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ON THE FOUR CONCURRENT CIRCLES

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ABSTRACT. In this paper, for general and special quadrilaterals, we study the position of their Euler centres.

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1. INTRODUCTION

Let $PQRS$ be an arbitrary quadrilateral. Note with E_{QRS}, E_{PRS}, E_{PQS}, E_{PQR} the nine-point circles of triangles QRS, PRS, PQS, PQR. It is well know that these four nine-point circles or Euler circles are concurrent in a point E and this point is called the *nine-point or Euler centre of the quadrilateral* (see [1]-[4]). In this paper we will show that this property is not an attribute of the quadrilaterals, but an attribute of the parallelograms.
Let O be the centre of symmetry of the parallelogram $ABCD$, the point M situated in its plan, N the symmetry of M with respect to O. The circles determined by the triplet of points (A, B, M), (B, C, N), (C, D, M), (D, A, N) we will call the *K-circles of the parallelogram ABCD*. In the first part of this paper we demonstrate that these K-circles are concurrent in a point. In the second part we show how this result can be applied for the quadrilaterals. We will consider general and special quadrilaterals too and we will examine the positions of their Euler centres and the geometric locus described by the Euler centres.

2. THE CONCURRENCE OF THE FOUR K-CIRCLES OF A PARALLELOGRAM

Theorem 1. *The K-circles of a parallelogram concur in a point.*

Proof. Let $AB = 2q$ and $BC = 2p$, where $p > 0$, $q > 0$. We attach to the parallelogram $ABCD$ an oblique coordinates system xOy such that its zero point is the point O, the x-axis parallel with the line BC and the y-axis with the line AB (Figure 1). Note the measure of angle ABC with θ. In this system, the coordinates of the points A, B, C, D are:

$$A = (-p, q),\ B = (-p, -q),\ C = (p, -q),\ D = (p, q).$$

If $M = (u, v)$, then $N = (-u, -v)$. Note the circles circumscribed to the triangles ABM, BCN, CDM, DAN with the symbols K_{ABM}, K_{BCN}, K_{CDM}, K_{DAN}.
In oblique coordinates system, the general equation of a circle Γ has the bellow form [5]:

$$x^2 + y^2 + 2xy\cos\theta + lx + my + n = 0. \tag{2.1}$$

Here, we give the conditions such that the circle Γ passes through the points A, B, C, D, M and N:

$$A \in \Gamma \Leftrightarrow p^2 + q^2 - 2pq\cos\theta - pl + qm + n = 0, \tag{2.2}$$

$$B \in \Gamma \Leftrightarrow p^2 + q^2 + 2pq\cos\theta - pl - qm + n = 0, \tag{2.3}$$

$$C \in \Gamma \Leftrightarrow p^2 + q^2 - 2pq\cos\theta + pl - qm + n = 0, \tag{2.4}$$

$$D \in \Gamma \Leftrightarrow p^2 + q^2 + 2pq\cos\theta + pl + qm + n = 0, \tag{2.5}$$

$$M \in \Gamma \Leftrightarrow u^2 + v^2 + 2uv\cos\theta + ul + vm + n = 0, \tag{2.6}$$

$$N \in \Gamma \Leftrightarrow u^2 + v^2 + 2uv\cos\theta - ul - vm + n = 0. \tag{2.7}$$

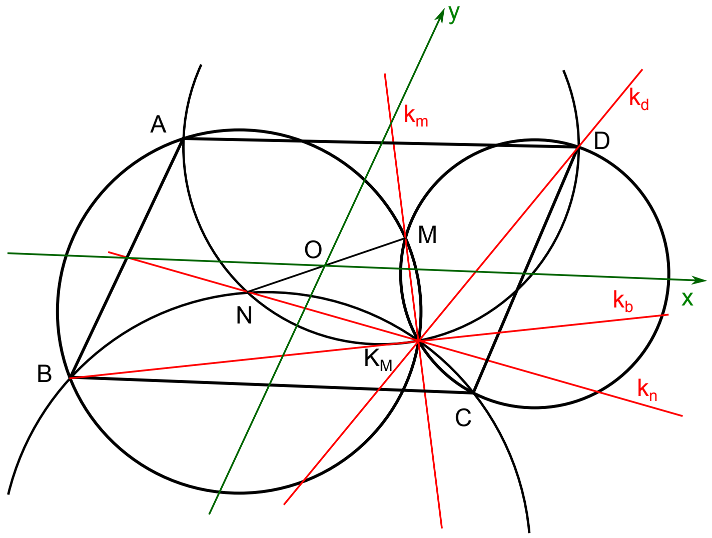

Figure 1

Introduce the following notation: $d = u^2 + v^2 - (p^2 + q^2)$. For example, we obtain the equation of the circle K_{BCN} by resolving the system formed with the equation (2.3), (2.4), (2.7). The equations of the circles K_{ABM}, K_{BCN}, K_{CDM}, K_{DAN} are

