SOME \((q, \alpha, \beta)\)-METRICS PROJECTIVELY RELATED TO A KROPINA METRIC

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ABSTRACT. The class of \((q, \alpha, \beta)\)-metrics is an important subclass of \((\alpha, \beta)\)-metrics which contains well-known metrics such as Randers, Berwald and Matsumoto metrics. In this paper, we find the necessary and sufficient conditions under which two classes of \((q, \alpha, \beta)\)-metrics are projectively related to a Kropina metric.

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1. INTRODUCTION

Two regular metrics are called projectively related if there is a diffeomorphism between them such that the pull-back metric is pointwise projective to another one. In Riemannian geometry, two Riemannian metrics \(\alpha\) and \(\bar{\alpha}\) on a manifold \(M\) are projectively related if and only if their spray coefficients have the relation 
\[
G^i_{\alpha} = \bar{G}^i_{\bar{\alpha}} + P_0 y^i,
\]
where \(P = P(x)\) is a scalar function on \(M\) and 
\[
P_0 = P(x, y).
\]
In Finsler geometry, two Finsler metrics \(F\) and \(\bar{F}\) on a manifold \(M\) are called projectively related if 
\[
G^i_F = \bar{G}^i_{\bar{F}} + P y^i,
\]
where \(G^i_F\) and \(\bar{G}^i_{\bar{F}}\) are the geodesic spray coefficients of \(F\) and \(\bar{F}\), respectively and 
\[
P = P(x, y)
\]
is a scalar function on the slit tangent bundle \(TM_0\). In this case, any geodesic of the first is also geodesic for the second and vice versa.

In order to find explicit examples of projectively related Finsler metrics, we consider \((\alpha, \beta)\)-metrics. An \((\alpha, \beta)\)-metric is defined by 
\[
F = \alpha \phi(s), \quad s = \beta/\alpha
\]
where \(\phi = \phi(s)\) is a \(C^\infty\) scalar function on \((-b_0, b_0)\) with certain regularity, 
\[
\alpha = \sqrt{a_{ij}(x)y^i y^j}
\]
is a Riemannian metric and 
\[
\beta = b_i(x)y^i
\]
is a 1-form on a manifold \(M\). The projective changes between two special \((\alpha, \beta)\)-metrics have been studied by many geometers [2, 6, 9]. Among the \((\alpha, \beta)\)-metrics, Randers metric \(F = \alpha + \beta\) and Kropina metric \(F = \alpha^2/\beta\) are important and have deep geometric meaning [3, 10]. Then, Cui-Shen find necessary and sufficient conditions under which the Berwald metric \(F = \alpha + 2\beta + \beta^2/\alpha\) and a Randers metric \(\bar{F} = \alpha + \beta\) are projectively related [2]. In [6], Mu-Cheng get the conditions that a Randers-Kropina metric \(F = \alpha + e\beta + \kappa\alpha^2/\beta\) is projectively equivalent to a Kropina metric \(\bar{F} = \alpha^2/\bar{\beta}\).

There exists a special subclass of \((\alpha, \beta)\)-metrics, namely \((q, \alpha, \beta)\)-metrics. Let \(\phi : [-1, 1] \to R\), \(\phi(s) = (1 + s)^q, 1 \leq q \leq 2\) and \(||\beta||_a < 1\). It is easy to see that 
\[
\phi' = q(1 + s)^{q-1}, \quad \phi'' = q(q - 1)(1 + s)^{q-2} > 0.
\]
Let us define $\phi(s) = (1+s)^q > 0$, then $\phi - s\phi' = (1+s)^q - [1 + s(1-q)] > 0$, ($|s| < 1$). Thus $F := \alpha\phi(\theta) = \frac{(\alpha + \beta)^q}{\alpha^q}$ is a Finsler metric. We call it $(q,\alpha,\beta)$-metric. When $q = 1$ or $q = 2$, $F$ becomes Randers metric and Berwald metric, respectively. If we substitute $\beta$ with $-\beta$ and take $q = -1$, the resulting metric is Matsumoto metric. In this paper, we are going to find the conditions under which the $(q,\alpha,\beta)$-metric $F = \frac{(\alpha + \beta)^q}{\alpha^q}$ and a Kropina metric $F = \frac{\alpha^2}{\beta}$ being projectively related.

**Theorem 1.** Let $F = \frac{(\alpha + \beta)^q}{\alpha^q}$ ($q \neq 1$) be a $(q,\alpha,\beta)$-metric and $\bar{F} = \frac{\alpha^2}{\beta}$ be a Kropina metric on a $n$-dimensional manifold $M$ ($n \geq 3$), where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of $\alpha$ and $\bar{\alpha}$ have the following relation

$$G_\alpha - \bar{G}_\alpha = \theta y^i + \frac{1}{2}b^2(\bar{a}^2\bar{s}^j + \bar{r}_{00}\bar{b}^j) - \frac{q(q-1)\alpha^2r_{00}}{2(1-q^2)\beta^2 + (2-q)\alpha\beta + [1 + (q^2 - q)\beta^2]\alpha^2}b^j,$$

where $b^i := a^i b_j, \ bar{b}^i := \bar{a}^i \bar{b}_j, \ \bar{b}^2 := ||\bar{\beta}||_\alpha$ and $\theta := \theta y^i$ is a 1-form on $M$.

