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# SOME $(q, \alpha, \beta)$-METRICS PROJECTIVELY RELATED TO A KROPINA METRIC

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ABSTRACT. The class of $(q, \alpha, \beta)$-metrics is an important subclass of $(\alpha, \beta)$-metrics which contains well-known metrics such as Randers, Berwald and Matsumoto metrics. In this paper, we find the necessary and sufficient conditions under which two classes of $(q, \alpha, \beta)$-metrics are projectively related to a Kropina metric.

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*Keywords:* $(q, \alpha, \beta)$-metric, Kropina metric, projectively related metrics.

## 1. INTRODUCTION

Two regular metrics are called projectively related if there is a diffeomorphism between them such that the pull-back metric is pointwise projective to another one. In Riemannian geometry, two Riemannian metrics $\alpha$ and $\bar{\alpha}$ on a manifold $M$ are projectively related if and only if their spray coefficients have the relation $G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + P_0 y^i$, where $P = P(x)$ is a scalar function on $M$ and $P_0 := P_{x^k} y^k$. In Finsler geometry, two Finsler metrics $F$ and $\bar{F}$ on a manifold $M$ are called projectively related if $G^i = \bar{G}^i + P y^i$, where $G^i$ and $\bar{G}^i$ are the geodesic spray coefficients of $F$ and $\bar{F}$, respectively and $P = P(x, y)$ is a scalar function on the slit tangent bundle $TM_0$. In this case, any geodesic of the first is also geodesic for the second and vice versa.
In order to find explicit examples of projectively related Finsler metrics, we consider $(\alpha, \beta)$-metrics. An $(\alpha, \beta)$-metric is defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$ where $\phi = \phi(s)$ is a $C^\infty$ scalar function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on a manifold $M$. The projective changes between two special $(\alpha, \beta)$-metrics have been studied by many geometers [2] [6] [9]. Among the $(\alpha, \beta)$-metrics, Randers metric $F = \alpha + \beta$ and Kropina metric $F = \alpha^2/\beta$ are important and have deep geometric meaning [3] [10]. Then, Cui-Shen find necessary and sufficient conditions under which the Berwald metric $F = \alpha + 2\beta + \beta^2/\alpha$ and a Randers metric $\bar{F} = \bar{\alpha} + \bar{\beta}$ are projectively related [2]. In [6], Mu-Cheng get the conditions that a Randers-Kropina metric $F = \alpha + \epsilon\beta + \kappa\alpha^2/\beta$ is projectively equivalent to a Kropina metric $F = \bar{\alpha}^2/\bar{\beta}$.
There exists a special subclass of $(\alpha, \beta)$-metrics, namely $(q, \alpha, \beta)$-metrics. Let $\phi : [-1, 1] \to R$, $\phi(s) = (1+s)^q$, $1 \le q \le 2$ and $||\beta||_\alpha < 1$. It is easy to see that

$$\phi' = q(1+s)^{q-1}, \quad \phi'' = q(q-1)(1+s)^{q-2} > 0.$$

Since $\phi(s) = (1+s)^q > 0$, then $\phi - s\phi' = (1+s)^{q-1}[1 + s(1-q)] > 0, \quad (|s| < 1)$. Thus $F := \alpha\phi(\frac{\beta}{\alpha}) = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$ is a Finsler metric. We call it $(q, \alpha, \beta)$-metric. When $q = 1$ or $q = 2$, $F$ becomes Randers metric and Berwald metric, respectively. If we substitute $\beta$ with $-\beta$ and take $q = -1$, the resulting metric is Matsumoto metric. In this paper, we are going to find the conditions under which the $(q, \alpha, \beta)$-metric $F = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$ and a Kropina metric $\bar{F} = \frac{\bar{\alpha}^2}{\bar{\beta}}$ being projectively related.

**Theorem 1.** *Let $F = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$ $(q \neq 1)$ be a $(q, \alpha, \beta)$-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of $\alpha$ and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha - \bar{G}^i_{\bar{\alpha}} = \theta y^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) - \frac{1}{2}\frac{q(q-1)\alpha^2 r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + [1 + (q^2-q)b^2]\alpha^2}b^i, \tag{1.1}$$

*where $b^i := a^{ij}b_j$, $\bar{b}^i := \bar{a}^{ij}\bar{b}_j$, $\bar{b}^2 := \|\bar{\beta}\|_{\bar{\alpha}}$ and $\theta := \theta_i y^i$ is a 1-form on $M$.*

Let us define $\phi(s) := s(\frac{s}{s-1})^{q-1}$. By a simple calculation, we get $\phi - s\phi' > 0$. Then $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$ is a Finsler metric. This metric is another $(q, \alpha, \beta)$-metric, also.