$$\begin{aligned} K_{ABM} : x^2 + y^2 + 2xy\cos\theta - \left(\frac{d}{u+p} + 2v\cos\theta\right) x + 2p\cos\theta \cdot y \\ - \frac{dp}{u+p} - (p^2+q^2) - 2vp\cos\theta = 0, \end{aligned} \tag{2.8}$$

$$\begin{aligned} K_{BCN} : x^2 + y^2 + 2xy\cos\theta + 2q\cos\theta \cdot x + \left(\frac{d}{v-q} + 2u\cos\theta\right) y \\ + \frac{dq}{v-q} - (p^2+q^2) + 2uq\cos\theta = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} K_{CDM} : x^2 + y^2 + 2xy\cos\theta - \left(\frac{d}{u-p} + 2v\cos\theta\right) x - 2p\cos\theta \cdot y \\ + \frac{dp}{u-p} - (p^2+q^2) + 2vp\cos\theta = 0, \end{aligned} \tag{2.10}$$

$$K_{DAN}: x^2+y^2+2xy\cos\theta-2q\cos\theta\cdot x+\left(\frac{d}{v+q}+2u\cos\theta\right)y$$
$$-\frac{dq}{v+d}-(p^2+q^2)-2uq\cos\theta=0. \quad (2.11)$$

Through each A, B, C, D, M and N points pass exactly two K-circles. For example, the circles K_{ABM} and K_{DAN} passes point A. Note their radical axis with the symbol k_a, etc. We obtain the equation of the radical axis of two circles by subtracting their equations [5]:

$$k_a: [(v+q)d+2(u+p)(v^2-q^2)\cos\theta]x+[(u+p)d+2(v+q)(u^2-p^2)\cos\theta]y$$
$$-(uq-vp)[d+2(u+p)(v+q)\cos\theta]=0, \quad (2.12)$$

$$k_b: [(v-q)d+2(u+p)(v^2-q^2)\cos\theta]x+[(u+p)d+2(v-q)(u^2-p^2)\cos\theta]y$$
$$+(uq+vp)[d+2(u+p)(v-q)\cos\theta]0, \quad (2.13)$$

$$k_c: [(v-q)d+2(u-p)(v^2-q^2)\cos\theta]x+[(u-p)d+2(v-q)(u^2-p^2)\cos\theta]y$$
$$+(uq-vp)[d+2(u-p)(v-q)\cos\theta]0, \quad (2.14)$$

$$k_d: [(v+q)d+2(u-p)(v^2-q^2)\cos\theta]x+[(u-p)d+2(v+q)(u^2-p^2)\cos\theta]y$$
$$-(uq+vp)[d+2(u-p)(v+q)\cos\theta]=0, \quad (2.15)$$

$$k_m: dx+2(u^2-p^2)\cos\theta\cdot y-ud-2v(u^2-p^2)\cos\theta=0, \quad (2.16)$$

$$k_n: 2(v^2-q^2)\cos\theta\cdot x+dy+vd+2u(v^2-q^2)\cos\theta=0. \quad (2.17)$$

If

$$\Delta=\begin{pmatrix} d & 2(u^2-p^2)\cos\theta \\ 2(v^2-q^2)\cos\theta & d \end{pmatrix}=d^2-4(u^2-p^2)(v^2-q^2)\cos^2\theta\neq 0,$$

then the radical axis k_m and k_n intersect in a point. It is easy to verify that

$$(v+q)k_m+(u+p)k_n-k_a=0,\ (v-q)k_m+(u+p)k_n-k_b=0,$$
$$(v-q)k_m+(u-p)k_n-k_c=0,\ (v+q)k_m+(u-p)k_n-k_d=0.$$

This involves that all radical axis k_a, k_b, k_c, k_d pass through the point of intersection of the radical axis k_m and k_n (Figure 1). Since these radical axis are concurrent, the K-circles are concurrent too.
Let K_M be their point of intersection, named in the following, the *K-circles centre*:

$$K_M=k_a\cap k_b\cap k_c\cap k_d\cap k_m\cap k_n=K_{ABM}\cap K_{BCN}\cap K_{CDM}\cap K_{DAN}.$$

2.1. **The coordinates and the position of the K-circles centre.**

To determine the coordinates x_K and y_K of the K-circles centre, we form a system with the equations of the radical axis k_m and k_n:

$$dx + 2(u^2 - p^2)\cos\theta \cdot y - du - 2v(u^2 - p^2)\cos\theta = 0,$$
$$2(v^2 - q^2)\cos\theta \cdot x + dy + vd + 2u(v^2 - q^2)\cos\theta = 0.$$

