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SEMI-SYMMETRIC LIGHTLIKE HYPERSURFACES IN A LORENTZIAN SPACE FORM

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ABSTRACT. In this paper, we study semi-symmetric lightlike hypersurface in a Lorentzian space form. We know ([11]), due to the degeneracy of the induced metric, that condition of semi-symmetry on a lightlike hypersurface of a Semi-Riemannian manifold, do not imply curvature condition of semi-symmetric type in general. We first show this implication in the case of a screen conformal lightlike hypersurface of a semi-Euclidean space. We give a necessary and sufficient condition for lightlike hypersurface to be semi-symmetric. We prove that a semi-symmetric screen conformal lightlike hypersurface of a semi-Euclidean space is totally umbilic. A local decomposition of a semi-symmetric screen conformal lightlike hypersurface of a Lorentzian manifold with constant curvature is obtained.

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Keywords: Lightlike hypersurface, screen conformal, screen shape operator, semi-symmetric lightlike hypersurface.

1. INTRODUCTION

A semi-Riemannian manifold is called semi-symmetric if $\mathcal{R} \cdot R = 0$, where R is curvature tensor and $\mathcal{R}$ the curvature operator corresponding to R. Semi-symmetric hypersurfaces of Euclidean spaces were classified by Nomizu [10] and a general study of semi-symmetric Riemannian manifolds was made by Szabo [12].
Semi-symmetric Lightlike hypersurfaces are introduced by Sahin in [11]. He showed that every screen conformal lightlike hypersurface of the Minkowski spacetime $\mathbb{R}^4_1$ is semi-symmetric. For $\mathbb{R}^{n+2}_q$, $n \geq 3$, he showed that semi-symmetry of a lightlike hypersurface depends on the geometry of a leaf of screen distribution.
In this paper, we study lightlike hypersurfaces of a Lorentzian manifold with constant curvature. It is known ([11]) that on lightlike hypersurfaces of a Semi-Riemannian manifolds, condition (30) do not imply condition (27) due to the degeneracy of the induced metric in general. We first show in section 3 that in the case of a screen conformal lightlike hypersurface of a Semi-Riemannian manifold, condition (30) imply condition (27) (Proposition 2). We give a necessary and sufficient condition for lightlike hypersurface to be semi-symmetric (Proposition3). We prove that a semi-symmetric lightlike hypersurface of a semi-Euclidean space is totally umbilic. At the end of the section, we obtain a local decomposition theorem for semi-symmetric screen conformal lightlike hypersurfaces of a Lorentzian space form.

2. PRELIMINARIES

2.1. **Lightlike hypersurfaces.** Let $(\overline{M},\overline{g})$ be a $(m+2)$-dimensional semi-Riemannian manifold of index ν, $(0<\nu<m+2)$. Consider a hypersurface M of $\overline{M}$ and denote by g the tensor field induced by $\overline{g}$ on M. We say that M is a lightlike (degenerate, null) hypersurface if $rank(g)=m$. Then the normal vector bundle $TM^{\perp}$ intersects the tangent bundle along a nonzero differentiable distribution called the radical distribution of M and denoted by $Rad(TM)$:

$$Rad(TM): x \mapsto Rad(T_xM) = T_xM \cap T_xM^{\perp}. \tag{1}$$

A *screen distribution* $S(TM)$ on M is a non-degenerate vector bundle complementary to $TM^{\perp}$. A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M,g,S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in the study of the geometry of lightlike hypersurfaces.

Theorem 2.1. *([8]) Let $(M,g,S(TM))$ be a lightlike hypersurface of $(\overline{M},\overline{g})$. Then there exists a unique vector bundle $tr(TM)$ of rank 1 over M, such that for any non zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U}\subset M$, there exists a unique section N of tr(TM) on $\mathcal{U}$ satisfying*

$$\overline{g}(N,\xi)=1 \text{ and } \overline{g}(N,N)=\overline{g}(N,W)=0, \tag{2}$$

for all $W\in\Gamma(S(TM)_{|\mathcal{U}})$.