**Theorem 2.** *Let $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$ $(q \neq 1, -1)$ be a $(q, \alpha, \beta)$-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of $\alpha$ and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha - \bar{G}^i_{\bar{\alpha}} = \theta y^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) - \frac{q\alpha^3 r_{00}}{2\big[\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha\big]}b^i, \tag{1.2}$$

*where $b^i := a^{ij}b_j$, $\bar{b}^i := \bar{a}^{ij}\bar{b}_j$, $\bar{b}^2 := \|\bar{\beta}\|_{\bar{\alpha}}$ and $\theta := \theta_i y^i$ is a 1-form on $M$.*

## 2. Preliminary

The geodesic curves of a Finsler metric $F = F(x, y)$ on a smooth manifold $M$, are determined by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where the local functions $G^i = G^i(x, y)$ are called the spray coefficients, and given by

$$G^i = \frac{1}{4}g^{il}\Big\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big\}.$$

A Finsler metric $F$ is called a Berwald metric, if $G^i$ are quadratic in $y \in T_xM$ for any $x \in M$.
Let

$$D^i_{j\,kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\big). \tag{2.1}$$

It is easy to verify that $\mathcal{D} := D^i_{j\ kl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle $TM_0$. We call $\mathcal{D}$ the Douglas tensor. The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics $F$ and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x,y)$ is positively $y$-homogeneous of degree one, then the Douglas tensor of $F$ is same as that of $\bar{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics [7] [8]. The notion of Douglas metrics was first proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [1].
An $(\alpha,\beta)$-metric is a Finsler metric on a manifold $M$ defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the interval $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. For an $(\alpha,\beta)$-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta^j_i$, where $\theta^i := dx^i$ and $\theta^j_i := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of $\alpha$. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, $\beta$ is closed if and only if $s_{ij} = 0$. An $(\alpha,\beta)$-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Put

$$r_{i0} := r_{ij}y^j, \ \ r_{00} := r_{ij}y^iy^j, \ \ r_j := b^ir_{ij}, \quad r_0 := r_jy^j$$
$$s_{i0} := s_{ij}y^j, \quad s_j := b^is_{ij}, \quad s_0 := s_jy^j.$$

For an $(\alpha,\beta)$-metric $F = \alpha\phi(s)$, put

$$Q := \frac{\phi'}{\phi - s\phi'}.$$

Let $G^i = G^i(x,y)$ and $\bar{G}^i_\alpha = \bar{G}^i_\alpha(x,y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. By definition, we have

$$G^i = G^i_\alpha + \alpha Q s^i_0 + (-2Q\alpha s_0 + r_{00})(\Theta\frac{y^i}{\alpha} + \Psi b^i), \tag{2.2}$$

where

$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi\Big[(\phi - s\phi') + (b^2 - s^2)\phi''\Big]}, \quad \Psi := \frac{1}{2}\frac{\phi''}{(\phi - s\phi') + (b^2 - s^2)\phi''}.$$

By (2.2), it follows that every trivial $(\alpha,\beta)$-metric satisfies $G^i = G^i_\alpha$ and then it reduces to a Berwald metric.

## 3. Proof of Theorem 1

For an $(q,\alpha,\beta)$-metric $F = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$, the following hold

$$\begin{aligned} Q &= \frac{q}{s(1-q)+1}, \\ \Theta &= \frac{1}{2}\frac{q(1-2(q-1)s)}{s^2(1-q^2) + s(2-q) + 1 + b^2q(q-1)}, \\ \Psi &:= \frac{1}{2}\frac{q(q-1)}{s^2(1-q^2) + s(2-q) + 1 + b^2q(q-1)}. \end{aligned} \tag{3.1}$$

In the following, we shall denote the quantities for $\bar{F}$ by the same letters with the bar and the corresponding indices. Then for a Kropina metric $\bar{F} = \bar{\alpha} + \bar{\beta}$, we have

$$\bar{Q} := -\frac{1}{2\bar{s}}, \quad \bar{\Theta} := -\frac{\bar{s}}{2\bar{b}^2}, \quad \bar{\Psi} := \frac{1}{2\bar{b}^2}. \tag{3.2}$$

To prove Theorem 1, we remark the following.

**Lemma 1.** *[5]* Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on a n-dimensional manifold $M$. Then
(1) $(n \geq 3)$ Kropina metric $F$ with $(b^2 \neq 0)$ is a Douglas metric if and only if

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i); \tag{3.3}$$

(2) $(n = 2)$ Kropina metric $F$ is a Douglas metric.

For an $(\alpha, \beta)$-metric , the Douglas tensor is determined by

$$D^i_{j\ kl} := \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\big(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\big), \tag{3.4}$$

where

$$T^i := \alpha Q s^i_0 + \Psi(r_{00} - 2\alpha Q s_0)b^i, \tag{3.5}$$

$$T^m_{y^m} = Q' s_0 + \Psi'\alpha^{-1}(b^2 - s^2)(r_{00} - 2\alpha Q s_0) + 2\Psi\big[r_0 - Q'(b^2 - s^2)s_0 - Qss_0\big]. \tag{3.6}$$

Now, let $F$ and $\bar{F}$ be two $(\alpha, \beta)$-metrics which have the same Douglas tensor, i.e., $D^i_{jkl} = \bar{D}^i_{jkl}$. From (2.1) and (3.4), we have

$$\frac{\partial^3}{\partial y^i \partial y^j \partial y^k}\Big[T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\Big] = 0. \tag{3.7}$$

Then there exists a class of scalar function $H^i_{jk} := H^i_{jk}(x)$ such that

$$T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i = H^i_{00}, \tag{3.8}$$

where $H^i_{00} = H^i_{jk}(x)y^iy^j$, $T^i$ and $T^m_{y^m}$ are given by (3.5) and (3.6) respectively. In this paper, we assume that $\lambda := \frac{1}{n+1}$.

