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ON THE TANGENT SPHERE BUNDLE OF THE PSEUDO HYPERBOLIC TWO SPACE

ISMET AYHAN

ABSTRACT. In this study, the Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε $T_\varepsilon H_1^2$ of the pseudo hyperbolic two space H_1^2 in semi Euclidean space E_1^3 is obtained. Moreover, the connection coefficients of the Levi Civita connection on the Sasaki semi Riemann manifold $(T_\varepsilon H_1^2, g^S)$ are found and then the non linear geodesic equations of $(T_\varepsilon H_1^2, g^S)$ are obtained. Moreover, the relations between geodesics of H_1^2 and $T_\varepsilon H_1^2$ are examined. Finally, the components of the Riemann curvature tensor of $(T_\varepsilon H_1^2, g^S)$ are calculated.

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1. INTRODUCTION

The geometry of the tangent sphere bundle of a manifold is a well known subject for the scientists related to bundle geometry. But the geometry of the tangent sphere bundle with a semi Riemann metric is a new subject.
The tangent sphere bundle of n dimensional manifold is defined as the disjoint union of the tangent vector space created by the unit tangent vectors at all points of this manifold. The first time was considered that the disjoint union of the tangent vector space created by the unit tangent vectors at all points of a geodesic circle of the unit 2-sphere gave a sphere and by moving this sphere along the geodesic circle was produced a torus by Klingenberg and Sasaki in [4]. Moreover, the authors studied on the torus family which contains produced all torus along each geodesic circle of the unit 2-sphere. The authors in their study proved that T_1S^2 was a Riemann manifold with constant sectional curvature. Nagy [5] calculated the components of the Riemann sectional curvature of tangent sphere bundle T_1M of a 2-dimensional Riemann manifold M. Moreover, he obtained that a curve $(x(t), y(t))$ in the tangent sphere bundle had the geodesic curve if and only if the geodesic curvature of $x(t)$ with Gaussian curvature of M must have been a constant rate or the parallel displacement of the vector component $y(t)$ along the curve $x(t)$ must have drawn a helical curve. Sasaki [8] classified the geodesics on the tangent sphere bundle of the unit n-sphere S^n and the hyperbolic n-space H^n by using the general formula of the Sasaki Riemann metric on T_1S^n and T_1H^n and taking regard of this classification, he obtained three different types geodesics on T_1S^3 and T_1H^2. Ayhan [1] obtained Sasaki Riemann metric of the tangent sphere bundle of the unit 3-sphere by

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using the geodesic polar coordinate of the unit 3-sphere. Furthermore, he calculated the general geodesic equations of the tangent sphere bundle of the unit 3-sphere. Ayhan [2] obtained the Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε, $T_\varepsilon S_1^2$ by using the parametric representation of the unit 2-sphere, S_1^2 in three dimensional semi Euclidean space with index one. Then, he calculated the connection coefficients of the Levi Civita connection and the coefficients of the Riemann curvature tensor of $(T_\varepsilon S_1^2, g^S)$ and then found out a non-linear differential equation's system which gives geodesics of $T_\varepsilon S_1^2$.
The aim of this study is to examine the geometry of the tangent sphere bundle with radius ε of a hyperboloid with one sheet in 3-dimensional semi Euclidean space with index one called pseudo hyperbolic 2-space. Firstly, the Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε $T_\varepsilon H_1^2$ of a pseudo hyperbolic two space H_1^2 is obtained. Then, the connection coefficients of the Levi Civita connection of $\left(T_\varepsilon H_1^2,\ g^S\right)$ have been calculated and then a differential equation's system which gives geodesics of $T_\varepsilon H_1^2$ has been obtained. Moreover, the components of the Riemann curvature tensor of $T_\varepsilon H_1^2$ are calculated. Finally, the condition providing the surface H_1^2 is totally geodesic submanifold of $T_\varepsilon H_1^2$ is examined and the lifting operation preserved the causal characters of geodesics from the surface H_1^2 to $T_\varepsilon H_1^2$ is considered.

