ON THE FOUR CONCURRENT CIRCLES

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ABSTRACT. In this paper, for general and special quadrilaterals, we study the position of their Euler centres.

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1. INTRODUCTION

Let $PQRS$ be an arbitrary quadrilateral. Note with $E_{QRS}, E_{PRS}, E_{PQS}, E_{PQR}$ the nine-point circles of triangles $QRS$, $PRS$, $PQS$, $PQR$. It is well known that these four nine-point circles or Euler circles are concurrent in a point $E$ and this point is called the nine-point or Euler centre of the quadrilateral (see [1]-[4]). In this paper we will show that this property is not an attribute of the quadrilaterals, but an attribute of the parallelograms.

Let $O$ be the centre of symmetry of the parallelogram $ABCD$, the point $M$ situated in its plan, $N$ the symmetry of $M$ with respect to $O$. The circles determined by the triplet of points $(A, B, M)$, $(B, C, N)$, $(C, D, M)$, $(D, A, N)$ we will call the $K$-circles of the parallelogram $ABCD$. In the first part of this paper we demonstrate that these $K$-circles are concurrent in a point. In the second part we show how this result can be applied for the quadrilaterals. We will consider general and special quadrilaterals too and we will examine the positions of their Euler centres and the geometric locus described by the Euler centres.

2. THE CONCURRENCE OF THE FOUR $K$-CIRCLES OF A PARALLELOGRAM

Theorem 1. The $K$-circles of a parallelogram concur in a point.

Proof. Let $AB = 2q$ and $BC = 2p$, where $p > 0, q > 0$. We attach to the parallelogram $ABCD$ an oblique coordinates system $xOy$ such that its zero point is the point $O$, the $x$-axis parallel with the line $BC$ and the $y$-axis with the line $AB$ (Figure 1). Note the measure of angle $ABC$ with $\theta$. In this system, the coordinates of the points $A, B, C, D$ are:

$$A = (-p, q), \ B = (-p, -q), \ C = (p, -q), \ D = (p, q).$$

If $M = (u, v)$, then $N = (-u, -v)$. Note the circles circumscribed to the triangles $ABM, BCN, CDM, DAN$ with the symbols $K_{ABM}, K_{BCN}, K_{CDM}, K_{DAN}$.

In oblique coordinates system, the general equation of a circle $\Gamma$ has the bellow form [5]:

$$x^2 + y^2 + 2xy \cos \theta + lx + my + n = 0. \quad (2.1)$$
Here, we give the conditions such that the circle $\Gamma$ passes through the points $A, B, C, D, M$ and $N$:

- $A \in \Gamma \iff p^2 + q^2 - 2pq \cos \theta - pl + qm + n = 0$, \hspace{1cm} (2.2)
- $B \in \Gamma \iff p^2 + q^2 + 2pq \cos \theta - pl - qm + n = 0$, \hspace{1cm} (2.3)
- $C \in \Gamma \iff p^2 + q^2 - 2pq \cos \theta + pl - qm + n = 0$, \hspace{1cm} (2.4)
- $D \in \Gamma \iff p^2 + q^2 + 2pq \cos \theta + pl + qm + n = 0$, \hspace{1cm} (2.5)
- $M \in \Gamma \iff u^2 + v^2 + 2uv \cos \theta + ul + vm + n = 0$, \hspace{1cm} (2.6)
- $N \in \Gamma \iff u^2 + v^2 + 2uv \cos \theta - ul - vm + n = 0$, \hspace{1cm} (2.7)

Introduce the following notation: $d = u^2 + v^2 - (p^2 + q^2)$. For example, we obtain the equation of the circle $K_{BCN}$ by resolving the system formed with the equation (2.3), (2.4), (2.7). The equations of the circles $K_{ABM}$, $K_{BCN}$, $K_{CDM}$, $K_{DAN}$ are