**Lemma 2.** *Let $F = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$ $(q \neq 1)$ be a $(q, \alpha, \beta)$-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics and $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ and $\bar{F}$ have the same Douglas tensor if and only if they are all Douglas metrics.*

**Proof.** The sufficiency is obvious. Suppose that $F$ and $\bar{F}$ have the same Douglas tensor on an n-dimensional manifold $M$ when $n \geq 3$. Then (3.8) holds. By plugging (3.1) and (3.2) into (3.8), we obtain

$$\frac{A^i\alpha^6 + B^i\alpha^5 + C^i\alpha^4 + D^i\alpha^3 + E^i\alpha^2 + F^i\alpha + H^i}{I\alpha^5 + J\alpha^4 + K\alpha^3 + L\alpha^2 + M\alpha + N} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} = H^i_{00}, \tag{3.9}$$

where

$$
\begin{aligned}
A^i :=& -2q(1-qb^2+q^2b^2)\Big[(1-qb^2+q^2b^2)s^i_0 - q(q-1)s_0b^i\Big],\\
B^i :=& q\Big[2(p-1)\lambda(1+qb^2)s_0y^i - 2q(q-1)(q-2)\beta s_0 b^i\\
&+4\beta(q-2)(q^2b^2-qb^2+1)s^i_0 + 2\lambda(q-1)(q^2b^2-qb^2+1)r_0y^i\\
&-(q-1)(q^2b^2-qb^2+1)r_{00}b^i\Big],\\
C^i :=& q\Big[(q-1)(q^3b^2-2q^2b^2+qb^2+2q-3)\beta r_{00}b^i + \lambda(q-1)(q-2)b^2r_{00}y^i\\
&-2(q-1)\lambda(q^3b^2-2q^2b^2+qb^2+2q-3)\beta r_0y^i\\
&+\big[4qb^2(q+1)(q-1)^2+(2q^2+8q-12)\big]\beta^2s^i_0\\
&-2\lambda(q-1)(3q^3b^2-2b^2q^2+2q-b^2q-3)\beta s_0y^i\\
&-2q(q+1)(q-1)^2\beta^2s_0b^i\Big],\\
D^i :=& q(q-1)\beta\Big[(2q^2-10q+6)\lambda\beta s_0y^i - 4(q+1)(q-2)\beta^2s^i_0\\
&-(q+4)(q-1)b^2r_{00}y^i + 3(q-1)\beta r_{00}b^i\Big],\\
E^i :=& -q(q-1)\beta^2\Big[2(q-1)(q+1)^2\beta^2s^i_0 + \lambda\big[2b^2(q+1)(q-1)^2+(q-2)\big]r_{00}y^i\\
&-2\lambda(q-1)(3q-1)(q+1)\beta s_0y^i - 2\lambda(q+1)(q-1)^2r_0y^i + (q+1)(q-1)^2\beta r_{00}b^i\Big],\\
F^i :=& -\lambda q(q+4)(q-1)^2\beta^3r_{00}y^i,\\
H^i :=& 2\lambda q(q+1)(p-1)^3\beta^4r_{00}y^i.
\end{aligned}
$$

and

$$
\begin{aligned}
I :=& -2(-qb^2+q^2b^2+1)^2,\\
J :=& 2\beta(-qb^2+q^2b^2+1)(qb^2-2q^2b^2+q^3b^2+3q-5),\\
K :=& 2\beta^2(-10-q^2+10q+6q^3b^2-12q^2b^2+6qb^2),\\
L :=& -2(q-1)\beta^3(2q^4b^2-2q^3b^2+3q^2-2q^2b^2+2q+2qb^2-10),\\
M :=& 2\beta^4(q+1)(q-5)(q-1)^2,\\
N :=& 2\beta^5(q+1)^2(q-1)^3.
\end{aligned}
$$

and

$$\bar{A}^i := \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0, \qquad \bar{B}^i := \bar{\beta}[2\lambda y^i(\bar{r}_0+\bar{s}_0) - \bar{b}^i\bar{r}_{00}].$$

(3.9) is equivalent to following

$$
\begin{aligned}
2\bar{b}^2\bar{\beta}(A^i\alpha^6 &+ B^i\alpha^5 + C^i\alpha^4 + D^i\alpha^3 + E^i\alpha^2 + F^i\alpha + H^i)\\
&+(\bar{A}^i\bar{\alpha}^2+\bar{B}^i)(I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N)\\
&= 2\bar{b}^2\bar{\beta}(I\alpha^5+J\alpha^4+K\alpha^3+L\alpha^2+M\alpha+N)H^i_{00}. \qquad (3.10)
\end{aligned}
$$

First we show that $\bar{A}^i$ can be divide by $\bar{\beta}$.