Let us define $\phi(s) := s\left(\frac{q}{s} - \frac{1}{q}\right)$. By a simple calculation, we get $\phi - s\phi' > 0$. Then $F = \frac{\beta^\alpha}{(\alpha - \beta)^q}$ is a Finsler metric. This metric is another $(q,\alpha,\beta)$-metric, also.

**Theorem 2.** Let $F = \frac{\beta^\alpha}{(\alpha - \beta)^q}$ ($q \neq 1, -1$) be a $(q,\alpha,\beta)$-metric and $\bar{F} = \frac{\alpha^2}{\beta}$ be a Kropina metric on a $n$-dimensional manifold $M$ ($n \geq 3$), where $\alpha$ and $\bar{\alpha}$ are two Riemannian metrics, $\beta$ and $\bar{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ is projectively related to $\bar{F}$ if and only if they are Douglas metrics and the geodesic coefficients of $\alpha$ and $\bar{\alpha}$ have the following relation

$$G_\alpha' - \bar{G}_\alpha' = \theta y^i + \frac{1}{2}b^2(\bar{a}^2\bar{s}^j + \bar{r}_{00}\bar{b}^j) - \frac{q\alpha^2r_{00}}{2[\beta^2(\alpha - \beta) + q(\beta^2 - \beta^2)\alpha]}b^i,$$

where $b^i := a^i b_j, \ bar{b}^i := \bar{a}^i \bar{b}_j, \ \bar{b}^2 := ||\bar{\beta}||_\alpha$ and $\theta := \theta y^i$ is a 1-form on $M$.

2. PRELIMINARY

The geodesic curves of a Finsler metric $F = F(x,y)$ on a smooth manifold $M$, are determined by the system of second order differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where the local functions $G^i = G^i(x,y)$ are called the spray coefficients, and given by

$$G^i = \frac{1}{4}g^{ij} \left\{ \frac{\partial^2F^2}{\partial x^j\partial y^k}y^k - \frac{\partial F^2}{\partial x^j} \right\}.$$

A Finsler metric $F$ is called a Berwald metric, if $G^i$ are quadratic in $y \in T_xM$ for any $x \in M$.

Let

$$D^i_{jkl} := \frac{\partial^3}{\partial y^j\partial y^k\partial y^l}(G^i - \frac{1}{n+1}\partial G^m \partial y^m y^i). \quad (2.1)$$
It is easy to verify that $D := D^{ij}_{kl} dx^i \otimes \partial_i \otimes dx^k \otimes dx^l$ is a well-defined tensor on slit tangent bundle $TM_0$. We call $D$ the Douglas tensor. The Douglas tensor $D$ is a non-Riemannian projective invariant, namely, if two Finsler metrics $F$ and $\tilde{F}$ are projectively equivalent, $G^i = \tilde{G}^i + P y^i$, where $P = P(x, y)$ is positively $y$-homogeneous of degree one, then the Douglas tensor of $F$ is same as that of $\tilde{F}$. Finsler metrics with vanishing Douglas tensor are called Douglas metrics $[7]$ $[8]$. The notion of Douglas metrics was first proposed by Bácso-Matsumoto as a generalization of Berwald metrics $[1]$. An $(\alpha, \beta)$-metric is a Finsler metric on a manifold $M$ defined by $F := \alpha \phi(s), s = \beta / \alpha$, where $\phi = \phi(s)$ is a $C^\infty$ function on the interval $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij} y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. For an $(\alpha, \beta)$-metric, let us define $b_{ij}$ by $b_{ij} \theta^j := db_i - b_j \theta^j$, where $\theta^i := dx^i$ and $\theta^j := \Gamma^j_{ik} dx^k$ denote the Levi-Civita connection form of $\alpha$. Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}).$$

Clearly, $\beta$ is closed if and only if $s_{ij} = 0$. An $(\alpha, \beta)$-metric is said to be trivial if $r_{ij} = s_{ij} = 0$. Put

$$r_{i0} := r_{ij} y^j, \quad r_{00} := r_{ij} y^i y^j, \quad r_j := b_i r_{ij}, \quad r_0 := r_j y^j,$$

$$s_{i0} := s_{ij} y^j, \quad s_j := b_i s_{ij}, \quad s_0 := s_j y^j.$$

For an $(\alpha, \beta)$-metric $F = \alpha \phi(s)$, put

$$Q := \frac{\phi'}{\phi - s \phi'}.$$

Let $G^i = G^i(x, y)$ and $\tilde{G}^i_a = \tilde{G}^i_a(x, y)$ denote the coefficients of $F$ and $\alpha$ respectively in the same coordinate system. By definition, we have

$$G^i = G^i_a + a Q s_0 + (-2Q a s_0 + r_{00}) (\Theta^i_\alpha + \Psi b^i), \quad (2.2)$$

where

$$\Theta := \frac{\phi \phi'' - s (\phi \phi''' + \phi' \phi'')}{2\phi \left( (\phi - s \phi') + (b^2 - s^2) \phi'' \right)}, \quad \Psi := \frac{1}{2} \frac{\phi''}{(\phi - s \phi') + (b^2 - s^2) \phi''}.$$

By (2.2), it follows that every trivial $(\alpha, \beta)$-metric satisfies $G^i = \tilde{G}^i_a$ and then it reduces to a Berwald metric.