2. THE PSEUDO HYPERBOLIC 2−SPACE

In this section, the parametric representation of the hyperboloid of one sheet in semi Euclidean space, the induced semi Riemann metric on H_1^2, the orthonormal base vectors of the tangent vector space at any point of H_1^2, the Christoffel symbols of H_1^2, a differential equation's system, which gives geodesics of H_1^2 are considered.

Definition 2.1. *Let $<,>$ be non degenerate, symmetric, bilinear form in semi Euclidean space E_1^3 defined by*

$$< u, v >= -u_1v_1 + u_2v_2 + u_3v_3, \tag{1}$$

for any vectors $u, v \in E_1^3$. H_1^2 is a surface in E_1^3 given by

$$H_1^2 = \left\{u = (x_1, x_2, x_3) :< u, u >= -1, u \in E_1^3\right\}. \tag{2}$$

H_1^2 is called as the hyperboloid of one sheet in semi Euclidean space or the pseudo hyperbolic 2-space. H_1^2 is represented by hyperboloid of two sheet in Euclidean space given by the following equation:

$$-x_1^2 + x_2^2 + x_3^2 = -1, \tag{3}$$

with respect to rectangular coordinate system. The parametric representation of H_1^2 are given by

$$\begin{aligned} x_1 &= \cosh a, \\ x_2 &= \sinh a \cos\theta, \\ x_3 &= \sinh a \sin\theta, \end{aligned} \tag{4}$$

and a curve on the surface H_1^2 is described by

$$c : t \to c(t) = (a(t), \theta(t)), \tag{5}$$

where (a, θ) is called as the generalized coordinates of H_1^2.

In order to find the arc length parameter of any curve on pseudo hyperbolic 2-space for $t_0 \leq t \leq t_1$, the covariant derivations of x_1, x_2, x_3 are used as follow:

$$\begin{aligned} dx_1 &= \sinh a da, \\ dx_2 &= \cosh a \cos\theta da - \sinh a \sin\theta d\theta, \\ dx_3 &= \cosh a \sin\theta da + \sinh a \cos\theta d\theta. \end{aligned} \tag{6}$$

Definition 2.2. *In semi Euclidean space E_1^3, the arc length parameter between different two point with infinitesimal distance on the surface H_1^2 (i.e. (x_1, x_2, x_3) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$) is calculated by*

$$\begin{aligned} ds^2 &=< (dx_1, dx_2, dx_3), (dx_1, dx_2, dx_3 > \\ &= -(dx_1)^2 + (dx_2)^2 + (dx_3)^2. \end{aligned} \tag{7}$$

By using the (6), we get

$$ds^2 = (da)^2 + \sinh^2 a\,(d\theta)^2 \tag{8}$$

and also the matrix representation of this equation has the following components:

$$g_{ik} : \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 a \end{pmatrix}, \textit{ for } i, k \in \{1, 2\}, \tag{9}$$

where g_{ik} is called as the induced metric on H_1^2 from E_1^3. The inverse of g_{ik} has the following matrix representation:

$$g^{kj} : \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sinh^2 a} \end{pmatrix}. \tag{10}$$

Assuming that $e_1(a, \theta)$ is any point on H_1^2 given by

$$e_1(a, \theta) = (\cosh a, \sinh a \cos\theta, \sinh a \sin\theta) \tag{11}$$

with respect to standard orthonormal base of E_1^3. Since a curve on the surface H_1^2 is described by $c : t \to c(t) = (a(t), \theta(t))$, the unit tangent vector of $a-$curves and $\theta-$curves passing through the point $e_1(a, \theta)$ must be expressed by

$$f_2 = \frac{\partial}{\partial a} \quad \text{and} \quad f_3 = \frac{1}{\sinh a}\frac{\partial}{\partial \theta}. \tag{12}$$

In addition, the unit tangent vectors f_2 and f_3 has the following local expression:

$$\begin{aligned} f_2(a, \theta) &= (\sinh a, \cosh a \cos\theta, \cosh a \sin\theta), \\ f_3(a, \theta) &= (0, -\sin\theta, \cos\theta), \end{aligned} \tag{13}$$

with respect to standard orthonormal base of E_1^3. Thus $\{e_1, f_2, f_3\}$ is another orthonormal base of E_1^3.