By replacing $y^i$ with $-y^i$ in (3.10), we get the following

$$\begin{aligned}-2\bar{b}^2\bar{\beta}(-A^i\alpha^6 + B^i\alpha^5 - C^i\alpha^4 + D^i\alpha^3 - E^i\alpha^2 + F^i\alpha - H^i)\\ -(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^5 - J\alpha^4 + K\alpha^3 - L\alpha^2 + M\alpha - N)\\ = -2\bar{b}^2\bar{\beta}(I\alpha^5 - J\alpha^4 + K\alpha^3 - L\alpha^2 + M\alpha - N)H^i_{00}.\end{aligned} \tag{3.11}$$

(3.10) + (3.11) yields

$$\begin{aligned}2\bar{b}^2\bar{\beta}(A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + H^i) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(J\alpha^4 + L\alpha^2 + N)\\ = 2\bar{b}^2\bar{\beta}(J\alpha^4 + L\alpha^2 + N)H^i_{00}.\end{aligned} \tag{3.12}$$

(3.10) − (3.11) implies that

$$\begin{aligned}(B^i\alpha^4 + D^i\alpha^2 + F^i)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^4 + K\alpha^2 + M)\\ = 2\bar{b}^2\bar{\beta}(I\alpha^4 + K\alpha^2 + M)H^i_{00}.\end{aligned} \tag{3.13}$$

If $q = -1$ then $H^i = N = M = 0$. Thus (3.12) and (3.13) are equivalent to

$$2\bar{b}^2\bar{\beta}(A^i\alpha^4 + C^i\alpha^2 + E^i) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(J\alpha^2 + L) = 2\bar{b}^2\bar{\beta}(J\alpha^2 + L)H^i_{00} \tag{3.14}$$

and

$$2\bar{b}^2\bar{\beta}(B^i\alpha^4 + D^i\alpha^2 + F^i) + (\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^4 + K\alpha^2) = 2\bar{b}^2\bar{\beta}(I\alpha^4 + K\alpha^2)H^i_{00}. \tag{3.15}$$

By (3.14) and (3.15), it results that $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(J\alpha^2 + L)$ and $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^4 + K\alpha^2)$ can be divided by $\bar{\beta}$. Thus $\beta = \mu\bar{\beta}$ and $\bar{A}^i\bar{\alpha}^2 I\alpha^4$ can be divided by $\bar{\beta}$. Since $\bar{\beta}$ is prime with respect to $\alpha$ and $\bar{\alpha}$, therefore $\bar{A}^i := \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. If $q \neq 1, -1$, then (3.12) and (3.13) implies that $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(J\alpha^4 + L\alpha^2 + N)$ and $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(I\alpha^4 + K\alpha^2 + M)$ can be divided by $\bar{\beta}$. Since $\bar{\beta}$ is prime with respect to $\alpha$ and $\bar{\alpha}$, then $\bar{A}^i := \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence, there is a scaler function $\psi^i(x)$ such that

$$\bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0 = \psi^i\bar{\beta}. \tag{3.16}$$

Contracting (3.16) with $\bar{y}_i := \bar{a}_{ij}y^j$ yields

$$\psi^i(x) = -\bar{s}^i.$$

Then we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i). \tag{3.17}$$

Now, suppose that $n \geq 3$. Then by Lemma 1, $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ is a Douglas metric. Since $F$ and $\bar{F}$ have the same Douglas tensor, then both of them are Douglas metrics.
If $n = 2$, then $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ is a Douglas metric by Lemma 1. Thus $F$ and $\bar{F}$ having the same Douglas tensors. This means that they are all Douglas metrics. This completes the proof of Lemma 2.

On the other hand, the following holds.

**Lemma 3.** *[4] Suppose that $Q/s \neq constant$ for an $(\alpha,\beta)$-metric $F = \phi(s)$ on a manifold $M$ of dimension $n \geq 3$. If $F$ is a Douglas metric and $b := \|\beta_x\|_\alpha \neq 0$, then $\beta$ is closed.*

Now, we are in the position to prove Theorem 1.