With this theorem we may write the following decomposition

$$T\overline{M}_{|M} = S(TM)\perp(TM^{\perp}\oplus tr(TM)) = TM\oplus tr(TM), \tag{3}$$

where $\perp$ denotes an orthogonal direct sum and $\oplus$ a direct sum. Throughout the paper, we denoted by $\Gamma(E)$ the $C^{\infty}(M)$-module of smooth sections of a vector bundle E over M, while $C^{\infty}(M)$ represents the algebra of a smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.
Let $(M,g,S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\overline{g})$, $\overline{\nabla}$ be the Levi-Civita connexion of $\overline{M}$, ∇ the induced connection on (M,g). Gauss and Weingarten formulas provide the following relations (see details in [8])

$$\overline{\nabla}_XY = \nabla_XY + h(X,Y), \tag{4}$$

$$\overline{\nabla}_XV = -A_VX + \nabla^t_XV, \tag{5}$$

for all $X,Y\in\Gamma(TM)$ and $V\in tr(TM)$, where ∇_XY and A_VX belong to $\Gamma(TM)$ while h is a $\Gamma(tr(TM))$-valued symmetric $C^{\infty}(M)$-bilinear form on $\Gamma(TM)$ and ∇^t is a linear connection on $tr(TM)$. It is easy to see that ∇ is a torsion-free connection. Define a symmetric $C^{\infty}(M)$-bilinear form B and a 1-form τ on the coordinate neighborhood $\mathcal{U}\subset M$ by

$$B(X,Y) = \overline{g}(h(X,Y),\xi), \tag{6}$$

$$\tau(X) = \overline{g}(\nabla^t_XN,\xi) \tag{7}$$

for all $X,Y\in\Gamma(TM_{|\mathcal{U}})$. Then, on $\mathcal{U}$, equations (4) and (5) become,

$$\overline{\nabla}_XY = \nabla_XY + B(X,Y)N, \tag{8}$$

$$\overline{\nabla}_XN = -A_NX + \tau(X)N, \tag{9}$$

respectively. It is important to stress the fact that the local second fundamental form B in Eq.(8) does not depend on the choice of the screen distribution and satisfies,

$$B(X,\xi) = 0, \tag{10}$$

for all $X \in \Gamma(TM|_{\mathcal{U}})$. Let P be the projection morphism of TM to $S(TM)$ with respect to the decomposition (2). We obtain: for all $X, Y \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$,

$$\nabla_X PY = \overset{*}{\nabla}_X PY + \overset{*}{h}(X, PY), \tag{11}$$

$$\nabla_X U = -\overset{*}{A}_U X + \overset{*}{\nabla}{}^t{}_X U, \tag{12}$$

where $\overset{*}{\nabla}_X PY$ and $\overset{*}{A}_U X$ belong to $\Gamma(S(TM))$, $\overset{*}{\nabla}$ and $\overset{*}{\nabla}{}^t$ are linear connections on $\Gamma(S(TM))$ and $\Gamma(TM^{\perp})$ respectively, $\overset{*}{h}$ is a $\Gamma(TM^{\perp})$-valued $C^{\infty}(M)$-bilinear form on $\Gamma(TM) \times \Gamma(S(TM))$, $\overset{*}{A}_U$ is a $\Gamma(S(TM))$-valued $C^{\infty}(M)$-linear operator on $\Gamma(S(TM))$. $\overset{*}{h}$ and $\overset{*}{A}_U$ are the second fundamental form and the shape operator of the screen distribution $S(TM)$ respectively. Define on $\mathcal{U}$ the following relations

$$C(X, PY) = \overline{g}(\overset{*}{h}(X, PY), N), \tag{13}$$

$$\epsilon(X) = \overline{g}(\overset{*}{\nabla}{}^t{}_X \xi, N). \tag{14}$$

One shows that $\epsilon(X) = -\tau(X)$. Thus, locally (11) and (12) become

$$\nabla_X PY = \overset{*}{\nabla}_X PY + C(X, PY)\xi, \tag{15}$$

$$\nabla_X \xi = -\overset{*}{A}_{\xi} X - \tau(X)\xi, \tag{16}$$

respectively. The linear connection $\overset{*}{\nabla}$ is a metric connection on $\Gamma(S(TM))$. But, in general, the induced connection ∇ on M is not compatible with the induced metric g. Indeed, we have:

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{17}$$

for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$, where

$$\eta(X) = \overline{g}(X, N), \tag{18}$$

for all $Y \in \Gamma(TM|_{\mathcal{U}})$. Finally, it is straightforward to verify that

$$B(X, Y) = g(\overset{*}{A}_{\xi} X, Y), \quad g(A_N Y, N) = 0, \tag{19}$$

$$C(X, PY) = g(A_N X, Y), \quad \overset{*}{A}_{\xi} \xi = 0, \tag{20}$$

for $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

We denote the curvature tensor associated with $\overline{\nabla}$ and ∇ by $\overline{R}$ and R, respectively. Then we have ([8]): for all $X, Y \in \Gamma(TM|_{\mathcal{U}})$

$$\overline{R}(X, Y)Z = R(X, Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X + (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z), \tag{21}$$

$$\begin{aligned} g\Big(R(X, Y)PZ, PW\Big) &= g\Big(\overset{*}{R}(X, Y)PZ, PW\Big) + C(X, PZ)B(Y, PW) \\ &\quad - C(Y, PZ)B(X, PW), \end{aligned} \tag{22}$$

$$\overline{g}\Big(\overline{R}(X, Y)\xi, N\Big) = C(Y, \overset{*}{A}_{\xi} X) - C(X, \overset{*}{A}_{\xi} Y) - 2d\tau(X, Y). \tag{23}$$

2.2. **Curvature condition of Semi-symmetry.** Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold. We denote its curvature operator by $\mathcal{R}(X,Y)$,

$$\overline{\mathcal{R}}(X,Y) = \overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]},$$

for all $X, Y \in \Gamma(T\overline{M})$, where $\overline{\nabla}$ denote the Livi-Civita connection on $\overline{M}$. Then the curvature tensor $\overline{R}$ and the Riemannian curvature tensor $\overline{\mathbf{R}}$ are defined by

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z, \tag{24}$$

$$\overline{\mathbf{R}}(X,Y,Z,W) = g(\overline{R}(X,Y)Z,W). \tag{25}$$

For a $(0,k)$-tensor field T on $\overline{M}$, $k \geq 1$, one defines a $(0,k+2)$-tensor field $\overline{R}\cdot T$ is defined by

$$\begin{aligned}(\overline{\mathcal{R}}\cdot T)(X_1,\ldots,X_k,X,Y) &= -T(\overline{R}(X,Y)X_1,X_2,\ldots,X_k)\\ &- \cdots - T(X_1,\ldots,X_{k-1},\overline{R}(X,Y)X_k)\end{aligned} \tag{26}$$

for $X, Y, X_1, \ldots, X_k \in \Gamma(T\overline{M})$. Curvature conditions, involving the form $\overline{\mathcal{R}}\cdot T = 0$, are called curvature conditions of semi-symmetric type ([7]).
A semi-Riemannian manifold $\overline{M}$ is said to be semi-symmetric if it satisfies the condition $\overline{\mathcal{R}}\cdot\overline{\mathbf{R}} = 0$. Thus, from properties of curvature tensor, we have

$$\begin{aligned}(\overline{\mathcal{R}}(X,Y)\cdot\overline{R})(U,V)W &= \overline{R}(X,Y)\overline{R}(U,V)W - \overline{R}(U,V)\overline{R}(X,Y)W\\ &-\overline{R}(\overline{R}(X,Y)U,V)W - \overline{R}(U,\overline{R}(X,Y)V)W = 0,\end{aligned} \tag{27}$$

for any $X, Y, U, V, W \in \Gamma(T\overline{M})$.

3. Semi-symmetric lightlike hypersurfaces in a Lorentzian space form

Let M be a lightlike hypersurface M of a semi-Riemannian manifold $(\overline{M}(k), \overline{g})$ of constant curvature k. We need the following proposition.