Resolving this system with the Cramer rules, we obtain:

$$\Delta = \begin{vmatrix} d & 2(u^2-p^2)\cos\theta \\ 2(v^2-q^2)\cos\theta & d \end{vmatrix} = d^2 - 4(u^2-p^2)(v^2-q^2)\cos^2\theta,$$

$$\Delta_x = \begin{vmatrix} du + 2v(u^2 - p^2)\cos\theta & 2(u^2 - p^2)\cos\theta \\ -vd - 2u(v^2 - q^2)\cos\theta & d \end{vmatrix}$$
$$= ud^2 + 4vd(u^2 - p^2)\cos\theta + 4u(u^2 - p^2)(v^2 - q^2)\cos^2\theta,$$

$$\Delta_y = \begin{vmatrix} d & du + 2v(u^2 - p^2)\cos\theta \\ 2(v^2 - q^2)\cos\theta & -vd - 2u(v^2 - q^2)\cos\theta \end{vmatrix}$$
$$= -vd^2 - 4ud(v^2 - q^2)\cos\theta - 4v(u^2 - p^2)(v^2 - q^2)\cos^2\theta.$$

Consequently, if $\Delta \neq 0$, then

$$x_K = \frac{\Delta_x}{\Delta} = \frac{ud^2 + 4vd(u^2-p^2)\cos\theta + 4u(u^2-p^2)(v^2-q^2)\cos^2\theta}{d^2 - 4(u^2 - p^2)(v^2 - q^2)\cos^2\theta}$$
$$y_K = \frac{\Delta_y}{\Delta} = -\frac{vd^2 + 4ud(v^2-q^2)\cos\theta + 4v(u^2-p^2)(v^2-q^2)\cos^2\theta}{d^2 - 4(u^2 - p^2)(v^2 - q^2)\cos^2\theta}.$$

If $u = 0$ and $v = 0$, then $M \equiv O \equiv N \equiv K_M = (0, 0)$.
If $u = -p$ and $v \neq \pm q$, then $M \in AB$, $M \neq A$, $M \neq B$ and $K_M = (-p, -v - 4u\cos\theta) \in AB$.
If $u \neq \pm p$ and $v = -q$, then $M \in BC$, $M \neq B$, $M \neq C$ and $K_M = (u + 4v\cos\theta, q) \in AD$.
If $u = p$ and $v \neq \pm q$, then $M \in CD$, $M \neq C$, $M \neq D$ and $K_M = (p, -v - 4u\cos\theta) \in CD$.
If $u \neq \pm p$ and $v = q$, then $M \in DA$, $M \neq D$, $M \neq A$ and $K_M = (u + 4v\cos\theta, -q) \in BC$.
In the following we suppose that $u^2 + v^2 \neq 0$, $u \neq \pm p$ and $v \neq \pm q$.

Theorem 2. *If the point M is on the diagonal AC of parallelogram $ABCD$, then K_M is situated on the other diagonal BD of the parallelogram $ABCD$ and conversely.*

Proof. The equation of the diagonals AC and BD are $qx + py = 0$ and $qx - py = 0$. If $M \in AC$, then $qu + pv = 0$. We want to demonstrate that $K_M \in BD$. We have

$$K_M \in BD \Leftrightarrow qx_K - py_K = 0 \Leftrightarrow q\Delta_x - p\Delta_y = 0 \Leftrightarrow v\Delta_x + u\Delta_y = 0$$
$$\Leftrightarrow 4d[v^2(u^2 - p^2) - u^2(v^2 - q^2)]\cos\theta = 0$$
$$\Leftrightarrow 4d(qu - pv)(qu + pv)\cos\theta = 0,$$

which is true.

If $M \in BD$, then $qu - pv = 0$ and

$$\begin{aligned} K_M \in AC &\Leftrightarrow qx_K + py_K = 0 \Leftrightarrow q\Delta_x + p\Delta_y = 0 \Leftrightarrow v\Delta_x + u\Delta_y = 0 \\ &\Leftrightarrow 4d[v^2(u^2 - p^2) - u^2(v^2 - q^2)] \cos\theta = 0 \\ &\Leftrightarrow 4d(qu - pv)(qu + pv) \cos\theta = 0, \end{aligned}$$

which is true, too.