3. PROOF OF THEOREM

For an $(q, \alpha, \beta)$-metric $F = \frac{(\alpha + \beta)^q}{\alpha r}$, the following hold

$$Q = \frac{q}{s(1 - q) + 1},$$

$$\Theta = \frac{1}{2} \frac{q(1 - 2(q - 1)s)}{s^2(1 - q^2) + s(2 - q) + 1 + b^2 q(q - 1)},$$

$$\Psi := \frac{q(q - 1)}{2 s^2(1 - q^2) + s(2 - q) + 1 + b^2 q(q - 1)}.$$

(3.1)
In the following, we shall denote the quantities for $\tilde{F}$ by the same letters with the bar and the corresponding indices. Then for a Kropina metric $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$, we have

$$Q := - \frac{1}{2\tilde{s}}, \quad \Theta := - \frac{s}{2\tilde{b}^2}, \quad \Psi := \frac{1}{2\tilde{b}^2}.$$  \hspace{1cm} (3.2)

To prove Theorem 1, we remark the following.

**Lemma 1.** [5] Let $F = \frac{q^2}{b^2}$ be a Kropina metric on a $n$-dimensional manifold $M$. Then

1. $(n \geq 3)$ Kropina metric $F$ with $(b^2 \neq 0)$ is a Douglas metric if and only if

$$s_{ij} = \frac{1}{b^2} (\tilde{b}_i \tilde{s}_j - \tilde{b}_j \tilde{s}_i); \hspace{1cm} (3.3)$$

2. $(n = 2)$ Kropina metric $F$ is a Douglas metric.

For an $(\alpha, \beta)$-metric, the Douglas tensor is determined by

$$D_{ijkl} := \frac{\partial^3}{\partial y^i \partial y^j \partial y^l} \left( T^i - \frac{1}{n+1} \frac{\partial T^m}{\partial y^m} y^j \right),$$  \hspace{1cm} (3.4)

where

$$T^i := \alpha Qs_0 + \Psi (r_{00} - 2\alpha Qs_0)b^i, \hspace{1cm} (3.5)$$

$$T^m_{ij} = \alpha Qs_0 + \Psi \alpha^{-1} (b_0 - s^2)(r_{00} - 2\alpha Qs_0) + 2\Psi [r_0 - \Psi (b^2 - s^2)s_0 - Qss_0].$$  \hspace{1cm} (3.6)

Now, let $F$ and $\tilde{F}$ be two $(\alpha, \beta)$-metrics which have the same Douglas tensor, i.e., $D_{ijkl} = \tilde{D}_{ijkl}$. From (2.1) and (3.4), we have

$$\frac{\partial^3}{\partial y^i \partial y^j \partial y^l} \left[ T^i - \tilde{T}^i - \frac{1}{n+1} (T^m_{ij} - \tilde{T}^m_{ij}) y^j \right] = 0.$$  \hspace{1cm} (3.7)

Then there exists a class of scalar function $H^i_{jk} := H^i_{jk}(x)$ such that

$$T^i - \tilde{T}^i - \frac{1}{n+1} (T^m_{ij} - \tilde{T}^m_{ij}) y^j = H^i_{00},$$  \hspace{1cm} (3.8)

where $H^i_{00} = H^i_{jk}(x)y^i y^j$, $T^i$ and $T^m_{ij}$ are given by (3.5) and (3.6) respectively. In this paper, we assume that $\lambda := \frac{1}{n+1}$.

**Lemma 2.** Let $F = \frac{(a + \beta)^q}{a^{\alpha} + b^2}$ be a $(q, \alpha, \beta)$-metric and $\tilde{F} = \frac{\tilde{\alpha}^2}{\tilde{b}}$ be a Kropina metric on a $n$-dimensional manifold $M$ $(n \geq 3)$, where $\alpha$ and $\tilde{\alpha}$ are two Riemannian metrics and $\beta$ and $\tilde{\beta}$ are two non-zero collinear 1-forms on $M$. Then $F$ and $\tilde{F}$ have the same Douglas tensor if and only if they are all Douglas metrics.

**Proof.** The sufficiency is obvious. Suppose that $F$ and $\tilde{F}$ have the same Douglas tensor on an $n$-dimensional manifold $M$ when $n \geq 3$. Then (3.8) holds. By plugging (3.1) and (3.2) into (3.8), we obtain

$$\frac{A\tilde{\alpha}^6 + B\alpha^5 + C\alpha^4 + D\tilde{\alpha}^3 + E\alpha^2 + F\alpha + H^i}{I\alpha^5 + f\alpha^4 + K\alpha^3 + L\alpha^2 + M\alpha + N} + \frac{\tilde{A}\tilde{\alpha}^2 + \tilde{B}^i}{2\tilde{b}^2\tilde{\beta}} = H^i_{00}.$$  \hspace{1cm} (3.9)
where

\[ A^i := -2q(1 - q^2b^2 + q^2b^2) \left[ (1 - q^2b^2 + q^2b^2)s_0^i - q(q - 1)s_0b^i \right], \]

\[ B^i := q \left[ 2(p - 1)\lambda(1 + q^2b^2)s_0y^i - 2q(q - 1)(q - 2)\beta s_0b^i \right. \]
\[ \left. + 4\beta(q - 2)(q^2b^2 - q^2b + 1) s_0^i + 2\lambda(q - 1)(q^2b^2 - q^2b + 1) r_0y^i \right] - (q - 1)(q^2b^2 - q^2b + 1)r_0b^i, \]