Proposition 1. *[2] Let $(\overline{M}(k), \overline{g})$ be a semi-Riemannian manifold of constant curvature k and M be a lightlike hypersurface of $\overline{M}(k)$. Let R the curvature tensor of the induced connection ∇ on M by the Levi-civita connection $\overline{\nabla}$. For any $X, Y, Z \in \Gamma(TM)$, we have:*
- (a) $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_NY + B(Y,Z)A_NX;$
- (b) $(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X);$
- (c) $B(A_NY,X) - B(A_NX,Y) = 2d\tau(X,Y);$
- (d) $(\nabla_Y A_N)(X) - (\nabla_X A_N)(Y) + k\{\eta(X)Y - \eta(Y)X\} = \tau(Y)A_NX - \tau(X)A_NY;$
- (e) $(\nabla_X \overset{*}{A}_\xi)(Y) - (\nabla_Y \overset{*}{A}_\xi)(X) = \tau(Y)\overset{*}{A}_\xi X - \tau(X)\overset{*}{A}_\xi Y - 2d\tau(X,Y)\xi;$
- (f) $\nabla_X PZ = \nabla_X Z - X\cdot\eta(Z)\xi + \eta(Z)\overset{*}{A}_\xi + \eta(Z)\tau(X)\xi.$

Now, we recall the definition of a screen conformal lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$.

Definition 3.1. *([1]). A lightlike hypersurface $(M, g, S(TM))$ of a semi-Riemannian manifold $\overline{M}$ is said to be screen globally) conformal if the shape operators A_N and $\overset{*}{A}_\xi$ of M and its screen distribution $S(TM)$ are related by*

$$A_N = \varphi\overset{*}{A}_\xi, \tag{28}$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M. In case $\mathcal{U} = M$) the screen conformality is said to be global.

It is easy to see that (28) is equivalent to

$$C(Y, PZ) = \varphi B(Y, Z), \tag{29}$$

for all $X, Y \in \Gamma(TM)$.
We note that there are many examples of screen conformal lightlike hypersurfaces of semi-Riemannian manifolds see [1]

Definition 3.2. *([11]) Let M be a lightlike hypersurface of an $(m+2)$-dimensional semi-Riemannian manifold $(\overline{M}(k), \bar{g})$. We say that M is semi-symmetric if the following condition is satisfied*

$$(\mathcal{R}(X, Y) \cdot \boldsymbol{R})(X_1, X_2, X_3, X_4) = 0, \tag{30}$$

for all $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$, where $\mathcal{R}$ and $\boldsymbol{R}$ are the induced curvature operator and the induced Riemann curvature on M.

We also note that on a lightlike hypersurface, condition (30) do not imply condition (27) as in the non degenerate case due to $g(R(X,Y)Z, W) \neq -g(Z, R(X,Y)W)$ in general for all $X, Y, Z, W \in \Gamma(TM)$. In the case of a screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}(k), \bar{g})$ of constant curvature k, we prove the following proposition.

Proposition 2. *Let M be a screen conformal lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}(k), \bar{g})$ of constant curvature k. Then,*

$$g\Big(R(X,Y)Z, W\Big) = -g\Big(Z, R(X,Y)W\Big), \tag{31}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Moreover, if $k = 0$, then

$$(\mathcal{R}(X, Y) \cdot \boldsymbol{R})(X_1, X_2, X_3, X_4) = 0$$

is equivalent to

$$(\mathcal{R}(X, Y) \cdot R)(X_1, X_2, X_3) = 0,$$

for all $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$.

Proof. From (1) in proposition 1 we have

$$R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} - B(X,Z)A_N Y + B(Y,Z)A_N X.$$

Then,we have by using equation (19) and (28):

$$\begin{aligned} R(X,Y)Z &= k\{g(Y,Z)X - g(X,Z)Y\} - \varphi g(\overset{*}{A}_\xi X, Z)\overset{*}{A}_\xi Y \\ &\quad + \varphi g(\overset{*}{A}_\xi Y, Z)\overset{*}{A}_\xi X. \end{aligned} \tag{32}$$