Theorem 3. *Let $L_{ABC}, L_{BCD}, L_{CDA}, L_{DAB}$ be the circles determined by the triplet of points (A, B, C), (B, C, D), (C, D, A), (D, A, B). If $M \in L_{ABC}$, then $K_M \equiv C$. If $M \in L_{BCD}$, then $K_M \equiv D$. If $M \in L_{CDA}$, then $K_M \equiv A$. If $M \in L_{DAB}$, then $K_M \equiv B$.*

Proof. We observe that if the parallelogram $ABCD$ is fix, then the circles L_{ABC}, L_{BCD}, L_{CDA}, L_{DAB} are fix, too. Suppose that $M \in L_{ABC}$. In this case $C \in K_{ABM}$ and $C \in K_{BCN}$, $C \in K_{CDM}$. Consequently, the circles K_{ABM}, K_{BCN}, K_{DAN} are concurrent in the point C.

3. The concurrence of the four Euler circles of a quadrilateral

If the quadrilateral $PQRS$ is given, let A, B, C, D be the midpoints of sides PQ, QR, RS, SP. The quadrilateral $ABCD$ is called the *Varignon parallelogram of PQRS*. We can determine the sides of Varignon parallelogram $ABCD$ and its angle ABC, which we note with θ. Let $AB = 2q$ and $BC = 2p$, where $p > 0$, $q > 0$. Here we bring up the following question: how much does the triplet (p, q, θ) characterize quadrilateral $PQRS$? We point out that if the triplet (p, q, θ) is given and the parallelogram $ABCD$ is fix, then there are an infinite number of quadrilaterals $PQRS$ with the same Varignon parallelogram $ABCD$. The premier question is how can we construct such a quadrilateral $PQRS$?
Briefly, the construction is the following: put an arbitrary point X in the plan of parallelogram $ABCD$. Construct the anticomplementary triangle of the triangle ABX and note its vertices with P, Q, R (let A be the midpoint of PQ, B the midpoint of QR and X the midpoint of PR). The point S will be the symmetric of R with respect to C (Figure 2).

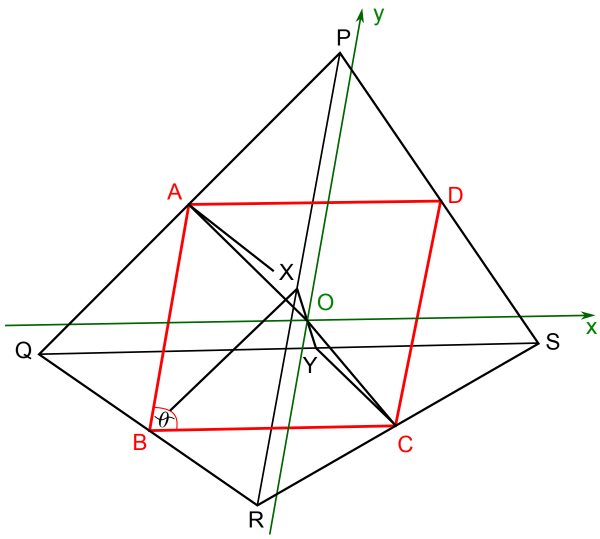

Figure 2

The justification of this construction is the following: let O be the intersection of the diagonals AC and BD, i.e. the *centerpoint of the quadrilateral PQRS* [4]. Note with Y the symmetric of X with respect to O. The triangles AXO and CYO are congruent, consequently $AX = CY$ and $AX \| CY$. The quadrilaterals $PABX$, $AQBX$, $ABRX$, $BCYQ$, $BYCR$ are parallelograms. Let $S = YQ \cap CR$. Since $YQ \| BC$ and $BY \| CR$, the quadrilateral $BCSY$ is a parallelogram too. Consequently $CR = BY = CS$, such that the point C is the midpoint of the segment $[RS]$. Since $YQ = BC = YS$, the point Y is the midpoint of the segment $[QS]$. The quadrilaterals $PXCD$, $BXDY$, $XCSD$ are parallelograms too, from this, that $PD \| XC \| DS$ and $PD = XC = DS$ follows. So, the points P, D, S are collinear and D is the midpoint of the segment $[PS]$.
We attach to the quadrilateral $PQRS$ an oblique coordinates system xOy such that its zero point is the point O, the x-axis parallel with the line BC and the y-axis with the line AB. If $X = (\alpha, \beta)$, then $Y = (-\alpha, -\beta)$ and $P = (\alpha, \beta + 2q)$, $Q = (-\alpha - 2p, -\beta)$, $R = (\alpha, \beta - 2q)$, $S = (-\alpha + 2p, -\beta)$. Let M be an arbitrary point situated in the plan of quadrilateral $PQRS$, N the symmetry of M with respect to O. If $M = (u, v)$, then $N = (-u, -v)$.