\[ C^i := q \left[ (q - 1)(q^2b^2 - 2q^2b^2 + q^2b + 2q - 3)\beta r_0b^i + \lambda(q - 1)(q - 2)b^2r_0y^i \right. \]
\[ - 2(q - 1)\lambda(q^3b^2 - 2q^2b^2 + q^2b + 2q - 3)\beta r_0y^i \]
\[ + [4qb^2(q + 1)(q - 1)^2 + (2q^2 + 8q - 12)]\beta s_0^i \]
\[ - 2\lambda(q - 1)(3q^3b^2 - 2b^2q^2 + 2q - b^2q - 3)\beta s_0y^i \]
\[ - 2q(q + 1)(q - 1)^2b^2s_0b^i, \]

\[ D^i := q(q - 1)\beta \left[ (2q^2 - 10q + 6)\lambda\beta s_0y^i - 4(q + 1)(q - 2)\beta^2s_0^i \right. \]
\[ - (q + 4)(q - 1)b^2r_0y^i + 3(q - 1)\beta r_0b^i, \]

\[ E^i := -q(q - 1)\beta^2 \left[ 2(q - 1)(q + 1)\beta^2s_0^i + \lambda [2b^2(q + 1)(q - 1)^2 + (q - 2)] r_0y^i \right. \]
\[ - 2\lambda(q - 1)(3q - 1)(q + 1)\beta s_0y^i - 2\lambda q + 1)(q - 1)^2r_0y^i + (q + 1)(q - 1)^2\beta r_0b^i \right], \]

\[ F^i := -\lambda q(q + 4)(q - 1)^2\beta^3r_0y^i, \]

\[ H^i := 2\lambda q(q + 1)(p - 1)^3\beta^4r_0y^i. \]

and

\[ I := -2(-q^2b^2 + q^2b^2 + 1)^2, \]
\[ J := 2\beta(-q^2b^2 + q^2b^2 + 1)(qb^2 - 2q^2b^2 + q^2b^2 + 3q - 5), \]
\[ K := 2\beta^2(-10 - q^2 + 10q + 6q^2b^2 - 12q^2b^2 + 6q^2b^2), \]
\[ L := -2(q - 1)\beta^3(2q^4b^2 - 2q^3b^2 + 3q^2 - 2q^2b^2 + 2q + 2qb^2 - 10), \]
\[ M := 2\beta^4(q + 1)(q - 5)(q - 1)^2, \]
\[ N := 2\beta^5(q + 1)^2(q - 1)^3. \]

and

\[ \bar{A}^i := \bar{b}^2s_0^i - \bar{b}is_0, \quad \bar{B}^i := \bar{\beta}[2\lambda(y^i(r_0 + s_0) - \bar{b}i\bar{r}_0)]. \]

\[ (3.9) \] is equivalent to following

\[
2\hat{\beta}^2\bar{\beta}(A'a^6 + B'a^5 + C'a^4 + D'a^3 + E'a^2 + F'a + H')
\]
\[ + (\bar{A}'a^2 + \bar{B}')(Ia^5 + Ja^4 + Ka^3 + La^2 + Ma + N)
\]
\[ = 2\hat{\beta}^2\bar{\beta}(Ia^5 + Ja^4 + Ka^3 + La^2 + Ma + N)H_0. \]

(3.10)

First we show that \( \bar{A}^i \) can be divide by \( \bar{\beta} \).
By replacing \( y' \) with \(-y'\) in (3.10), we get the following
\[
-2b^2 \bar{\beta} (-A' \alpha^6 + B' \alpha^5 - C' \alpha^4 + D' \alpha^3 - E' \alpha^2 + F' \alpha - H')
- (\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^5 - J \alpha^4 + K \alpha^3 - L \alpha^2 + M \alpha - N)
= -2b^2 \bar{\beta} (I \alpha^5 - J \alpha^4 + K \alpha^3 - L \alpha^2 + M \alpha - N) H_{00}.
\]
(3.10) + (3.11) yields
\[
2b^2 \bar{\beta} (A' \alpha^6 + C' \alpha^4 + E' \alpha^2 + H') + (\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + L \alpha^2 + N)
= 2b^2 \bar{\beta} (I \alpha^4 + L \alpha^2 + N) H_{00}.
\]
(3.10) - (3.11) implies that
\[
(B' \alpha^4 + D' \alpha^2 + F') (2b^2 \bar{\beta}) + (\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + K \alpha^2 + M)
= 2b^2 \bar{\beta} (I \alpha^4 + K \alpha^2 + M) H_{00}.
\]
If \( q = -1 \) then \( H^i = N = M = 0 \). Thus (3.12) and (3.13) are equivalent to
\[
2b^2 \bar{\beta} (A' \alpha^6 + C' \alpha^4 + E') + (\bar{A}' \bar{\alpha}^2 + \bar{B}') (J \alpha^2 + L) = 2b^2 \bar{\beta} (J \alpha^2 + L) H_{00}
\]
and
\[
2b^2 \bar{\beta} (B' \alpha^4 + D' \alpha^2 + F') + (\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + K \alpha^2) = 2b^2 \bar{\beta} (I \alpha^4 + K \alpha^2) H_{00}.
\]
By (3.14) and (3.15), it results that \((\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + L)\) and \((\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + K \alpha^2)\) can be divided by \( \bar{\beta} \). Thus \( \bar{\beta} = \mu \bar{\beta} \) and \( \bar{A}' \bar{\alpha}^2 \bar{\alpha}^4 \) can be divided by \( \bar{\beta} \). Since \( \bar{\beta} \) is prime with respect to \( \alpha \) and \( \bar{\alpha} \), therefore \( \bar{A}' := \bar{\beta}^2 \bar{s}^0_0 - \bar{\beta} \bar{s}_0 \) can be divided by \( \bar{\beta} \). If \( q \neq 1, -1 \), then (3.12) and (3.13) implies that \((\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + L \alpha^2 + N)\) and \((\bar{A}' \bar{\alpha}^2 + \bar{B}') (I \alpha^4 + K \alpha^2 + M)\) can be divided by \( \bar{\beta} \). Since \( \bar{\beta} \) is prime with respect to \( \alpha \) and \( \bar{\alpha} \), then \( \bar{A}' := \bar{\beta}^2 \bar{s}^0_0 - \bar{\beta} \bar{s}_0 \) can be divided by \( \bar{\beta} \). Hence, there is a scaler function \( \psi^i(x) \) such that
\[
\bar{\beta}^2 \bar{s}^0_0 - \bar{\beta} \bar{s}_0 = \psi^i \bar{\beta}.
\]
Contracting (3.16) with \( \bar{g}_i := a_{ij}y^j \) yields
\[
\psi^i(x) = -\bar{s}^i.
\]
Then we have
\[
\bar{s}_{ij} = \frac{1}{\bar{\beta}^2} (\bar{b}_{ij} \bar{s}_j - \bar{b}_j \bar{s}_i).
\]
Now, suppose that \( n \geq 3 \). Then by Lemma 1 \( \bar{F} = \bar{\alpha}^2 / \bar{\beta} \) is a Douglas metric. Since \( F \) and \( \bar{F} \) have the same Douglas tensor, then both of them are Douglas metrics.