Thus, using the equation (32), we obtain (31) by direct computation. To prove the equivalence we have

$$
\begin{aligned}
&(\mathcal{R}(X,Y)\cdot\mathbf{R})(X_1,X_2,X_3,X_4) = -\mathbf{R}\Big(R(X,Y)X_1,X_2,X_3,X_4\Big)\\
&-\mathbf{R}\Big(X_1,R(X,Y)X_2,X_3,X_4\Big) - \mathbf{R}\Big(X_1,X_2,R(X,Y)X_3,X_4\Big)\\
&-\mathbf{R}\Big(X_1,X_2,X_3,R(X,Y)X_4\Big)\\
&= -g\Big(R(R(X,Y)X_1,X_2)X_3,X_4\Big) - g\Big(R(X_1,R(X,Y)X_2)X_3,X_4\Big)\\
&-g\Big(R(X_1,X_2)R(X,Y)X_3,X_4\Big) - g\Big(R(X_1,X_2)X_3,(R(X,Y)X_4\Big)\\
&\overset{(31)}{=} -g\Big(R(R(X,Y)X_1,X_2)X_3,X_4\Big) - g\Big(R(X_1,R(X,Y)X_2)X_3,X_4\Big)\\
&-g\Big(R(X_1,X_2)R(X,Y)X_3,X_4\Big) + g\Big(R(X,Y)\cdot R(X_1,X_2)X_3,X_4\Big)\\
&= g\Big((\mathcal{R}(X,Y)\cdot R)(X_1,X_2,X_3),X_4\Big). \qquad (33)
\end{aligned}
$$

It follows that

$$(\mathcal{R}(X,Y)\cdot\mathbf{R})(X_1,X_2,X_3,X_4) = g\Big((\mathcal{R}(X,Y)\cdot R)(X_1,X_2,X_3),X_4\Big). \qquad (34)$$

We can easily see from (32) that $(\mathcal{R}(X,Y)\cdot R)(X_1,X_2,X_3) \in \Gamma(S(TM))$ if $k = 0$, for all $X, Y, X_1, X_2, X_3, X_4 \in \Gamma(TM)$. Thus equivalence follows from (34).
Note that for a class of screen conformal lightlike hypersurface M, the screen distribution $S(TM)$ is Riemannian, integrable and the induced Ricci tensor on M is symmetric ([1]). Then, according to Proposition 3.4 in [8], there exists a canonical null pair $\{\xi, N\}$ satisfying (2) such that the corresponding 1-form τ from (9) vanishes. Since ξ is an eigenvector field of $\overset{*}{A}_\xi$ corresponding to the eigenvalue 0 and $\overset{*}{A}_\xi$ is $\Gamma(S(TM))$-valued real symmetric, $\overset{*}{A}_\xi$ has m orthonormal eigenvector fields in $S(TM)$ and is diagonalizable. Consider a frame field of eigenvectors $\{\xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then, $\overset{*}{A}_\xi\ E_i = \lambda_i E_i, \quad 1 \le i \le m$. We call the eigenvalues λ_i the *screen principal curvatures* for all i.
We now proceed to investigate the effect of semi-symmetry condition on the geometry of lightlike hypersurfaces in a Lorentzian space form. We have

Proposition 3. *Let M be a screen conformal lightlike hypersurface of a $(m+2)$ dimensional Lorentz manifold $(\overline{M}(k), \bar{g})$ of constant curvature k. Then, M is semi-symmetric if and only if for distinct i, j, r, the screen principal curvatures satisfy*

$$(k + \varphi\lambda_i\lambda_j)(\lambda_i - \lambda_j)\lambda_r = 0. \qquad (35)$$

Proof. Consider the frame field of eigenvectors $\{E_0 = \xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then $\overset{*}{A}_\xi\ E_i = \lambda_i E_i$, $1 \le i \le m$. If i, j, r are distinct, we use (32) to get

$$R(E_i, E_j)E_j = (k + \varphi\lambda_i\lambda_j)E_i. \qquad (36)$$

By using this equation we have:

$$\begin{aligned}\mathbf{R}\Big(R(E_i,E_j)E_i,E_r,E_r,E_j\Big) &= g\Big(R(R(E_i,E_j)E_i,E_r)E_r,E_j\Big)\\ &= -g\Big(R(R(E_j,E_i)E_i,E_r)E_r,E_j\Big)\\ &= -(k+\varphi\lambda_i\lambda_j)g\Big(R(E_j,E_r)E_r,E_j\Big)\\ &= -(k+\varphi\lambda_i\lambda_j)(k+\varphi\lambda_j\lambda_r). \qquad (37)\end{aligned}$$