- $K_{ABM} : x^2 + y^2 + 2xy \cos \theta - \left( \frac{d}{u+p} + 2v \cos \theta \right) x + 2p \cos \theta \cdot y - \frac{dp}{u+p} - (p^2 + q^2) - 2vp \cos \theta = 0$, \hspace{1cm} (2.8)
- $K_{BCN} : x^2 + y^2 + 2xy \cos \theta + 2q \cos \theta \cdot x + \left( \frac{d}{v-q} + 2u \cos \theta \right) y + \frac{dq}{v-q} - (p^2 + q^2) + 2uv \cos \theta = 0$, \hspace{1cm} (2.9)
- $K_{CDM} : x^2 + y^2 + 2xy \cos \theta - \left( \frac{d}{u-p} + 2v \cos \theta \right) x - 2p \cos \theta \cdot y + \frac{dp}{u-p} - (p^2 + q^2) + 2vp \cos \theta = 0$, \hspace{1cm} (2.10)
On the Four Concurrent Circles

\[ K_{\text{DAN}} : x^2 + y^2 + 2xy \cos \theta - 2q \cos \theta \cdot x + \left( \frac{d}{v + q} + 2u \cos \theta \right) y - \frac{dq}{v + d} - (p^2 + q^2) - 2uq \cos \theta = 0. \] (2.11)

Through each \( A, B, C, D, M \) and \( N \) points pass exactly two \( K \)-circles. For example, the circles \( K_{ABM} \) and \( K_{DAN} \) passes point \( A \). Note their radical axis with the symbol \( k_a \), etc.

We obtain the equation of the radical axis of two circles by subtracting their equations [5]:

\[ k_a : [(v+q)d + 2(u+p)(v^2 - q^2) \cos \theta]x + [(u+p)d + 2(v+q)(u^2 - p^2) \cos \theta]y - (uq - vp)[d + 2(u + p)(v + q) \cos \theta] = 0, \] (2.12)

\[ k_b : [(v-q)d + 2(u+p)(v^2 - q^2) \cos \theta]x + [(u+p)d + 2(v-q)(u^2 - p^2) \cos \theta]y + (uq + vp)[d + 2(u + p)(v - q) \cos \theta] = 0, \] (2.13)

\[ k_c : [(v-q)d + 2(u-p)(v^2 - q^2) \cos \theta]x + [(u-p)d + 2(v-q)(u^2 - p^2) \cos \theta]y + (uq - vp)[d + 2(u - p)(v - q) \cos \theta] = 0, \] (2.14)

\[ k_d : [(v+q)d + 2(u-p)(v^2 - q^2) \cos \theta]x + [(u-p)d + 2(v+q)(u^2 - p^2) \cos \theta]y - (uq + vp)[d + 2(u - p)(v + q) \cos \theta] = 0, \] (2.15)

\[ k_m : dx + 2(u^2 - p^2) \cos \theta \cdot y - ud - 2v(u^2 - p^2) \cos \theta = 0, \] (2.16)

\[ k_n : 2(v^2 - q^2) \cos \theta \cdot x + dy + vd + 2u(v^2 - q^2) \cos \theta = 0. \] (2.17)

If

\[ \Delta = \left( \frac{d}{2(v^2 - q^2) \cos \theta} \right)^2 - \frac{2(u^2 - p^2) \cos \theta}{d} = d^2 - 4(u^2 - p^2)(v^2 - q^2) \cos^2 \theta \neq 0, \]

then the radical axis \( k_m \) and \( k_n \) intersect in a point. It is easy to verify that

\[(v + q)k_m + (u + p)k_n - k_d = 0, \quad (v - q)k_m + (u + p)k_n - k_b = 0, \]

\[(v - q)k_m + (u - p)k_n - k_c = 0, \quad (v + q)k_m + (u - p)k_n - k_d = 0. \]

This involves that all radical axis \( k_a, k_b, k_c, k_d \) pass through the point of intersection of the radical axis \( k_m \) and \( k_n \) (Figure 1). Since these radical axis are concurrent, the \( K \)-circles are concurrent too.