**Proof of Theorem 1:** We prove the theorem in two cases, as follows.

**Case (1):** $q = -1$.
First we proof the necessity. If $F$ is projectively equivalent to $\bar{F}$, then they have the same Douglas tensor. By Lemma 2, $F$ and $\bar{F}$ are both Douglas metrics. $F = \frac{\alpha^2}{\alpha+\beta}$ is a Douglas metric if and only if $b_{i|j} = 0$. Thus by (2.2), we have

$$G^i = G^i_\alpha. \tag{3.18}$$

On the other hand, plugging (3.2) and (3.17) in (2.2) yields

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}\big[-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + 2\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\big]. \tag{3.19}$$

By the projective equivalence of $F$ and $\bar{F}$ again, there is a scalar function $P = P(x, y)$ on $TM_0$ such that $G^i = \bar{G}^i + Py^i$. From (3.18) and (3.19) we have

$$\big[P - \frac{1}{\bar{b}^2}(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2})\big]y^i = G^i_\alpha - \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i). \tag{3.20}$$

Note that the right side of (3.20) is a quadratic in $y$. Then there exists a 1-form $\theta = \theta_i(x)y^i$ on $M$ such that

$$P - \frac{1}{\bar{b}^2}(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2}) = \theta. \tag{3.21}$$

Thus we have

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i. \tag{3.22}$$

This completes the proof of the necessity.
Conversely, because of $r_{00} = 0$ and from (1.1), (3.18) and (3.19) we have

$$G^i = \bar{G}^i + \big[\theta + \frac{1}{2\bar{b}^2}(\bar{s}_0 + \frac{\bar{r}_{00}\bar{\beta}}{\bar{\alpha}^2})\big]y^i. \tag{3.23}$$

In this case, $F$ is projectively related to $\bar{F}$.

**Case (2):** $q \neq 1, -1$.
First we proof the necessity. If $F$ is projectively equivalent to $\bar{F}$, then they have the same Douglas tensor. By Lemma 2, we know that $F$ and $\bar{F}$ are both Douglas metrics. If $q = 1, -1$, then it is easy to prove that $\phi(s) = (1+s)^q$ satisfies $Q/s \neq constant$. By Lemma 3, we have $s_{ij} = 0$. By (2.2), it follows that

$$\begin{aligned} G^i = G^i_\alpha &+ \frac{1}{2}\frac{q(\alpha - 2(q-1)\beta)r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + (1+q(q-1)b^2)\alpha^2}y^i \\ &+ \frac{1}{2}\frac{q(q-1)\alpha^2 r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + (1+q(q-1)b^2)\alpha^2}b^i. \end{aligned} \tag{3.24}$$

Plugging (3.2) and (3.17) in (2.2) implies that

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}\big[-\bar{\alpha}^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + 2\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\big]. \tag{3.25}$$

By the projective equivalence of $F$ and $\bar{F}$ again, there is a scalar function $P = P(x,y)$ on $TM_0$ such that $G^i = \bar{G}^i + Py^i$. By (3.24) and (3.25), we have

$$\begin{aligned}\bar{G}^i_{\bar{\alpha}} - G^i_{\alpha} &+ \big[P - \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) - \frac{1}{2}\frac{q[\alpha - 2(q-1)\beta]r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + [1+(q^2-q)b^2]\alpha^2}\big]y^i \\ &= \frac{1}{2}\frac{(q^2-q)\alpha^2 r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + [1+q(q-1)b^2]\alpha^2}b^i - \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i). \quad (3.26)\end{aligned}$$

Note that the right side of (3.26) is a quadratic in $y$. Then there exists a 1-form $\theta = \theta_i(x)y^i$ on $M$ such that

$$P - \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) - \frac{1}{2}\frac{q(\alpha - 2(q-1)\beta)r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + (1+q(q-1)b^2)\alpha^2} = \theta. \quad (3.27)$$

Thus we get

$$G^i_{\alpha} + \frac{1}{2}\frac{q(q-1)\alpha^2 r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + (1+q(q-1)b^2)\alpha^2}b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i.$$

This completes the proof of the necessity.
Conversely, by (1.1), (3.18) and (3.19) we have

$$G^i - \bar{G}^i = \big[\theta + \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) + \frac{1}{2}\frac{q[\alpha - 2(q-1)\beta]r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + [1+(q^2-q)b^2]\alpha^2}\big]y^i.$$

Thus $F$ is projectively equivalent to $\bar{F}$. This completes the proof.

By Lemma 1, Lemma 3 and Theorem 1, we have the following.

**Corollary 1.** *Let $F = \frac{(\alpha+\beta)^q}{\alpha^{q-1}}$ $(q \neq 1)$ be a $(q,\alpha,\beta)$ metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a $n$-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two nonzero collinear 1-forms on $M$. Then $F$ is projectively equivalent to $\bar{F}$ if and only if .*

$$\begin{aligned}&G^i_{\alpha} + \frac{1}{2}\frac{q(q-1)\alpha^2 r_{00}}{(1-q^2)\beta^2 + (2-q)\alpha\beta + [1+q(q-1)b^2]\alpha^2}b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i, \\ &s_{ij} = 0, \\ &\bar{s}_{ij} := \frac{1}{\bar{b}^2}\{\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\}.\end{aligned}$$

*where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta$ with respect to $\alpha$.*

It is well known that the Berwald metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ on a manifold $M$ is a Douglas metric if and only if

$$b_{i|j} = 2\tau[(1+2b^2)a_{ij} - 3b_ib_j], \quad (3.28)$$

where $\tau = \tau(x)$ is a scalar function on $M$. Thus by (3.28) and Theorem 1, we get the following.