$$\mathbf{R}\Big(E_i,R(E_i,E_j)E_r,E_r,E_j\Big) = \mathbf{R}\Big(E_i,E_r,R(E_i,E_j)E_r,E_j\Big)=0 \qquad (38)$$

$$\mathbf{R}\Big(E_i,E_r,E_r,R(E_i,E_j)E_j\Big) = (k+\varphi\lambda_i\lambda_r)(k+\varphi\lambda_i\lambda_j). \qquad (39)$$

Thus, if M is semi-symmetric i.e $\mathcal{R}(X,Y)\cdot\mathbf{R}=0$, for all X and Y (condition (30)), we get

$$\begin{aligned}&(\mathcal{R}(E_i,E_j)\cdot\mathbf{R})(E_i,E_r,E_r,E_j)=0\\ &= -\mathbf{R}\Big(R(E_i,E_j)E_i,E_r,E_r,E_j\Big)-\mathbf{R}\Big(E_i,R(E_i,E_j)E_r,E_r,E_j\Big)\\ &-\mathbf{R}\Big(E_i,E_r,R(E_i,E_j)E_r,E_j\Big)-\mathbf{R}\Big(E_i,E_r,E_r,R(E_i,E_j)E_j\Big)\\ &=(k+\varphi\lambda_i\lambda_j)(k+\varphi\lambda_j\lambda_r)-(k+\varphi\lambda_i\lambda_r)(k+\varphi\lambda_i\lambda_j)\\ &=-\varphi(k+\varphi\lambda_i\lambda_j)(\lambda_i-\lambda_j)\lambda_r. \qquad (40)\end{aligned}$$

Thus, we have (35) since φ is a non vanishing smooth function. Conversely, suppose that this condition holds. It is sufficient to verify $\mathcal{R}(E_i,E_j)\cdot\mathbf{R}=0$ for $i\neq j$. If i,j,k,l,r, and s are all distinct, then $(\mathcal{R}(E_i,E_j)\cdot\mathbf{R})(E_r,E_l,E_r,E_s)=(\mathcal{R}(E_i,E_j)\cdot\mathbf{R})(E_j,E_r,E_l,E_s)=(\mathcal{R}(E_i,E_j)\cdot\mathbf{R})(E_r,E_l,E_l,E_s)=(\mathcal{R}(E_i,E_i)\cdot\mathbf{R})(E_j,E_r,E_r,E_r)=0$. By assumption, $(\mathcal{R}(E_i,E_j)\cdot\mathbf{R})(E_i,E_r,E_r,E_j)=0$. Finally, the skew symmetry takes care of the rest. ■
We need the following definition.

Definition 3.3. *([8]) Let M be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M},\bar{g})$. The **type number** $t(u)$ of M at a point u is the rank of the shape operator A_N at u.*

Remark 1. If M is screen conformal, then $\mathrm{rank}A_N=\mathrm{rank}\overset{*}{A}_\xi$. Thus, the type number $t(u)$ of M at a point u is the rank of the screen shape operator $\overset{*}{A}_\xi$ at u.

Next, we say that M is totally umbilical there exists a smooth function ρ such that

$$B(X,Y)=\rho g(X,Y), \qquad (41)$$

for all $X,Y\in\Gamma(TM)$, or equivalently,

$$\overset{*}{A}_\xi\, X=\rho PX, \qquad (42)$$

Theorem 3.1. *Let M be a screen conformal lightlike hypersurface in $\mathbb{R}^{m+2}_1$ and $u\in M$. Then, the following assertions hold.*

1. *If M is a semi-symmetric lightlike hypersurface such that $t(u)\geq 3$, then the non-zero screen principal curvatures are equal. Moreover, if $t(u)=m$, then M is totally umbilic.*
2. *Every screen conformal lightlike hypersurface M of $\mathbb{R}^{m+2}_1$ such that $t(u)\leq 2$ is a semi-symmetric lightlike hypersurface.*