Let \( K_M \) be their point of intersection, named in the following, the \( K \)-circles centre:

\[ K_M = k_a \cap k_b \cap k_c \cap k_d \cap k_m \cap k_n = K_{ABM} \cap K_{BCN} \cap K_{CDM} \cap K_{DAN}. \]
2.1. The coordinates and the position of the K-circles centre.

To determine the coordinates \( x_K \) and \( y_K \) of the K-circles centre, we form a system with the equations of the radical axis \( k_m \) and \( k_n \):

\[
\begin{align*}
dx + 2(u^2 - p^2) \cos \theta \cdot y - du - 2v(u^2 - p^2) \cos \theta &= 0, \\
2(v^2 - q^2) \cos \theta \cdot x + dy + vd + 2u(v^2 - q^2) \cos \theta &= 0.
\end{align*}
\]

Resolving this system with the Cramer rules, we obtain:

\[
\Delta = \begin{vmatrix}
d & 2(u^2 - p^2) \cos \theta \\
2(v^2 - q^2) \cos \theta & d
\end{vmatrix} = d^2 - 4(u^2 - p^2)(v^2 - q^2) \cos^2 \theta,
\]

\[
\Delta_x = \begin{vmatrix}
du + 2v(u^2 - p^2) \cos \theta & 2(u^2 - p^2) \cos \theta \\
-\vd - 2u(v^2 - q^2) \cos \theta & d
\end{vmatrix} = ud^2 + 4vd(u^2 - p^2) \cos \theta + 4u(u^2 - p^2)(v^2 - q^2) \cos^2 \theta,
\]

\[
\Delta_y = \begin{vmatrix}
d & du + 2v(u^2 - p^2) \cos \theta \\
2(v^2 - q^2) \cos \theta & -\vd - 2u(v^2 - q^2) \cos \theta
\end{vmatrix} = -\vd^2 - 4ud(v^2 - q^2) \cos \theta - 4v(u^2 - p^2)(v^2 - q^2) \cos^2 \theta.
\]

Consequently, if \( \Delta \neq 0 \), then

\[
x_K = \frac{\Delta_x}{\Delta}, \quad y_K = \frac{\Delta_y}{\Delta}.
\]

If \( u = 0 \) and \( v = 0 \), then \( M \equiv O \equiv N \equiv K_M = (0, 0) \).

If \( u = -p \) and \( v \neq \pm q \), then \( M \in AB, M \neq A, M \neq B \) and \( K_M = (-p, -v - 4u \cos \theta) \in AB \).

If \( u \neq \pm p \) and \( v = -q \), then \( M \in BC, M \neq B, M \neq C \) and \( K_M = (u + 4v \cos \theta, q) \in AD \).

If \( u = p \) and \( v \neq \pm q \), then \( M \in CD, M \neq C, M \neq D \) and \( K_M = (p, -v - 4u \cos \theta) \in CD \).

If \( u \neq \pm p \) and \( v = q \), then \( M \in DA, M \neq D, M \neq A \) and \( K_M = (u + 4v \cos \theta, -q) \in BC \).

In the following we suppose that \( u^2 + v^2 \neq 0 \), \( u \neq \pm p \) and \( v \neq \pm q \).

**Theorem 2.** If the point \( M \) is on the diagonal \( AC \) of parallelogram \( ABCD \), then \( K_M \) is situated on the other diagonal \( BD \) of the parallelogram \( ABCD \) and conversely.