**Corollary 2.** *Let $F = \frac{(\alpha+\beta)^2}{\alpha}$ be a Berwald metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the following holds*

$$G^i_\alpha - \bar{G}^i_{\bar{\alpha}} = \theta y^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) - 2\tau\alpha^2 b^i,$$

*where $\tau = \tau(x)$ is a scalar function on $M$.*

## 4. Proof of Theorem 2

In this section, we are going to prove the Theorem 2. More precisely, we find the conditions that an $(q,\alpha,\beta)$-metric $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$ being projectively equivalent to a Kropina metric. For the $(q,\alpha,\beta)$-metric $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$, the following are hold

$$Q = \frac{s-q}{(q-1)s}, \quad \Psi = \frac{q}{2[s^2(s-1)+q(b^2-s^2)]}, \quad \Theta = \frac{s(s-2q)}{2[s^2(s-1)+q(b^2-s^2)]}. \tag{4.1}$$

First we prove the following.

**Lemma 4.** *Let $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$ be an $(q,\alpha,\beta)$-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$ where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics and $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms. Then $F$ and $\bar{F}$ have the same Douglas tensor if and only if they are all Douglas metrics.*

**Proof.** The sufficiency is obvious. Suppose that $F$ and $\bar{F}$ have the same Douglas tensor on a manifold $M$ of dimension $n \geq 3$. Then (3.8) holds. By plugging (3.2) and (4.1) into (3.8), we obtain

$$\frac{\sum_{j=1}^{8} A^i_j \alpha^j}{\sum_{j=0}^{6} B_j \alpha^j} + \frac{\bar{A}^i\bar{\alpha}^2 + \bar{B}^i}{2\bar{b}^2\bar{\beta}} = H^i_{00}, \tag{4.2}$$

where

$$\begin{aligned}
A^i_1 &= 2\beta^7 s^i_0 - 3\lambda q(q-1)\beta^5 r_{00}y^i\\
A^i_2 &= 6\lambda\beta^5 s_0 y^i - 2(3q+2)\beta^6 s^i_0 + 2q(q^2-1)\lambda\beta^4 r_{00}y^i\\
A^i_3 &= 2[(q+1)^2 + 2q(q+1)]\beta^5 s^i_0 - 2q(6q+1)\lambda\beta^4 s_0 y^i + q(q-1)\beta^4 r_{00}b^i\\
&\quad + q(q-1)[3\lambda b^2\beta^3 r_{00}y^i - 2\lambda\beta^4 r_0 y^i],\\
A^i_4 &= [2q(q+1)(3q-1) - 6qb^2]\lambda\beta^3 s_0 y^i + 2q(q^2-1)\lambda\beta^2[\beta r_0 - b^2 r_{00}]y^i\\
&\quad - 2q\beta^4[(q+1)^2 - 2b^2]s^i_0 - q\beta^3[2\beta s_0 + (q^2-1)r_{00}]b^i,\\
A^i_5 &= 2q(5q+2)\lambda b^2\beta^2 s_0 y^i - 2q(2q+1)\beta^3[2b^2 s^i_0 - s_0 b^i],\\
A^i_6 &= 2q^2(q+1)\beta^2[2b^2 s^i_0 - s_0 b^i] - 2q^2(3q+1)\lambda b^2\beta s_0 y^i\\
&\quad - q^2(q-1)\lambda b^2\beta[2r_0 y^i - r_{00}b^i],\\
A^i_7 &= 2q^2 b^2\beta(b^2 s^i_0 - s_0 b^i),\\
A^i_8 &= -2q^3 b^2(b^2 s^i_0 - s_0 b^i).
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
B_0 &= 2(q-1)\beta^7,\\
B_1 &= -4(q-1)(q+1)\beta^6,\\
B_2 &= 2(q-1)(q+1)^2\beta^5,\\
B_3 &= 4q(q-1)b^2\beta^4,\\
B_4 &= -4q(q^2-1)b^2\beta^3,\\
B_5 &= 0,\\
B_6 &= 2q^2(q-1)b^4\beta,
\end{aligned} \tag{4.4}$$

and

$$\bar{A}^i := \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0, \quad \bar{B}^i := \bar{\beta}[2\lambda y^i(\bar{r}_0+\bar{s}_0) - \bar{b}^i\bar{r}_{00}].$$

(4.2) is equivalent to

$$(\sum_{j=1}^{8} A^i_j\alpha^j)(2\bar{b}^2\bar{\beta}) + (\bar{A}^i\bar{\alpha}^2+\bar{B}^i)(\sum_{j=0}^{6} B_j\alpha^j) = (2\bar{b}^2\bar{\beta})(\sum_{j=0}^{6} B_j\alpha^j)H^i_{00}. \tag{4.5}$$