Proof. Since $k = 0$ and φ is a non vanishing smooth function, the formula (35) reduces to $\lambda_i\lambda_j(\lambda_i - \lambda_j)\lambda_r = 0$. Thus if $t(u) \leq 2$ the formula is always satisfied, then M is semi-symmetric which proves (2). If $t(u) \geq 3$, then if λ_i and λ_j are non-zero, we can find $\lambda_r \neq 0$ and by the formula, it comes that $\lambda_i = \lambda_j$. Then, the non-zero eigenfuntions are equal. In the case when $t(u) = m$, the screen principal curvatures $\lambda_i, \lambda_j, \lambda_r$ are non-zero for all $i, j, k \in \{1, \ldots, m\}$ and then the formula (35) imply that $\lambda_i = \lambda_j$, for all i, j. This means that all λ_i are equal, says λ. Consider the frame field of eigenvectors $\{E_0 = \xi, E_1, \ldots, E_m\}$ of $\overset{*}{A}_\xi$ such that $\{E_1, \ldots, E_m\}$ is an orthonormal frame field of $S(TM)$. Then $\overset{*}{A}_\xi E_i = \lambda_i E_i$, $1 \leq i \leq m$. For any $X \in \Gamma(TM)$, we have $X = \sum_1^m g(X, E_i)E_i + \eta(X)\xi$. Thus,

$$\begin{aligned}
\overset{*}{A}_\xi X &= \overset{*}{A}_\xi \Big(\sum_1^m g(X, E_i)E_i + \eta(X)\xi \Big) \qquad (43)\\
&= \sum_1^m g(X, E_i) \overset{*}{A}_\xi E_i + \eta(X) \overset{*}{A}_\xi \xi \\
&= \sum_1^m g(X, E_i)\lambda_i E_i \quad (\text{since } \overset{*}{A}_\xi \xi = 0) \\
&= \sum_1^m g(X, E_i)\lambda E_i \\
&= \lambda \sum_1^m g(X, E_i)E_i \\
&= \lambda PX. \qquad (44)
\end{aligned}$$

Which prove that M is totally umbilic ■

Remark 2. Conclusion (2) generalize Proposition 3.2 in [11]. In [1], Atindogbe and Duggal showed that a screen conformal lightlike hypersurface M of a semi-Riemannian manifold $\overline{M}$ is totally geodesic, totally umbilic if and only if any leaf M' of its integrable distribution is so immersed in $\overline{M}$ as a codimension 2 non-degenerate submanifold. Thus conclusion (1) of Theorem 3.1 says that if $t(u) = m$, then any leaf M' of $S(TM)$ of a semi-symmetric screen conformal lightlike hypersurface in $\mathbb{R}_1^{m+2}$ is a totally umbilic submanifold of $\mathbb{R}_1^{m+2}$ of codimension 2. In fact, we can see from proof of Theorem 3.1 that if $t(u) = m$, all screen principal curvatures of a semi-symmetric lightlike hypersurface M are equal.

Let us now deal with the case $k \neq 0$.

Proposition 4. *Let M be a semi-symmetric screen conformal lightlike hypersurface of a $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \bar{g})$ of constant curvature $k \neq 0$, $m > 2$. Then, for any $u \in M$ either $t(u) = m$ or $t(u) \leq 1$.*

Proof. Let $u \in M$, we know that $\overset{*}{A}_{\xi_u} \xi_u = 0$, then $t(u) < m + 1$. Suppose $t(u) \neq m$, thus $\lambda_i = 0$ for some i. Then for any $j \neq r$ distinct from i, the formula (35) reduces to $\lambda_j\lambda_r k = 0$. Thus λ_j is non-zero for at most one j and $t(u) \leq 1$ ■

Now, we prove the following result.