**Proof.** The equation of the diagonals \( AC \) and \( BD \) are \( qx + py = 0 \) and \( qx - py = 0 \). If \( M \in AC \), then \( qu + pv = 0 \). We want to demonstrate that \( K_M \in BD \). We have

\[
K_M \in BD \iff qx_K - py_K = 0 \iff q\Delta_x - p\Delta_y = 0 \iff v\Delta_x + u\Delta_y = 0
\]

\[
\iff 4d[v^2(u^2 - p^2) - u^2(v^2 - q^2)] \cos \theta = 0
\]

\[
\iff 4d(qu - pv)(qu + pv) \cos \theta = 0,
\]

which is true.
If \( M \in BD \), then \( qu - pv = 0 \) and

\[
K_M \in AC \iff qx_K + py_K = 0 \iff q\Delta_x + p\Delta_y = 0 \iff v\Delta_x + u\Delta_y = 0
\]

\[
\iff 4d\left[u^2(v^2 - p^2) - u^2(v^2 - q^2)\right] \cos \theta = 0
\]

\[
\iff 4d(qu - pv)(qu + pv) \cos \theta = 0,
\]

which is true, too.

**Theorem 3.** Let \( L_{ABC}, L_{BCD}, L_{CDA}, L_{DAB} \) be the circles determined by the triplet of points (\( A, B, C \)), (\( B, C, D \)), (\( C, D, A \)), (\( D, A, B \)). If \( M \in L_{ABC} \), then \( K_M \equiv C \). If \( M \in L_{BCD} \), then \( K_M \equiv D \). If \( M \in L_{CDA} \), then \( K_M \equiv A \). If \( M \in L_{DAB} \), then \( K_M \equiv B \).

**Proof.** We observe that if the parallelogram \( ABCD \) is fix, then the circles \( L_{ABC}, L_{BCD}, L_{CDA}, L_{DAB} \) are fix, too. Suppose that \( M \in L_{ABC} \). In this case \( C \in K_{ABM} \) and \( C \in K_{BCN} \), \( C \in K_{CDO} \). Consequently, the circles \( K_{ABM}, K_{BCN}, K_{CDO} \) are concurrent in the point \( C \).

3. **The Concurrence of the Four Euler Circles of a Quadrilateral**

If the quadrilateral \( PQRS \) is given, let \( A, B, C, D \) be the midpoints of sides \( PQ, QR, RS, SP \). The quadrilateral \( ABCD \) is called the Varignon parallelogram of \( PQRS \). We can determine the sides of Varignon parallelogram \( ABCD \) and its angle \( ABC \), which we note with \( \theta \). Let \( AB = 2q \) and \( BC = 2p \), where \( p > 0, q > 0 \). Here we bring up the following question: how much does the triplet \( (p, q, \theta) \) characterize quadrilateral \( PQRS \)? We point out that if the triplet \( (p, q, \theta) \) is given and the parallelogram \( ABCD \) is fix, then there are an infinite number of quadrilaterals \( PQRS \) with the same Varignon parallelogram \( ABCD \). The premier question is how can we construct such a quadrilateral \( PQRS \)? Briefly, the construction is the following: put an arbitrary point \( X \) in the plan of parallelogram \( ABCD \). Construct the anticomplementary triangle of the triangle \( ABX \) and note its vertices with \( P, Q, R \) (let \( A \) be the midpoint of \( PQ \), \( B \) the midpoint of \( QR \) and \( X \) the midpoint of \( PR \)). The point \( S \) will be the symmetric of \( R \) with respect to \( C \) (Figure 2).

![Figure 2](image-url)
The justification of this construction is the following: let O be the intersection of the diagonals AC and BD, i.e. the centerpoint of the quadrilateral PQRS [4]. Note with Y the symmetric of X with respect to O. The triangles AXO and CYO are congruent, consequently AX = CY and AX ∥ CY. The quadrilaterals PABX, AQBX, ABX, BCYQ, BYCR are parallelograms. Let S = YQ ∩ CR. Since YQ ∥ BC and BY ∥ CR, the quadrilateral BCSY is a parallelogram too. Consequently CR = BY = CS, such that the point C is the midpoint of the segment [RS]. Since YQ = BC = YS, the point Y is the midpoint of the segment [QS]. The quadrilaterals PXCD, BXDY, XCSD are parallelograms too, from this, that PD ∥ XC ∥ DS and PD = XC = DS follows. So, the points P, D, S are collinear and D is the midpoint of the segment [PS].