Theorem 2.1. *Let H_1^2 be pseudo hyperbolic 2-space. If $T_{e_1}H_1^2$ is a tangent vector space at any point $e_1(a, \theta)$ on H_1^2, g is semi Riemann metric on H_1^2 defined by*

$$\begin{aligned} g : \quad T_{e_1}H_1^2 \times T_{e_1}H_1^2 &\to \quad IR. \\ (X, Y) &\to g(X, Y) \end{aligned} \tag{14}$$

Proof. Let $X = af_2 + bf_3$, $Y = cf_2 + df_3$ and $Z = pf_2 + qf_3$ be the tangent vectors at any point on H_1^2 where $\{f_2, f_3\}$ is orthonormal base of $T_{e_1}H_1^2$. For all X, Y, $Z \in T_{e_1}S_1^2$ and α, $\beta \in IR$, we get

$$\begin{aligned} g(\alpha X+\beta Y,Z) &= g(\alpha\,[af_2+bf_3]+\beta\,[cf_2+df_3]\,,[pf_2+qf_3]) \\ &= \alpha g(X,Z)+\beta g(Y,Z). \end{aligned}$$

Similarly we get $g(X, \alpha Y + \beta Z) = \alpha g(X,Y) + \beta g(X,Z)$. Thus g is bilinear transformation. Furthermore g must be symmetric map because the following equation is hold:

$$\begin{aligned} g(X,Y) &= g(af_2+bf_3,cf_2+df_3) \\ &= g(Y,X). \end{aligned}$$

Finally, g is a non degenerate map such that

$$g(X,Y)=0 \Longleftrightarrow Y=0 \quad \text{for all } X \in T_{e_1}H_1^2.$$

Since g is non degenerate, symmetric, bilinear form, g must be a semi Riemann metric on the surface H_1^2.

Theorem 2.2. *Let H_1^2 be pseudo hyperbolic 2-space. Let $\{e_1, f_2, f_3\}$ be an another orthonormal base in E_1^3 and f_2, f_3 be the base vectors of the tangent space $T_{e_1}H_1^2$ at a point e_1 of H_1^2 given by the equations* (11), (12) *and* (13). *e_1 is the time like and f_2 and f_3 the space like unit vectors of E_1^3.*

Proof. Since the value of the unit vectors e_1, f_2 and f_3 given by (11) and (13) under the semi Euclidean metric $<,>$ in E_1^3 have the following expression:

$$\begin{aligned} &< e_1,e_1 >= -\cosh^2 a+\sinh^2 a\cos^2\theta+\sinh^2 a\sin^2\theta=-1, \\ &< f_2,f_2 >= -\sinh^2 a+\cosh^2 a\cos^2\theta+\cosh^2 a\sin^2\theta=1, \\ &< f_3,f_3 >= \sin^2\theta+\cos^2\theta=1, \end{aligned}$$

e_1 must be the time like unit vector and f_2, f_3 must be the space like unit vectors, respectively. If we consider the unit tangent vectors f_2 and f_3 given by (12), we must use the induced metric on H_1^2 from E_1^3 given by (9). As a consequence of this fact, we get

$$\begin{aligned} g(f_2,f_2) &= \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=1, \\ g(f_3,f_3) &= \begin{pmatrix} 0 & \frac{1}{\sinh a} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sinh^2 a \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{\sinh a} \end{pmatrix}=1. \end{aligned}$$

Thus, f_2 and f_3 are the space like unit vectors.

Theorem 2.3. *Let H_1^2 be pseudo hyperbolic 2-space and $\{e_1, f_2, f_3\}$ be an another orthonormal base of E_1^3. The covariant derivations of these unit-orthogonal vectors are given by*

$$\begin{aligned} de_1 &= daf_2+\sinh a d\theta f_3, \\ df_2 &= dae_1+\cosh a d\theta f_3, \\ df_3 &= \sinh a d\theta e_1-\cosh a d\theta f_2. \end{aligned}$$

Proof. We use the covariant derivations of orthonormal vectors e_1, f_2, f_3 in order to examine the change of the base vectors on different two points with infinitesimal distance on