Corollary 3.1. *If $u = \alpha$ and $v = \beta$, then $M \equiv X$ and $N \equiv Y$, i.e. the four Euler circles E_{QRS}, E_{PRS}, E_{PQS}, E_{PQR} of the quadrilateral PQRS are special K-circles of its Varignon parallelogram ABCD, consequently the Euler circles are concurrent.*

Note with E_X the point of intersection of the Euler circles, which is called the *Euler centre*:

$$E_X = E_{QRS} \cap E_{PRS} \cap E_{PQS} \cap E_{PQR}.$$

If $\alpha = 0$ and $\beta = 0$, then $X \equiv O \equiv Y \equiv E_X = (0, 0)$ and the quadrilateral $PQRS$ is parallelogram.
If $\alpha = -p$ and $\beta \neq \pm q$, then $X \in AB$, $X \neq A$, $X \neq B$ and $Q \in PR$, $E_X = (-p, -\beta - 4\alpha \cos\theta) \in AB$.
If $\alpha \neq \pm p$ and $\beta = -q$, then $X \in BC$, $X \neq B$, $X \neq C$ and $S \in PQ$, $E_X = (\alpha + 4\beta \cos\theta, q) \in AD$.
If $\alpha = p$ and $\beta \neq \pm q$, then $X \in CD$, $X \neq C$, $X \neq D$ and $S \in PR$, $E_X = (p, -\beta - 4\alpha \cos\theta) \in CD$.
If $\alpha \neq \pm p$ and $\beta = q$, then $X \in DA$, $X \neq D$, $X \neq A$ and $R \in QS$, $E_X = (\alpha + 4\beta \cos\theta, -q) \in BC$.
In the following we suppose that $\alpha^2 + \beta^2 \neq 0$, $\alpha \neq \pm p$ and $\beta \neq \pm q$.

3.1. **The Euler centres of special quadrilaterals.**

a) **Orthodiagonal quadrilaterals**: its diagonals are perpendicular [4], i.e. $\theta = 90^\circ$ (Figure 3).

Corollary 3.2. *The Euler centre of a quadrilateral orthodiagonal coincides with the point of intersection of its diagonals, which have the coordinates $(\alpha, -\beta)$.*

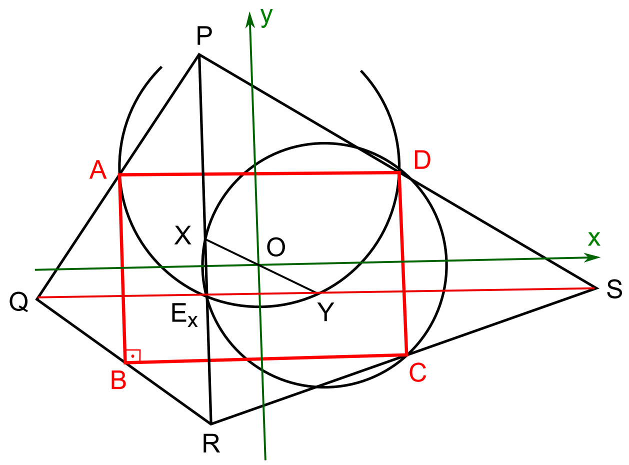

Figure 3

If X describes the circle circumscribed to the rectangle $ABCD$, then the Euler centre describes the same circle. From the correspondence $X \to E_X$, i.e. $(\alpha, \beta) \to (\alpha, -\beta)$ it results that if X describes a line or an circle, then the Euler centre describes the symmetric line or circle with respect to the axis Ox.

b) **Cyclic quadrilaterals**: are inscribed in a circle (see [6]).

Theorem 4. *The quadrilateral $PQRS$ is cyclic if and only if*

$$\alpha^2 - \beta^2 = p^2 - q^2.$$

Proof. Let l and l' be two lines with slopes m and m'. In oblique coordinates systems the condition of perpendicularity of lines l and l' is $m \cdot m' + 1 + (m + m')\cos\theta = 0$ [5]. The slopes of sides PQ, QR, RS, SP are

$$m_{PQ} = \frac{\beta + q}{\alpha + p}, \; m_{QR} = \frac{\beta - q}{\alpha + p}, \; m_{RS} = \frac{\beta - q}{\alpha - p}, \; m_{SP} = \frac{\beta + q}{\alpha - p}.$$

Note the mid-perpendicular of sides PQ, QR, RS, SP with t_a, t_b, t_c, t_d and their slopes with the symbols m_a, m_b, m_c, m_d:

$$m_a = -\frac{1 + m_{PQ}\cos\theta}{m_{PQ} + \cos\theta} = -\frac{\alpha + p + (\beta + q)\cos\theta}{\beta + q + (\alpha + p)\cos\theta},$$

$$m_b = -\frac{1 + m_{QR}\cos\theta}{m_{QR} + \cos\theta} = -\frac{\alpha + p + (\beta - q)\cos\theta}{\beta - q + (\alpha + p)\cos\theta},$$

$$m_c = -\frac{1 + m_{RS}\cos\theta}{m_{RS} + \cos\theta} = -\frac{\alpha - p + (\beta - q)\cos\theta}{\beta - q + (\alpha - p)\cos\theta},$$

$$m_d = -\frac{1 + m_{SP}\cos\theta}{m_{SP} + \cos\theta} = -\frac{\alpha - p + (\beta + q)\cos\theta}{\beta + q + (\alpha - p)\cos\theta}.$$