If \( n = 2 \), then \( \bar{F} = \bar{\alpha}^2 / \bar{\beta} \) is a Douglas metric by Lemma 1. Thus \( F \) and \( \bar{F} \) having the same Douglas tensors. This means that they are all Douglas metrics. This completes the proof of Lemma 2.

On the other hand, the following holds.

**Lemma 3.** \( \text{[4]} \) Suppose that \( Q / s \neq \text{constant} \) for an \((\alpha, \beta)\)-metric \( F = \phi(s) \) on a manifold \( M \) of dimension \( n \geq 3 \). If \( F \) is a Douglas metric and \( b := \| \beta_x \|_s \neq 0 \), then \( \beta \) is closed.
Now, we are in the position to prove Theorem 1.

**Proof of Theorem 1** We prove the theorem in two cases, as follows.

**Case (1):** \( q = -1 \).
First we prove the necessity. If \( F \) is projectively equivalent to \( \bar{F} \), then they have the same Douglas tensor. By Lemma 2, \( F \) and \( \bar{F} \) are both Douglas metrics. \( F = \frac{\alpha^2}{\alpha + \beta} \) is a Douglas metric if and only if \( b_{ij} = 0 \). Thus by (2.2), we have

\[ G^i = G^i_\alpha. \tag{3.18} \]

On the other hand, plugging (3.2) and (3.17) in (2.2) yields

\[ \bar{G}^i = \bar{C}^i - \frac{1}{2b^2} \left[ -\bar{\alpha}^2 \bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00} \bar{b}^i) + 2\bar{\theta}_{00} \bar{\beta} y^i \right]. \tag{3.19} \]

By the projective equivalence of \( F \) and \( \bar{F} \) again, there is a scalar function \( P = P(x, y) \) on \( TM_0 \) such that \( \bar{G}^i = \bar{G}^i + Py^i \). From (3.18) and (3.19) we have

\[ [P - \frac{1}{b^2}(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2})]y^i = G^i_\alpha - G^i_\bar{\alpha} - \frac{1}{2b^2}(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i). \tag{3.20} \]

Note that the right side of (3.20) is a quadratic in \( y \). Then there exists a 1-form \( \theta = \theta_i(x)y^i \) on \( M \) such that

\[ P - \frac{1}{b^2}(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2}) = \theta. \tag{3.21} \]

Thus we have

\[ G^i_\alpha = \bar{C}^i + \frac{1}{2b^2}(\bar{\alpha}^2 \bar{s}^i + \bar{r}_{00} \bar{b}^i) + \theta y^i. \tag{3.22} \]

This completes the proof of the necessity.
Conversely, because of \( r_{00} = 0 \) and from (1.1), (3.18) and (3.19) we have

\[ G^i = \bar{C}^i + \left[ \theta + \frac{1}{2b^2}(\bar{s}_0 + \frac{\bar{r}_{00} \bar{\beta}}{\bar{\alpha}^2}) \right]y^i. \tag{3.23} \]

In this case, \( F \) is projectively related to \( \bar{F} \).