We attach to the quadrilateral PQRS an oblique coordinates system xOy such that its zero point is the point O, the x-axis parallel with the line BC and the y-axis with the line AB. If X = (α, β), then Y = (−α, −β) and P = (α, β + 2q), Q = (−α − 2p, −β), R = (α, β − 2q), S = (−α + 2p, −β). Let M be an arbitrary point situated in the plan of quadrilateral PQRS, N the symmetry of M with respect to O. If M = (u, v), then N = (−u, −v).

**Corollary 3.1.** If u = α and v = β, then M ≡ X and N ≡ Y, i.e. the four Euler circles E_{QRS}, E_{PRS}, E_{PQS}, E_{PQR} of the quadrilateral PQRS are special K-circles of its Varignon parallelogram ABCD, consequently the Euler circles are concurrent.

Note with Ex the point of intersection of the Euler circles, which is called the Euler centre:

\[ E_X = E_{QRS} \cap E_{PRS} \cap E_{PQS} \cap E_{PQR}. \]

If α = 0 and β = 0, then X ≡ O ≡ Y ≡ E_X = (0,0) and the quadrilateral PQRS is parallelogram.
If α = −p and β ≠ ±q, then X ∈ AB, X ≠ A, X ≠ B and Q ∈ PR, E_X = (−p, −β − 4α cos θ) ∈ AB.
If α ≠ ±p and β = −q, then X ∈ BC, X ≠ B, X ≠ C and S ∈ PQ, E_X = (α + 4β cos θ, q) ∈ AD.
If α = p and β ≠ ±q, then X ∈ CD, X ≠ C, X ≠ D and S ∈ PR, E_X = (p, −β − 4α cos θ) ∈ CD.
If α ≠ ±p and β = q, then X ∈ DA, X ≠ D, X ≠ A and R ∈ QS, E_X = (α + 4β cos θ, −q) ∈ BC.

In the following we suppose that \( \alpha^2 + \beta^2 \neq 0, \alpha \neq \pm p \) and \( \beta \neq \pm q \).

### 3.1. The Euler centres of special quadrilaterals.

a) Orthodiagonal quadrilaterals: its diagonals are perpendicular [4], i.e. \( \theta = 90^\circ \) (Figure 3).

**Corollary 3.2.** The Euler centre of a quadrilateral orthodiagonal coincides with the point of intersection of its diagonals, which have the coordinates \( (\alpha, −\beta) \).
If \( X \) describes the circle circumscribed to the rectangle \( ABCD \), then the Euler centre describes the same circle. From the correspondence \( X \rightarrow E_{X} \), i.e. \( (\alpha, \beta) \rightarrow (\alpha, -\beta) \) it results that if \( X \) describes a line or an circle, then the Euler centre describes the symmetric line or circle with respect to the axis \( O \). x

b) \textbf{Cyclic quadrilaterals}: are inscribed in a circle (see [6]).

\textbf{Theorem 4.} \textit{The quadrilateral PQRS is cyclic if and only if}

\[ \alpha^2 - \beta^2 = p^2 - q^2. \]

\textbf{Proof.} Let \( l \) and \( l' \) be two lines with slopes \( m \) and \( m' \). In oblique coordinates systems the condition of perpendicularity of lines \( l \) and \( l' \) is \( m \cdot m' + 1 + (m + m') \cos \theta = 0 \) [5]. The slopes of sides \( PQ, QR, RS, SP \) are

\[ m_{PQ} = \frac{\beta + q}{\alpha + p}, \quad m_{QR} = \frac{\beta - q}{\alpha + p}, \quad m_{RS} = \frac{\beta - q}{\alpha - p}, \quad m_{SP} = \frac{\beta + q}{\alpha - p}. \]