The equations of mid-perpendiculars are

$$\begin{aligned} t_a : y - q = m_a(x+p) &\Leftrightarrow -m_a \cdot x + y - pm_a - q = 0 \\ &\Leftrightarrow [\alpha + p + (\beta + q)\cos\theta]x + [\beta + q + (\alpha + p)\cos\theta]y \\ &+ [\alpha + p + (\beta + q)\cos\theta]p - [\beta + q + (\alpha + p)\cos\theta]q = 0 \end{aligned} \tag{3.1}$$

$$\begin{aligned} t_b : y + q = m_b(x+p) &\Leftrightarrow -m_b \cdot x + y - pm_b + q = 0 \\ &\Leftrightarrow [\alpha + p + (\beta - q)\cos\theta]x + [\beta - q + (\alpha + p)\cos\theta]y \\ &+ [\alpha + p + (\beta - q)\cos\theta]p + [\beta - q + (\alpha + p)\cos\theta]q = 0 \end{aligned} \tag{3.2}$$

$$\begin{aligned} t_c : y + q = m_c(x-p) &\Leftrightarrow -m_c \cdot x + y + pm_c + q = 0 \\ &\Leftrightarrow [\alpha - p + (\beta - q)\cos\theta]x + [\beta - q + (\alpha - p)\cos\theta]y \\ &- [\alpha - p + (\beta - q)\cos\theta]p + [\beta - q + (\alpha - p)\cos\theta]q = 0 \end{aligned} \tag{3.3}$$

$$\begin{aligned} t_d : y - q = m_d(x-p) &\Leftrightarrow -m_d \cdot x + y + pm_d - q = 0 \\ &\Leftrightarrow [\alpha - p + (\beta + q)\cos\theta]x + [\beta + q + (\alpha - p)\cos\theta]y \\ &- [\alpha - p + (\beta + q)\cos\theta]p - [\beta + q + (\alpha - p)\cos\theta]q = 0. \end{aligned} \tag{3.4}$$

The mid-perpendiculars t_a, t_b, t_c are concurrent if and only if

$$\begin{vmatrix} \alpha+p+(\beta+q)\cos\theta & \beta+q+(\alpha+p)\cos\theta & [\alpha+p+(\beta+q)\cos\theta]p-\dots \\ \alpha+p+(\beta-q)\cos\theta & \beta-q+(\alpha+p)\cos\theta & [\alpha+p+(\beta-q)\cos\theta]p+\dots \\ \alpha-p+(\beta-q)\cos\theta & \beta-q+(\alpha-p)\cos\theta & -[\alpha-p+(\beta-q)\cos\theta]p+\dots \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} \alpha+p+(\beta+q)\cos\theta & \beta+q+(\alpha+p)\cos\theta & 2[\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p+(\beta-q)\cos\theta & \beta-q+(\alpha+p)\cos\theta & 0 \\ \alpha-p+(\beta-q)\cos\theta & \beta-q+(\alpha-p)\cos\theta & 2[\alpha-p+(\beta-q)\cos\theta]p \end{vmatrix} = 0$$

$$\Leftrightarrow (1-\cos^2\theta) \begin{vmatrix} \alpha+p & \beta+q+(\alpha+p)\cos\theta & [\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p & \beta-q+(\alpha+p)\cos\theta & 0 \\ \alpha-p & \beta-q+(\alpha-p)\cos\theta & [\alpha-p+(\beta-q)\cos\theta]p \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 0 & 2q & [\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p & \beta-q+(\alpha+p)\cos\theta & 0 \\ \alpha-p & \beta-q+(\alpha-p)\cos\theta & [\alpha-p+(\beta-q)\cos\theta]p \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 0 & 2q & [\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p & \beta-q & 0 \\ \alpha-p & \beta-q & [\alpha-p+(\beta-q)\cos\theta]p \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{vmatrix} 0 & 2q & [\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p & \beta-q & 0 \\ -2p & 0 & [\alpha-p+(\beta-q)\cos\theta]p \end{vmatrix} = 0$$

$$\Leftrightarrow pq(\beta-q)[\beta+q+(\alpha+p)\cos\theta] - pq(\alpha+p)[\alpha-p+(\beta-q)\cos\theta] = 0$$

$$\Leftrightarrow \alpha^2 - \beta^2 = p^2 - q^2.$$

In case of the cyclic quadrilaterals $d = 2(\alpha^2 - p^2) = 2(\beta^2 - q^2)$ and the coordinates (x_E, y_E) of the Euler centre are

$$x_E = \frac{\alpha + 2\beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta},\ y_E = -\frac{\beta + 2\alpha\cos\theta + \beta\cos^2\theta}{\sin^2\theta}.$$