**Case (2):** \( q \neq 1, -1 \).
First we prove the necessity. If \( F \) is projectively equivalent to \( \bar{F} \), then they have the same Douglas tensor. By Lemma 2, we know that \( F \) and \( \bar{F} \) are both Douglas metrics. If \( q = 1, -1 \), then it is easy to prove that \( \phi(s) = (1 + s)^q \) satisfies \( Q/s \neq constant \). By Lemma 3, we have \( s_{ij} = 0 \). By (2.2), it follows that

\[ G^i = C^i_\alpha + \frac{1}{2} \frac{q(\alpha - 2(\alpha - 1)\beta) r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha \beta + (1 + q(1 - b^2))\alpha^2 y^i} \]

\[ + \frac{1}{2} \frac{q(q - 1)\alpha^2 r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha \beta + (1 + q(1 - b^2))\alpha^2 b^i}. \tag{3.24} \]

Plugging (3.2) and (3.17) in (2.2) implies that

\[ \bar{G}^i = \bar{C}^i_\bar{\alpha} - \frac{1}{2b^2} \left[ -\bar{\alpha}^2 \bar{s}^i + (2\bar{s}_0 y^i - \bar{r}_{00} \bar{b}^i) + 2\bar{\theta}_{00} \bar{\beta} y^i \right]. \tag{3.25} \]
Corollary 1. By Lemma 1, Lemma 3 and Theorem 1, we have the following.

Thus we get

\[
G^i - \tilde{G}^i + \left[ P - \frac{1}{2b^2}(s_0 + r_{00}b^i) - \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha\beta + [1 + (q^2 - q)b^2]\alpha^2} b^j \right] y^j = \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha\beta + [1 + (q^2 - q)b^2]\alpha^2} b^j - \frac{1}{2b^2}(\alpha^2 b^j + r_{00}b^j). \tag{3.26}
\]

Note that the right side of (3.26) is a quadratic in \( y \). Then there exists a 1-form \( \theta = \theta_i(x)y^i \) on \( M \) such that

\[
P - \frac{1}{2b^2}(s_0 + r_{00}b^i) - \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha\beta + [1 + (q^2 - q)b^2]\alpha^2} = \theta. \tag{3.27}
\]

Thus we get

\[
G^i - \tilde{G}^i = \left[ \theta + \frac{1}{2b^2}(s_0 + r_{00}b^i) + \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha\beta + [1 + (q^2 - q)b^2]\alpha^2} b^j \right] y^j.
\]

This completes the proof of the necessity. Conversely, by (1.1), (3.18) and (3.19) we have

\[
G^i - \tilde{G}^i = \left[ \theta + \frac{1}{2b^2}(s_0 + r_{00}b^i) + \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{(1 - q^2)\beta^2 + (2 - q)\alpha\beta + [1 + (q^2 - q)b^2]\alpha^2} b^j \right] y^j.
\]

Thus \( F \) is projectively equivalent to \( \tilde{F} \). This completes the proof.

By Lemma 1, Lemma 2, and Theorem 1, we have the following.

Corollary 1. Let \( F = \frac{(\alpha + \beta)^2}{\alpha^2} \) (\( q \neq 1 \)) be a \((q, \alpha, \beta)\) metric and \( \tilde{F} = \tilde{\alpha}^2 / \tilde{\beta} \) be a Kropina metric on a \( n \)-dimensional manifold \( M \) \((n \geq 3)\), where \( \alpha \) and \( \tilde{\alpha} \) are two Riemannian metrics, \( \beta \) and \( \tilde{\beta} \) are two nonzero collinear 1-forms on \( M \). Then \( F \) is projectively equivalent to \( \tilde{F} \) if and only if

\[
G^i = \tilde{G}^i + \left[ \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{\alpha^2} b^j \right] y^j = \tilde{G}^i + \left[ \frac{1}{2} \frac{q(\alpha - 2(q - 1)\beta)r_{00}}{\alpha^2} b^j \right] y^j,
\]

\[
s_{ij} = 0,
\]

\[
s_{ij} = \frac{1}{b^2} \{ b_i s_j - b_j s_i \}.
\]

where \( b_{ij} \) denote the coefficients of the covariant derivatives of \( \beta \) with respect to \( \alpha \).

It is well known that the Berwald metric \( F = \frac{(\alpha + \beta)^2}{\alpha^2} \) on a manifold \( M \) is a Douglas metric if and only if

\[
b_{ij} = 2\tau[(1 + 2b^2)a_{ij} - 3b_i b_j], \tag{3.28}
\]

where \( \tau = \tau(x) \) is a scalar function on \( M \). Thus by (3.28) and Theorem 1, we get the following.
Corollary 2. Let \( F = \frac{(a + \beta)}{\tilde{a}} \) be a Berwald metric and \( \tilde{F} = \bar{a}^2 / \tilde{\beta} \) be a Kropina metric on a \( n \)-dimensional manifold \( M \) (\( n \geq 3 \)), where \( a \) and \( \tilde{a} \) are two Riemannian metrics, \( \beta \) and \( \tilde{\beta} \) are two non-zero collinear 1-forms on \( M \). Then \( F \) is projectively related to \( \tilde{F} \) if and only if they are Douglas metrics and the following holds

\[
C_{\tilde{a}} - C_{\bar{a}} = \theta y^i + \frac{1}{2b^2}(\tilde{a}^2 s^i + \bar{r}_{00}\tilde{b}' - 2\tau a^2 b'),
\]

where \( \tau = \tau(x) \) is a scalar function on \( M \).