Note the mid-perpendicular of sides \( PQ, QR, RS, SP \) with \( t_a, t_b, t_c, t_d \) and their slopes with the symbols \( m_a, m_b, m_c, m_d \):

\[ m_a = -\frac{1 + m_{PQ} \cos \theta}{m_{PQ} + \cos \theta} = -\frac{\alpha + p + (\beta + q) \cos \theta}{\beta + q + (\alpha + p) \cos \theta}, \]

\[ m_b = -\frac{1 + m_{QR} \cos \theta}{m_{QR} + \cos \theta} = -\frac{\alpha + p + (\beta - q) \cos \theta}{\beta - q + (\alpha + p) \cos \theta}, \]

\[ m_c = -\frac{1 + m_{RS} \cos \theta}{m_{RS} + \cos \theta} = -\frac{\alpha - p + (\beta - q) \cos \theta}{\beta - q + (\alpha - p) \cos \theta}, \]

\[ m_d = -\frac{1 + m_{SP} \cos \theta}{m_{SP} + \cos \theta} = -\frac{\alpha - p + (\beta + q) \cos \theta}{\beta + q + (\alpha - p) \cos \theta}. \]
The equations of mid-perpendiculars are

\[ t_a : y - q = m_a(x + p) \iff -m_a \cdot x + y - pm_a - q = 0 \]
\[ \iff [a + p + (\beta + q) \cos \theta]x + [\beta + q + (\alpha + p) \cos \theta]y \]
\[ + [a + p + (\beta + q) \cos \theta]p - [\beta + q + (\alpha + p) \cos \theta]q = 0 \quad (3.1) \]

\[ t_b : y + q = m_b(x + p) \iff -m_b \cdot x + y - pm_b + q = 0 \]
\[ \iff [a + p + (\beta - q) \cos \theta]x + [\beta - q + (\alpha + p) \cos \theta]y \]
\[ + [a + p + (\beta - q) \cos \theta]p + [\beta - q + (\alpha + p) \cos \theta]q = 0 \quad (3.2) \]

\[ t_c : y + q = m_c(x - p) \iff -m_c \cdot x + y + pm_c + q = 0 \]
\[ \iff [a - p + (\beta - q) \cos \theta]x + [\beta - q + (\alpha - p) \cos \theta]y \]
\[ - [a - p + (\beta - q) \cos \theta]p + [\beta - q + (\alpha - p) \cos \theta]q = 0 \quad (3.3) \]

\[ t_d : y - q = m_d(x - p) \iff -m_d \cdot x + y + pm_d - q = 0 \]
\[ \iff [a - p + (\beta + q) \cos \theta]x + [\beta + q + (\alpha - p) \cos \theta]y \]
\[ - [a - p + (\beta + q) \cos \theta]p - [\beta + q + (\alpha - p) \cos \theta]q = 0. \quad (3.4) \]

The mid-perpendiculars \( t_a, t_b, t_c \) are concurrent if and only if

\[ \begin{vmatrix}
  [a + p + (\beta + q) \cos \theta] & [\beta + q + (\alpha + p) \cos \theta] & [a + p + (\beta + q) \cos \theta]p - \ldots \\
  [a + p + (\beta - q) \cos \theta] & [\beta - q + (\alpha + p) \cos \theta] & [a + p + (\beta - q) \cos \theta]p + \ldots \\
  [a - p + (\beta + q) \cos \theta] & [\beta + q + (\alpha - p) \cos \theta] & [a - p + (\beta + q) \cos \theta]p + \ldots \\
\end{vmatrix} = 0 \]

\[ \iff \begin{vmatrix}
  [\beta + q + (\alpha + p) \cos \theta]p - \ldots \\
  [\beta - q + (\alpha + p) \cos \theta]p + \ldots \\
  [\beta + q + (\alpha - p) \cos \theta]p + \ldots \\
\end{vmatrix} = 0 \]

\[ \iff 0 = 0 \]