By replacing $y^i$ with $-y^i$ in (4.5), we get

$$\begin{aligned}
&(\sum_{j=0}^{3} A^i_{(2j+1)}\alpha^{(2j+1)} - \sum_{j=1}^{4} A^i_{(2j)}\alpha^{(2j)})(-2\bar{b}^2\bar{\beta})\\
&\qquad -(\bar{A}^i\bar{\alpha}^2+\bar{B}^i)(\sum_{j=0}^{1} B_{(2j+1)}\alpha^{(2j+1)} - \sum_{j=0}^{3} B_{(2j)}\alpha^{(2j)}) =\\
&\qquad -2\bar{b}^2\bar{\beta}(\sum_{j=0}^{1} B_{(2j+1)}\alpha^{(2j+1)} - \sum_{j=0}^{3} B_{(2j)}\alpha^{(2j)})H^i_{00}.
\end{aligned} \tag{4.6}$$

$(4.5)-(4.6)$ implies that

$$2\bar{b}^2\bar{\beta}(\sum_{j=1}^{4} A^i_{(2j)}\alpha^{(2j)}) + (\bar{A}^i\bar{\alpha}^2+\bar{B}^i)\sum_{j=0}^{3} B_{(2j)}\alpha^{(2j)} = 2\bar{b}^2\bar{\beta}\sum_{j=0}^{3} B_{(2j)}\alpha^{(2j)}H^i_{00}. \tag{4.7}$$

$(4.5)+(4.6)$ yields

$$\begin{aligned}
&2\bar{b}^2\bar{\beta}\sum_{j=0}^{3} A^i_{(2j+1)}\alpha^{(2j+1)} + (\bar{A}^i\bar{\alpha}^2+\bar{B}^i)\quad\sum_{j=0}^{1} B_{(2j+1)}\alpha^{(2j+1)}\\
&\qquad = 2\bar{b}^2\bar{\beta}\sum_{j=0}^{1} B_{(2j+1)}\alpha^{(2j+1)}H^i_{00}.
\end{aligned} \tag{4.8}$$

$[\frac{\beta}{q} \times (4.7)] + [(4.8) \times \alpha]$ give us the following

$$(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\Big[(\frac{\beta}{q})(B_6\alpha^6) + (\frac{\beta}{q}B_4 + B_3)\alpha^4 + (\frac{\beta}{q}B_2 + B_1)\alpha^2 + \frac{\beta}{q}B_0\Big]$$
$$+(2\bar{b}^2\bar{\beta})\Big[(\frac{\beta}{q}A^i_6 + A^i_5)\alpha^6 + (\frac{\beta}{q}A^i_4 + A^i_3)\alpha^4 + (\frac{\beta}{q}A^i_2 + A^i_1)\alpha^2\Big] =$$
$$(2\bar{b}^2\bar{\beta})H^i_{00}\Big[(\frac{\beta}{q})(B_6\alpha^6) + (\frac{\beta}{q}B_4 + B_3)\alpha^4 + (\frac{\beta}{q}B_2 + B_1)\alpha^2 + \frac{\beta}{q}B_0\Big]. \tag{4.9}$$

All of members of set $\Big\{(\frac{\beta}{q}A^i_j + A^i_{(j-1)}), (\frac{\beta}{q}B_k + B_{(k-1)}), \frac{\beta}{q}B_6, \frac{\beta}{q}B_0 \ \ |j = 6,4,2, \ \ k = 4,2\Big\}$ have the factor $\beta^2$. Let us put

$$\begin{aligned}
D^i_j &:= \frac{1}{\beta^2}(\frac{\beta}{q}A^i_j + A^i_{(j-1)}), \quad j = 6,4,2\\
C_k &:= \frac{1}{\beta^2}(\frac{\beta}{q}B_k + B_{(k-1)}), \quad k = 4,2\\
C_6 &:= \frac{1}{q\beta}B_6 = 2q(q-1)b^4,\\
C_0 &:= \frac{1}{q\beta}B_0.
\end{aligned} \tag{4.10}$$

Then $\frac{1}{\beta^2} \times (4.9)$ yields

$$\begin{aligned}
(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\ \big[C_6\alpha^6 + C_4\alpha^4 + C_2\alpha^2 + C_0\big] + 2\bar{b}^2\bar{\beta}\big[D^i_6\alpha^6 + D^i_4\alpha^4 + D^i_2\alpha^2\big]&\\
= 2\bar{b}^2\bar{\beta}H^i_{00}\big[C_6\alpha^6 + C_4\alpha^4 + C_2\alpha^2 + C_0\big].&
\end{aligned} \tag{4.11}$$