4. PROOF OF THEOREM 2

In this section, we are going to prove the Theorem 2. More precisely, we find the conditions that an \((q, a, \beta)\)-metric \( F = \frac{\beta^i}{(\beta - a)^{s-1}} \) being projectively equivalent to a Kropina metric. For the \((q, a, \beta)\)-metric \( F = \frac{\beta^i}{(\beta - a)^{s-1}} \), the following are hold

\[
Q = \frac{s - q}{(q - 1)s}, \quad \Psi = \frac{q}{2[s^2(s - 1) + q(b^2 - s^2)]}, \quad \Theta = \frac{s(s - 2q)}{2[s^2(s - 1) + q(b^2 - s^2)]}. (4.1)
\]

First we prove the following.

Lemma 4. Let \( F = \frac{\beta^i}{(\beta - a)^{s-1}} \) be an \((q, a, \beta)\)-metric and \( \tilde{F} = \bar{a}^2 / \tilde{\beta} \) be a Kropina metric on a \( n \)-dimensional manifold \( M \) (\( n \geq 3 \)) where \( a \) and \( \tilde{a} \) are two Riemannian metrics and \( \beta \) and \( \tilde{\beta} \) are two non-zero collinear 1-forms. Then \( F \) and \( \tilde{F} \) have the same Douglas tensor if and only if they are all Douglas metrics.

Proof. The sufficiency is obvious. Suppose that \( F \) and \( \tilde{F} \) have the same Douglas tensor on a manifold \( M \) of dimension \( n \geq 3 \). Then \((3.8)\) holds. By plugging \((3.2)\) and \((4.1)\) into \((3.8)\), we obtain

\[
\frac{\sum_{i=1}^{8} A_{i}^j a^j}{\sum_{j=1}^{6} B_{i} a^j} + \frac{\tilde{A}_{i}^j \tilde{a}^j + \tilde{B}_{i}^j}{2b^2 \tilde{\beta}} = H_{00}, (4.2)
\]

where

\[
A_{1}^j = 2\beta^2 s^j_0 - 3\lambda q(q - 1)\beta^5 r_{00}y^j,
A_{2}^j = 6\lambda \beta^2 s^j_0 - 2(3q + 2)\beta^6 s^j_0 + 2q(q^2 - 1)\lambda \beta^4 r_{00}y^j,
A_{3}^j = 2((q + 1)^2 + 2q(q + 1))\beta^3 s^j_0 - 2q(6q + 1)\lambda \beta^3 s^j_0 + q(q - 1)\beta^4 r_{00}b^j,
A_{4}^j = q(q + 1)(3q + 1) - 6q^2 \lambda^3 s^j_0 y^j + 2q(q^2 - 1)\lambda \beta^2 r_{00} - b^2 r_{00} y^j,
A_{5}^j = 2q(q + 1)(3q + 1) - 6q^2 \lambda^3 s^j_0 y^j + 2q(q^2 - 1)\lambda \beta^2 r_{00} - b^2 r_{00} y^j,
A_{6}^j = 2q^2(q + 1)\beta^2 [2b^2 s^j_0 - s^j_0 b^j] - q^2(3q + 1)\lambda b^2 \beta s^j_0 y^j,
A_{7}^j = q^2(q + 1)\lambda b^2 \beta [2b^2 y^j - r_{00} b^j],
A_{8}^j = -2q^3 b^2(2b^2 s^j_0 - s^j_0 b^j),
A_{9}^j = -2q^3 b^2(2b^2 s^j_0 - s^j_0 b^j). (4.3)
\]
and

\[
B_0 = 2(q - 1)\beta^7, \\
B_1 = -4(q - 1)(q + 1)\beta^6, \\
B_2 = 2(q - 1)(q + 1)^2\beta^5, \\
B_3 = 4q(q - 1)b^2\beta^4, \\
B_4 = -4q(q^2 - 1)b^2\beta^3, \\
B_5 = 0, \\
B_6 = 2q^2(q - 1)b^4\beta, 
\]

(4.4)

and

\[
\bar{A}^i := B^2s^i_0 - \bar{B}s_0, 
\bar{B}^i := \beta[2\lambda y^i(r_0 + s_0) - B^i\bar{r}_{00}].
\]

(4.2) is equivalent to

\[
\left( \sum_{j=1}^{8} A^i_{(j)} \alpha^{(j)} \right) \left( 2\bar{B}^2 \bar{B} \right) + \left( \bar{A}^i \bar{\alpha}^2 + \bar{B}^i \right) \left( \sum_{j=0}^{6} B_j \alpha^j \right) = \left( 2\bar{B}^2 \bar{\beta} \right) \left( \sum_{j=0}^{6} B_j \alpha^j \right) H_{00}. 
\]

(4.5)

By replacing \( y' \) with \(-y'\) in (4.5), we get

\[
\left( \sum_{j=0}^{3} A^i_{(2j+1)} \alpha^{(2j+1)} \right) - \sum_{j=1}^{4} A^i_{(2j)} \alpha^{(2j)} \left( -2\bar{B}^2 \bar{\beta} \right) \\
- \left( \bar{A}^i \bar{\alpha}^2 + \bar{B}^i \right) \left( \sum_{j=0}^{1} B_{(2j+1)} \alpha^{(2j+1)} - \sum_{j=0}^{3} B_{(2j)} \alpha^{(2j)} \right) = \\
-2\bar{B}^2 \bar{\beta} \left( \sum_{j=0}^{1} B_{(2j+1)} \alpha^{(2j+1)} - \sum_{j=0}^{3} B_{(2j)} \alpha^{(2j)} \right) H_{00}. 
\]

(4.6)