\[ \iff \begin{vmatrix}
  2q & [\beta + q + (\alpha + p) \cos \theta]p \\
  0 & [\beta + q + (\alpha - p) \cos \theta]p \\
\end{vmatrix} = 0 \]

\[ \iff \begin{vmatrix}
  0 & [\beta + q + (\alpha + p) \cos \theta]p \\
  0 & [\beta + q + (\alpha - p) \cos \theta]p \\
\end{vmatrix} = 0 \]

\[ \iff pq(\beta - q)(\beta + q + (\alpha + p) \cos \theta) - pq(\alpha + p)(\alpha - p + (\beta - q) \cos \theta) = 0 \]
\[ \iff \alpha^2 - \beta^2 = p^2 - q^2. \]
We observe that points \( E \) and \( m \) altitudes quadrilateral to the opposite sides (called the quadrilateral). Consequently, by resolving this system with the Cramer rule, we obtain:

\[
\begin{align*}
[\alpha + p + (\beta + q) \cos \theta]x + [\beta + q + (\alpha + p) \cos \theta]y & = 0 \\
[\alpha + p + (\beta - q) \cos \theta]x + [\beta - q + (\alpha + p) \cos \theta]y & = 0
\end{align*}
\]

By resolving this system with the Cramer rules, we obtain:

\[
\Delta = \begin{vmatrix}
\alpha + p + (\beta + q) \cos \theta & \beta + q + (\alpha + p) \cos \theta \\
\alpha + p + (\beta - q) \cos \theta & \beta - q + (\alpha + p) \cos \theta
\end{vmatrix} = -2q(\alpha + p)(1 - \cos^2 \theta)
\]

\[
\Delta_x = \begin{vmatrix}
-x + p + (\beta + q) \cos \theta & x + [\beta + q + (\alpha + p) \cos \theta]q \\
-x + p + (\beta - q) \cos \theta & x - [\beta - q + (\alpha + p) \cos \theta]q
\end{vmatrix} = 2q(\alpha + p)(\alpha + 2\beta \cos \theta + \alpha \cos^2 \theta)
\]

\[
\Delta_y = \begin{vmatrix}
-x + p + (\beta + q) \cos \theta & -x + p + (\beta + q) \cos \theta & [\beta + q + (\alpha + p) \cos \theta]q \\
-x + p + (\beta - q) \cos \theta & -x + p + (\beta - q) \cos \theta & -[\beta - q + (\alpha + p) \cos \theta]q
\end{vmatrix} = -2q(\alpha + p)(\beta + 2\alpha \cos \theta + \beta \cos^2 \theta)
\]

Consequently,

\[
x_U = \frac{\Delta_x}{\Delta} = \frac{\alpha + 2\beta \cos \theta + \alpha \cos^2 \theta}{\sin^2 \theta}
\]

\[
y_U = \frac{\Delta_y}{\Delta} = \frac{\beta + 2\alpha \cos \theta + \beta \cos^2 \theta}{\sin^2 \theta}
\]

We observe that points \( E \) and \( U \) are symmetric with respect to \( O \), the centerpoint of the quadrilateral \( PQRS \).

It is well-known that the perpendiculars from the midpoints of the sides of a cyclic quadrilateral to the opposite sides (called the maltitudes (see [1], [2], or [6])) are concurrent in a point \( L \), called the Mathot point or anticenter of the cyclic quadrilateral (see [1], [2], or [6]).

