligned}$$

$$\frac{MA_2}{MB_2} = \frac{B_1A_1}{AB} = \frac{B_1A_1}{BA} = \frac{C_1A_1}{CA} = \frac{C_1A_1}{AC} = \frac{NA_2}{NC_2}.$$

Thus, triangles A_2MB_2 and A_2NC_2 are similar in the same direction.
Therefore, triangles A_2MN and $A_2B_2C_2$ are similar in the same direction (2).
From (1) and (2), deduce that $A_1B_1C_1$ and $A_2B_2C_2$ are similar in the same direction.
In other words, triangle $A_2B_2C_2$ are similar to triangles ABC and $A_1B_1C_1$ in the same direction.

Return to the proof of theorem 1.3.

Let A_2, B_2 be the intersections of PC_1 and BC, CA respectively. Let A_3, B_3, C_3 be the intersections of PA_1, PB_1, PC_1 and the lines parallel to Δ, passing through A, B, C respectively. Let A_0, B_0, C_0 be the intersections of Δ and AA_3, BB_3, CC_3 respectively (see f.3).

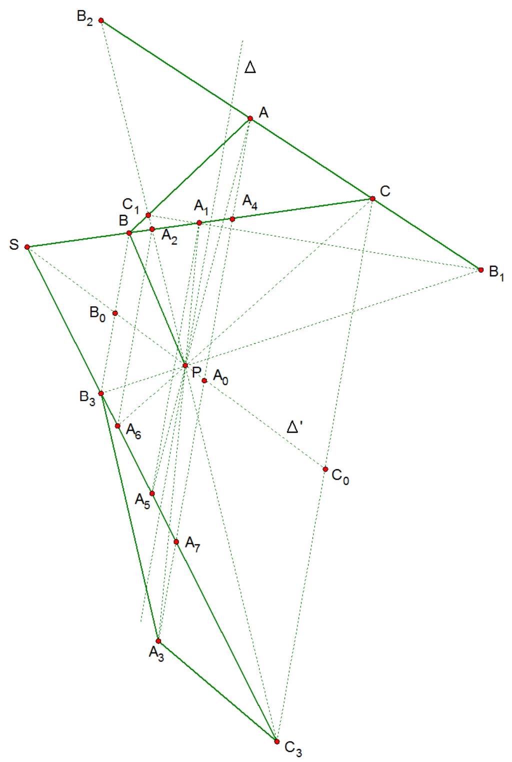

Figure 3

Because $(PA,\ PA_1, \Delta,\ \Delta')=(PB,\ PB_1,\ \Delta,\ \Delta')=(PC,\ PC_1, \Delta,\ \Delta')=-1$,
$(PA,\ PA_3,\ PA_0,\ \Delta)=(PB,\ PB_3,\ PB_0, \Delta)=(PC,\ PC_3,\ PC_0,\ \Delta)=-1$.
Then, combined with the fact that AA_3, BB_3, CC_3 are all parallel to Δ, we can deduce that A_0, B_0, C_0 are the midpoints of AA_3, BB_3, CC_3 respectively.
If BC, CA, AB are parallel to B_3C_3, C_3A_3, A_3B_3 respectively, then by lemma 3.1, triangles ABC and $A_3B_3C_3$ are similar in the same direction. From this, noting that A_0, B_0, C_0 are the midpoints of AA_3, BB_3, CC_3 respectively, by lemma 3.2, we can deduce that A_0, B_0, C_0 are not collinear, contradiction. Thus, BC, CA, AB are not respectively parallel to B_3C_3, C_3A_3, A_3B_3. Without the loss of generality, assume that BC and B_3C_3 are not parallel.
Let S be the intersection of BC and B_3C_3. Let A_2, A_4 be the intersections of BC and C_1P, AA_3 respectively. Let A_5, A_6, A_7 be the intersections of B_3C_3 and AP, CP, AA_3 respectively.
Apparently, S belongs to Δ'.
Applying Cevas theorem to triangle SC_3C, noting that SC_0, C_3A_2, CA_6 are concurrent (at P), we have

$$\frac{\overline{C_0C_3}}{\overline{C_0C}}.\frac{\overline{A_2C}}{\overline{A_2S}}.\frac{\overline{A_6S}}{\overline{A_6C_3}}=-1.$$

Combined with the fact that C_0 is the midpoint of C_3C, we have $\frac{\overline{A_2S}}{\overline{A_2C}}=\frac{\overline{A_6S}}{\overline{A_6C_3}}$.
Therefore, by Thales theorem, $A_2A_6\ //\ CC_3$ (3).

Applying Menelaus theorem to triangles A_0SA_4 and A_0SA_7, noting that A_1, A_3, P are collinear and A_5, A, P are collinear, we have

$$\frac{\overline{A_1S}}{\overline{A_1A_4}}.\frac{\overline{A_3A_4}}{\overline{A_3A_0}}.\frac{\overline{PA_0}}{\overline{PS}} = 1 = \frac{\overline{A_5S}}{\overline{A_5A_7}}.\frac{\overline{AA_7}}{\overline{AA_0}}.\frac{\overline{PA_0}}{\overline{PS}}.$$

From this, noting that A_0 is the midpoint of both AA_3 and A_4A_7, deduce that

$$\frac{\overline{A_1S}}{\overline{A_1A_4}} = \frac{\overline{A_5S}}{\overline{A_5A_7}}.$$

Therefore, by Thales theorem, $A_1A_5 \,//\, A_4A_7$ (4).
From (3) and (4), deduce that $BB_3//CC_3//A_1A_5//A_2A_6$.
Hence,

$$\frac{\overline{A_1B}}{\overline{A_1C}} : \frac{\overline{A_2B}}{\overline{A_2C}} = (BCA_1A_2) = (B_3C_3A_5A_6) = P(B_3C_3A_5A_6)$$
$$= P(B_1B_2AC) = P(ACB_1B_2) = (ACB_1B_2) = \frac{\overline{B_1A}}{\overline{B_1C}} : \frac{\overline{B_2A}}{\overline{B_2C}}.$$

From this, noting that A_2, B_2, C_1 are collinear, by Menelaus theorem, deduce that

$$\frac{\overline{A_1B}}{\overline{A_1C}}.\frac{\overline{B_1C}}{\overline{B_1A}}.\frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{A_2B}}{\overline{A_2C}}.\frac{\overline{B_2C}}{\overline{B_2A}}.\frac{\overline{C_1A}}{\overline{C_1B}} = 1.$$

Thus, by Menelaus theorem, A_1, B_1, C_1 are collinear.

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