(4.5) \(-\) (4.6) implies that

\[
2\bar{B}^2 \bar{\beta} \left( \sum_{j=1}^{4} A^i_{(2j)} \alpha^{(2j)} \right) + \left( \bar{A}^i \bar{\alpha}^2 + \bar{B}^i \right) \sum_{j=0}^{3} B_{(2j)} \alpha^{(2j)} = 2\bar{B}^2 \bar{\beta} \sum_{j=0}^{3} B_{(2j)} \alpha^{(2j)} H_{00}. 
\]

(4.7)

(4.5) \(+\) (4.6) yields

\[
2\bar{B}^2 \bar{\beta} \sum_{j=0}^{3} A^i_{(2j+1)} \alpha^{(2j+1)} + \left( \bar{A}^i \bar{\alpha}^2 + \bar{B}^i \right) \sum_{j=0}^{1} B_{(2j+1)} \alpha^{(2j+1)} \\
= 2\bar{B}^2 \bar{\beta} \sum_{j=0}^{1} B_{(2j+1)} \alpha^{(2j+1)} H_{00}. 
\]

(4.8)
By Lemma 4, Proof of Theorem 2:

Now, we are in the position to prove Theorem 2.

In the case that $n = 3$, by Lemma 1, $\bar{F} = \bar{\alpha}/\bar{\beta}$ is a Douglas metric. Since $F$ and $\bar{F}$ have the same Douglas tensor, both of them are Douglas metrics.

In the case $n = 2$, by Lemma 1, $\bar{F} = \bar{\alpha}/\bar{\beta}$ is a Douglas metric. Thus $F$ and $\bar{F}$ having the same Douglas tensor means that they are all Douglas metrics. This completes the proof of Lemma 4.

Now, we are in the position to prove Theorem 2.

**Proof of Theorem 2**: First, we proof the necessity. If $F$ is projectively related to $\bar{F}$, then they have the same Douglas tensor. By Lemma 1 $F$ and $\bar{F}$ are both Douglas metrics. If
\( q = 1, -1 \), then it is easy to prove that \( \phi(s) = \frac{s^q}{(s-1)^{q+1}} \) satisfies \( \bar{q} \neq \text{constant} \). By Lemma 3 it results that \( s_{ij} = 0 \). By (2.2), we get
\[
G_i^j = G_{i}^j + \frac{\beta(\beta - 2qa\alpha)r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}y^j + \frac{q\alpha^3r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}b^j. \tag{4.14}
\]
Plugging (3.2) and (4.13) in (2.2) yields
\[
G_i^j = G_{i}^j - \frac{q\alpha^3r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}b^j. \tag{4.15}
\]
By assumption, there is a scalar function \( P = P(x, y) \) on \( TM_0 \) such that \( G_i^j = \bar{G}_i^j + Py^j \). Then by (4.14) and (4.15) we have
\[
[ P - \frac{1}{2}\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha - \frac{1}{2\beta^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) ]y^j = G_{i}^j - \bar{G}_{i}^j - \frac{q\alpha^3r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}b^j. \tag{4.16}
\]
The right side of (4.16) is a quadratic in \( y \). Then there exists a 1-form \( \theta = \theta_i(x)y^j \) on \( M \) such that
\[
P - \frac{1}{2}\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha - \frac{1}{2\beta^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) = \theta. \tag{4.17}
\]
Thus we get
\[
G_i^j + \frac{q\alpha^3r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}b^j = \bar{G}_{i}^j + \frac{1}{2\beta^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) + \theta y^j. \tag{4.18}
\]
This completes the proof of the necessity.

Conversely, from (1.2), (4.14) and (4.15) it follows that
\[
G_i^j = \bar{G}_i^j + [ \theta + \frac{1}{2}\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha - \frac{1}{2\beta^2}(\bar{s}_0 + \bar{r}_{00}\bar{b}^i) ]y^j. \tag{4.19}
\]
Thus \( F \) is projectively equivalent to \( \bar{F} \). This completes the proof.

By Lemmas 1, 3 and Theorem 2 we can conclude the following.

**Corollary 3.** Let \( F = \frac{1}{(\beta - \alpha)^{q+1}} \) \((q \neq 1, -1)\) be a \((q, \alpha, \beta)\)-metric and \( \bar{F} = \bar{\alpha}/\bar{\beta} \) be a Kropina metric on a \( n \)-dimensional manifold \( M \) \((n \geq 3)\) where \( \alpha \) and \( \bar{\alpha} \) are two Riemannian metrics, \( \beta \) and \( \bar{\beta} \) are two nonzero collinear \( 1 \)-forms. Then \( F \) is projectively related to \( \bar{F} \) if and only if the following holds
\[
G_i^j - \bar{G}_{i}^j = \theta y^j + \frac{1}{2\beta^2}(\bar{s}^i + \bar{r}_{00}\bar{b}^i) - \frac{1}{2}\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha \frac{q\alpha^3r_{00}}{2\beta^2(\beta - \alpha) + q(b^2a^2 - \beta^2)\alpha}b^j, \tag{4.20}
\]
\[
s_{ij} = 0, \tag{4.21}
\]
\[
\bar{s}_{ij} = \frac{1}{b^2}(\bar{b}_j\bar{s}_i - \bar{b}_i\bar{s}_j), \tag{4.22}
\]
where \( b_{ij} \) denote the coefficients of the covariant derivatives of \( \beta \) with respect to \( \alpha \).
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