**Theorem 5.** The Euler centre and the anticenter of a cyclic quadrilateral coincide.

**Proof.** Note with \( s_a, s_b, s_c, s_d \) the maltitudes of the cyclic quadrilateral \( PQRS \), and let \( s_a \parallel t_a, s_b \parallel t_b, s_c \parallel t_c, s_d \parallel t_d \) (\( C \in s_a, D \in s_b, A \in s_c, B \in s_d \)) (Figure 4).
The equations of the maltitudes are the following:

\[ s_a : y + q = m_a (x - p) \iff -m_a \cdot x + y + pm_a + q = 0 \]
\[ \iff [a + p + (\beta + q) \cos \theta] x + [\beta + q + (a + p) \cos \theta] y - [a + p + (\beta + q) \cos \theta] p - [\beta + q + (a + p) \cos \theta] q = 0 \quad (3.5) \]

\[ s_b : y - q = m_b (x - p) \iff -m_b \cdot x + y + pm_b - q = 0 \]
\[ \iff [a + p + (\beta - q) \cos \theta] x + [\beta - q + (a + p) \cos \theta] y - [a + p + (\beta - q) \cos \theta] p - [\beta - q + (a + p) \cos \theta] q = 0 \quad (3.6) \]

\[ s_c : y - q = m_c (x + p) \iff -m_c \cdot x + y - pm_c - q = 0 \]
\[ \iff [a - p + (\beta - q) \cos \theta] x + [\beta - q + (a - p) \cos \theta] y + [a - p + (\beta - q) \cos \theta] p - [\beta - q + (a - p) \cos \theta] q = 0 \quad (3.7) \]

\[ s_d : y + q = m_d (x + p) \iff -m_d \cdot x + y - pm_d + q = 0 \]
\[ \iff [a - p + (\beta + q) \cos \theta] x + [\beta + q + (a - p) \cos \theta] y + [a - p + (\beta + q) \cos \theta] p + [\beta + q + (a - p) \cos \theta] q = 0. \quad (3.8) \]

These equations are the same as the equations of mid-perpendiculars, except for the free terms, which have opposite signs. Consequently, the coordinates of the anticenter \( L \) are

\[ x_L = \frac{a + 2\beta \cos \theta + a \cos^2 \theta}{\sin^2 \theta} = x_E, \quad y_L = -\frac{\beta + 2\alpha \cos \theta + \beta \cos^2 \theta}{\sin^2 \theta} = y_E. \]
**Remark 1.** The equation of the line perpendicular to the line \( lx + my + n = 0 \) and which passes through the point \((x', y')\) is
\[
\begin{vmatrix}
x & y & 1 \\
x' & y' & 1 \\
l - m \cos \theta & m - l \cos \theta & 0
\end{vmatrix} = 0.
\]
Consequently, the equation of the line \( s_x \) perpendicular to the diagonal \( QS \) with equation \( y + \beta = 0 \) and which passes through the point \( X = (\alpha, \beta) \) is
\[
\begin{vmatrix}
x & y & 1 \\
\alpha & \beta & 1 \\
-\cos \theta & 1 & 0
\end{vmatrix} = 0 \iff x + \cos \theta \cdot y = \alpha + \beta \cos \theta.
\]
Similarly, we obtain the equation of the line \( s_y \) perpendicular to the diagonal \( PR \) with equation \( x - \alpha = 0 \), and which passes through the point \( Y = (-\alpha, -\beta) \):
\[
\begin{vmatrix}
x & y & 1 \\
-\alpha & -\beta & 1 \\
1 & -\cos \theta & 0
\end{vmatrix} = 0 \iff \cos \theta \cdot x + y = - (\alpha \cos \theta + \beta).
\]
By resolving the system formed with the equations of the lines \( s_x \) and \( s_y \), we obtain the coordinates of the anticenter. So, the lines \( s_x \) and \( s_y \) pass through the anticecenter of the quadrilateral \( PQRS \). If the point \( Z \) is the intersection of the diagonals \( PR \) and \( QS \), then the orthocenter of the triangle \( XYZ \) is the anticecenter \( L \) of the quadrilateral.

4. **Further research**

With the computer programs it is possible to investigate another geometrical locus of the Euler centre. For example what describe the point \( E_X \), if the point \( X \) describe a line, a circle, an conics, the incircle of the triangle \( ABC \), a line remarkable of the triangle \( ABC \) etc.

**References**