Let U be the centre of the circle circumscribed to the quadrilateral $PQRS$. To determine the coordinates x_U and y_U of the centre U, we form a system with the equation of the mid-perpendiculars t_a and t_b:

$$[\alpha + p + (\beta + q)\cos\theta]x + [\beta + q + (\alpha + p)\cos\theta]y$$
$$+[\alpha + p + (\beta + q)\cos\theta]p - [\beta + q + (\alpha + p)\cos\theta]q = 0$$

$$[\alpha + p + (\beta - q)\cos\theta]x + [\beta - q + (\alpha + p)\cos\theta]y$$
$$+[\alpha + p + (\beta - q)\cos\theta]p + [\beta - q + (\alpha + p)\cos\theta]q = 0.$$

By resolving this system with the Cramer rules, we obtain:

$$\Delta = \begin{vmatrix} \alpha+p+(\beta+q)\cos\theta & \beta+q+(\alpha+p)\cos\theta \\ \alpha+p+(\beta-q)\cos\theta & \beta-q+(\alpha+p)\cos\theta \end{vmatrix} = -2q(\alpha+p)(1-\cos^2\theta),$$

$$\Delta_x = \begin{vmatrix} -[\alpha+p+(\beta+q)\cos\theta]p+[\beta+q+(\alpha+p)\cos\theta]q & \beta+q+(\alpha+p)\cos\theta \\ -[\alpha+p+(\beta-q)\cos\theta]p-[\beta-q+(\alpha+p)\cos\theta]q & \beta-q+(\alpha+p)\cos\theta \end{vmatrix}$$
$$= 2q(\alpha + p)(\alpha + 2\beta\cos\theta + \alpha\cos^2\theta)$$

$$\Delta_y = \begin{vmatrix} \alpha+p+(\beta+q)\cos\theta & -[\alpha+p+(\beta+q)\cos\theta]p+[\beta+q+(\alpha+p)\cos\theta]q \\ \alpha+p+(\beta-q)\cos\theta & -[\alpha+p+(\beta-q)\cos\theta]p-[\beta-q+(\alpha+p)\cos\theta]q \end{vmatrix}$$
$$= -2q(\alpha + p)(\beta + 2\alpha\cos\theta + \beta\cos^2\theta).$$

Consequently,

$$x_U = \frac{\Delta_x}{\Delta} = -\frac{\alpha + 2\beta\cos\theta + \alpha\cos^2\theta}{\sin^2\theta}$$
$$y_U = \frac{\Delta_y}{\Delta} = \frac{\beta + 2\alpha\cos\theta + \beta\cos^2\theta}{\sin^2\theta}.$$

We observe that points E_X and U are symmetric with respect to O, the centerpoint of the quadrilateral $PQRS$.
It is well-known that the perpendiculars from the midpoints of the sides of a cyclic quadrilateral to the opposite sides (called the *maltitudes* (see [1], [2], or [6]) are concurrent in a point L, called the *Mathot point or anticenter* of the cyclic quadrilateral (see [1], [2], or [6]).

Theorem 5. *The Euler centre and the anticenter of a cyclic quadrilateral coincide.*

Proof. Note with s_a, s_b, s_c, s_d the maltitudes of the cyclic quadrilateral $PQRS$, and let $s_a \| t_a$, $s_b \| t_b$, $s_c \| t_c$, $s_d \| t_d$ ($C \in s_a$, $D \in s_b$, $A \in s_c$, $B \in s_d$) (Figure 4).

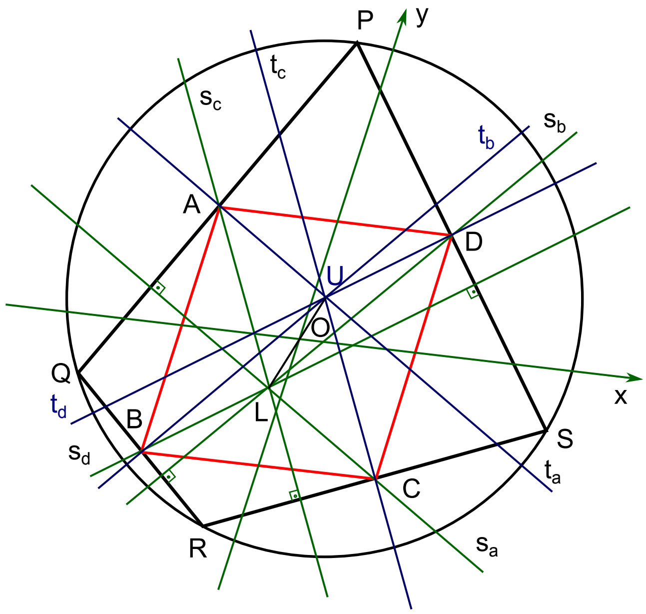

Figure 4

The equations of the maltitudes are the following:

$$\begin{aligned} s_a : y+q = m_a(x-p) &\Leftrightarrow -m_a \cdot x + y + pm_a + q = 0 \\ &\Leftrightarrow [\alpha+p+(\beta+q)\cos\theta]x + [\beta+q+(\alpha+p)\cos\theta]y \\ &- [\alpha+p+(\beta+q)\cos\theta]p - [\beta+q+(\alpha+p)\cos\theta]q = 0 \end{aligned} \tag{3.5}$$

$$\begin{aligned} s_b : y-q = m_b(x-p) &\Leftrightarrow -m_b \cdot x + y + pm_b - q = 0 \\ &\Leftrightarrow [\alpha+p+(\beta-q)\cos\theta]x + [\beta-q+(\alpha+p)\cos\theta]y \\ &- [\alpha+p+(\beta-q)\cos\theta]p - [\beta-q+(\alpha+p)\cos\theta]q = 0 \end{aligned} \tag{3.6}$$

$$\begin{aligned} s_c : y-q = m_c(x+p) &\Leftrightarrow -m_c \cdot x + y - pm_c - q = 0 \\ &\Leftrightarrow [\alpha-p+(\beta-q)\cos\theta]x + [\beta-q+(\alpha-p)\cos\theta]y \\ &+ [\alpha-p+(\beta-q)\cos\theta]p - [\beta-q+(\alpha-p)\cos\theta]q = 0 \end{aligned} \tag{3.7}$$

$$\begin{aligned} s_d : y+q = m_d(x+p) &\Leftrightarrow -m_d \cdot x + y - pm_d + q = 0 \\ &\Leftrightarrow [\alpha-p+(\beta+q)\cos\theta]x + [\beta+q+(\alpha-p)\cos\theta]y \\ &+ [\alpha-p+(\beta+q)\cos\theta]p + [\beta+q+(\alpha-p)\cos\theta]q = 0. \end{aligned} \tag{3.8}$$

These equations are the same as the equations of mid-perpendiculars, except for the free terms, which have opposite signs. Consequently, the coordinates of the anticenter L are

$$x_L = \frac{\alpha+2\beta\cos\theta+\alpha\cos^2\theta}{\sin^2\theta} = x_E, y_L = -\frac{\beta+2\alpha\cos\theta+\beta\cos^2\theta}{\sin^2\theta} = y_E\,.$$

Remark 1. The equation of the line perpendicular to the line $lx + my + n = 0$ and which passes through the point (x', y') is

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ l - m\cos\theta & m - l\cos\theta & 0 \end{vmatrix} = 0.$$

Consequently, the equation of the line s_x perpendicular to the diagonal QS with equation $y + \beta = 0$ and which passes through the point $X = (\alpha, \beta)$ is

$$\begin{vmatrix} x & y & 1 \\ \alpha & \beta & 1 \\ -\cos\theta & 1 & 0 \end{vmatrix} = 0 \Leftrightarrow x + \cos\theta \cdot y = \alpha + \beta\cos\theta.$$

Similarly, we obtain the equation of the line s_y perpendicular to the diagonal PR with equation $x - \alpha = 0$, and which passes through the point $Y = (-\alpha, -\beta)$:

$$\begin{vmatrix} x & y & 1 \\ -\alpha & -\beta & 1 \\ 1 & -\cos\theta & 0 \end{vmatrix} = 0 \Leftrightarrow \cos\theta \cdot x + y = -(\alpha\cos\theta + \beta).$$

By resolving the system formed with the equations of the lines s_x and s_y, we obtain the coordinates of the anticenter. So, the lines s_x and s_y pass through the anticenter of the quadrilateral $PQRS$. If the point Z is the intersection of the diagonals PR and QS, then the orthocenter of the triangle XYZ is the anticenter L of the quadrilateral.

4. Further research

With the computer programs it is possible to investigate another geometrical locus of the Euler centre. For example what describe the point E_X, if the point X describe a line, a circle, an conics, the incircle of the triangle ABC, a line remarkable of the triangle ABC etc.

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