By (4.7) and (4.11), it follows that $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)(\sum_{j=0}^{3} B_{(2j)}\alpha^{(2j)})$ and $(\bar{A}^i\bar{\alpha}^2 + \bar{B}^i)\big[C_6\alpha^6 + C_4\alpha^4 + C_2\alpha^2 + C_0\big]$ can be divided by $\bar{\beta}$. Thus $\beta = \mu\bar{\beta}$ and $\bar{A}^i\bar{\alpha}^2C_6\alpha^4$ can be divided by $\bar{\beta}$. Since $\bar{\beta}$ is prime with respect to $\alpha$ and $\bar{\alpha}$, then $\bar{A}^i := \bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0$ can be divided by $\bar{\beta}$. Hence, there is a scaler function $\psi^i(x)$ such that

$$\bar{b}^2\bar{s}^i_0 - \bar{b}^i\bar{s}_0 = \psi^i\bar{\beta}. \tag{4.12}$$

Contracting (4.12) with $\bar{y}_i := \bar{a}_{ij}y^j$ yields $\psi^i(x) = -\bar{s}^i$. Then we have

$$\bar{s}_{ij} = \frac{1}{\bar{b}^2}(\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i). \tag{4.13}$$

Now, suppose that $n \geq 3$. Then by Lemma 1, $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ is a Douglas metric. Since $F$ and $\bar{F}$ have the same Douglas tensor, both of them are Douglas metrics.
In the case $n = 2$, by Lemma 1, $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ is a Douglas metric. Thus $F$ and $\bar{F}$ having the same Douglas tensor means that they are all Douglas metrics. This completes the proof of Lemma 4.

Now, we are in the position to prove Theorem 2.

**Proof of Theorem 2:** First, we proof the necessity. If $F$ is projectively related to $\bar{F}$, then they have the same Douglas tensor. By Lemma 4, $F$ and $\bar{F}$ are both Douglas metrics. If

$q = 1, -1$, then it is easy to prove that $\phi(s) = \frac{s^q}{(s-1)^{q-1}}$ satisfies $\frac{Q}{s} \neq constant$. By Lemma 3, it results that $s_{ij} = 0$. By (2.2), we get

$$G^i = G^i_\alpha + \frac{1}{2}\frac{\beta(\beta - 2q\alpha)r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha}y^i + \frac{1}{2}\frac{q\alpha^3 r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha}b^i. \quad (4.14)$$

Plugging (3.2) and (4.13) in (2.2) yields

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{1}{2\bar{b}^2}\Big[ - \bar{\alpha}^2\bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00}\bar{b}^i) + 2\frac{\bar{r}_{00}\bar{\beta}y^i}{\bar{\alpha}^2}\Big]. \quad (4.15)$$

By assumption, there is a scalar function $P = P(x, y)$ on $TM_0$ such that $G^i = \bar{G}^i + Py^i$. Then by (4.14) and (4.15) we have

$$\begin{aligned}\Big[P - \frac{1}{2}\frac{\beta(\beta - 2q\alpha)r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha} - \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i)\Big]y^i &= G^i_\alpha - \bar{G}^i_{\bar{\alpha}} \\ - \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \frac{1}{2}\frac{q\alpha^3 r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha}b^i. \quad (4.16)\end{aligned}$$

The right side of (4.16) is a quadratic in $y$. Then there exists a 1-form $\theta = \theta_i(x)y^i$ on $M$ such that

$$P - \frac{1}{2}\frac{\beta(\beta - 2q\alpha)r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha} - \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) = \theta. \quad (4.17)$$

Thus we get

$$G^i_\alpha + \frac{1}{2}\frac{q\alpha^3 r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha}b^i = \bar{G}^i_{\bar{\alpha}} + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) + \theta y^i. \quad (4.18)$$

This completes the proof of the necessity.
Conversely, from (1.2), (4.14) and (4.15) it follows that

$$G^i = \bar{G}^i + \Big[\theta + \frac{1}{2}\frac{\beta(\beta - 2q\alpha)r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha} + \frac{1}{2\bar{b}^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i)\Big]y^i. \quad (4.19)$$

Thus $F$ is projectively equivalent to $\bar{F}$. This completes the proof.

By Lemmas 1, 3 and Theorem 2, we can conclude the following.

**Corollary 3.** *Let $F = \frac{\beta^q}{(\beta-\alpha)^{q-1}}$ $(q \neq 1, -1)$ be a $(q, \alpha, \beta)$-metric and $\bar{F} = \bar{\alpha}^2/\bar{\beta}$ be a Kropina metric on a n-dimensional manifold $M$ $(n \geq 3)$ where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two nonzero collinear 1-forms. Then $F$ is projectively related to $\bar{F}$ if and only if the following holds*

$$\begin{aligned} G^i_\alpha - \bar{G}^i_{\bar{\alpha}} &= \theta y^i + \frac{1}{2\bar{b}^2}(\bar{\alpha}^2\bar{s}^i + \bar{r}_{00}\bar{b}^i) - \frac{1}{2}\frac{q\alpha^3 r_{00}}{\beta^2(\beta-\alpha) + q(b^2\alpha^2 - \beta^2)\alpha}b^i, && (4.20)\\ s_{ij} &= 0, && (4.21)\\ \bar{s}_{ij} &= \frac{1}{\bar{b}^2}\{\bar{b}_i\bar{s}_j - \bar{b}_j\bar{s}_i\}. && (4.22)\end{aligned}$$

*where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta$ with respect to $\alpha$.*

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