Theorem 3.2. *Let M be a semi-symmetric screen conformal lightlike hypersurface of a $(m+2)$-dimensional Lorentzian manifold $(\overline{M}(k), \bar{g})$ of constant curvature $k \neq 0$ ($m \geq 2$) such that* *for any $u \in M$, $t(u) = m$. Then $\overset{*}{A}_\xi$ has at most two distinct screen curvatures at each point. Moreover, M is locally a product manifold $M = L \times M_\lambda \times M_\mu$, where L is a null curve and M_λ and M_μ are totally umbilical leaves of some integrables distributions of M.*

Proof. If $t(u) = m$, for $i = 1$, from (35) we have for any j, $(k + \varphi\lambda_1\lambda_j)(\lambda_1 - \lambda_j) = 0$ so $\lambda_j = \lambda_1$ or $\lambda_j = -\frac{k}{\varphi\lambda_1}$. Thus at most two distinct screen curvatures are distinct at each point. If we note $\lambda_1 = \lambda$ and $\lambda_j = \mu$, we define the distributions $T_\lambda = \{X \in \Gamma(S(TM))\backslash \overset{*}{A}_\xi\ X = \lambda X\}$ and $T_\mu = \{X \in \Gamma(S(TM))\backslash \overset{*}{A}_\xi\ X = \mu X\}$. Since $(\nabla_X \overset{*}{A}_\xi)(Y) - (\nabla_Y \overset{*}{A}_\xi)(X)$=0, it follows that if $X, Y \in \Gamma(T_\lambda)$, $\overset{*}{A}_\xi\ [X,Y] = \overset{*}{A}_\xi\ (\nabla_X Y) - \overset{*}{A}_\xi\ (\nabla_Y X) = \nabla_X(\overset{*}{A}_\xi\ Y) - \nabla_Y(\overset{*}{A}_\xi\ X)$. However, $\overset{*}{A}_\xi\ X = \lambda X$ and $\overset{*}{A}_\xi\ Y = \lambda Y$ so $\overset{*}{A}_\xi\ [X,Y] = (X\cdot\lambda)X - (Y\cdot\lambda)X + \lambda[X,Y]$, thus

$$(\overset{*}{A}_\xi - \lambda)[X,Y] = (X\cdot\lambda)Y - (Y\cdot\lambda)X$$

The left side of the above equation lies in T_λ and the right side in T_μ, then $(\overset{*}{A}_\xi - \lambda)[X,Y] = 0$ and $(X\cdot\lambda)Y - (Y\cdot\lambda)X = 0$. $(\overset{*}{A}_\xi - \lambda)[X,Y] = 0$ implies that $[X,Y] \in T_\lambda$ which proves that T_λ is integrable. Also, since $(X\cdot\lambda)Y - (Y\cdot\lambda)X = 0$, if $\dim T_\lambda > 1$, we may choose X and Y to be linearly independent. Thus $X\cdot\lambda = 0$. However $\mu = -\frac{k}{\varphi\lambda}$ so $X\cdot\mu = -k\frac{(X\cdot\varphi)\lambda - \varphi(X\cdot\lambda)}{(\varphi\lambda)^2} = 0$. If we choose $X, Y \in \Gamma(T_\mu)$, by the same way, we have T_μ involutive and $X\cdot\lambda = X\cdot\mu = 0$. Hence λ and μ are constant along the screen distribution. Then By Lemma 3.4 in [2], if $X \in \Gamma(T_\lambda)$, $Y \in \Gamma(T_\mu)$, then $\nabla_X Y \in \Gamma(T_\mu)$, $\nabla_Y X \in \Gamma(T_\lambda)$, which shows that T_λ and T_μ are parallel along their normals in $S(TM)$. From ([1]) a conformal lightlike hypersurface M is locally a product manifold $C \times M'$, where C is a null curve and M' is a leaf of $S(TM)$. Since the leaf M' of $S(TM)$ is Riemannian and $S(TM) = T^s_\lambda \oplus_{orth} T^s_\mu$, where T^s_λ and T^s_μ are parallel distributions with respect to the induced connection $\overset{*}{\nabla}$ of M', by the decomposition theorem of de Rham ([6]) we have $M' = M_\lambda \times M_\mu$, where M_λ and M_μ are some leaves of T^s_λ and T^s_μ, respectively. It follows that $M = C \times M' = C \times M_\lambda \times M_\mu$ and the result follows. ■

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