H_1^2 (i.e. (e_1, f_2, f_3) and $(e_1 + de_1, f_2 + df_2, f_3 + df_3)$). The covariant derivatives of these vectors are calculated by using the partial derivation as follow:

$$de_1 = \frac{\partial e_1}{\partial a}da + \frac{\partial e_1}{\partial \theta}d\theta = da f_2 + \sinh a d\theta f_3,$$
$$df_2 = \frac{\partial f_2}{\partial a}da + \frac{\partial f_2}{\partial \theta}d\theta = da e_1 + \cosh a d\theta f_3,$$
$$df_3 = \frac{\partial f_3}{\partial a}da + \frac{\partial f_3}{\partial \theta}d\theta = \sinh a d\theta e_1 - \cosh a d\theta f_2.$$

Theorem 2.4. *Let $\left(H_1^2, g\right)$ be a semi Riemann manifold. Let D be Levi Civita connection of $\left(H_1^2, g\right)$ and $\phi_{ij}^k; i, j, k \in \{1, 2\}$ be Christoffel symbols with respect to the semi Riemann metric g. Then the non-zero the Christoffel symbols of $\left(H_1^2, g\right)$ have the following components:*

$$\phi_{22}^1 = -\sinh a \cosh a, \quad \phi_{12}^2 = \coth a,$$

where $\phi_{ij}^k = \phi_{ji}^k$ for all $i, j, k \in \{1, 2\}$.

Proof. On the semi Riemann manifold $\left(H_1^2, g\right)$, there is a unique connection D such that D is torsion free and compatible with semi Riemann metric g. This connection is called as Levi Civita connection and characterized by the Kozsul formula:

$$2g\left(D_{\partial_a}\partial_\theta, \partial_\theta\right) = \partial_a g\left(\partial_\theta, \partial_\theta\right) + \partial_\theta g\left(\partial_\theta, \partial_a\right) - \partial_\theta g\left(\partial_a, \partial_\theta\right) + \\ - g\left(\left[\partial_a, \partial_\theta\right], \partial_\theta\right) + g\left(\left[\partial_\theta, \partial_\theta\right], \partial_a\right) + g\left(\left[\partial_\theta, \partial_a\right], \partial_\theta\right),$$

where $\partial_a = \frac{\partial}{\partial a} = \partial_1$, and $\partial_\theta = \frac{\partial}{\partial \theta} = \partial_2$. Since D is symmetric, $[\partial_a, \partial_\theta]$ must be zero. If we get $D_{\partial_a}\partial_\theta = \phi_{12}^1\partial_a + \phi_{12}^2\partial_\theta$, from Kozsul formula, it is obtained by

$$\phi_{12}^1 = \frac{1}{2}g^{1m}\left(\partial_1 g_{m2} + \partial_2 g_{2m} - \partial_m g_{12}\right) = 0,$$
$$\phi_{12}^2 = \frac{1}{2}g^{2m}\left(\partial_1 g_{m2} + \partial_2 g_{2m} - \partial_m g_{12}\right) = \coth a,$$

where $m \in \{1, 2\}$. The other Christoffel symbols can be obtained by using the similar method.

Theorem 2.5. *Let $\left(H_1^2, g\right)$ be semi Riemann manifold and $c : t \in R \to c(t) = (a(t), \theta(t)) \in H_1^2$ be a curve on the pseudo Hyperbolic 2-space H_1^2. c is a geodesic if and only if the following differential equation's system has been provided:*

$$\ddot{a} - \sinh a \cosh a \dot{\theta}^2 = 0, \tag{15}$$
$$\ddot{\theta} + 2\coth a \dot{a}\dot{\theta} = 0. \tag{16}$$

Proof. $c(t) = (a(t), \theta(t))$ is geodesic if and only if $D_{\dot{c}}\dot{c}$ must be zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a + \dot{\theta}\partial_\theta$, $D_{\dot{c}}\dot{c}$ is equal to $D_{\dot{a}\partial_a}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta\right) + D_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta\right)$. For $D_{\dot{c}}\dot{c} = 0$,

$$D_{\dot{c}}\dot{c} = \left(\ddot{a} - \sinh a \cosh a \dot{\theta}^2\right)\partial_a + \left(\ddot{\theta} + 2\coth a \dot{a}\dot{\theta}\right)\partial_\theta$$

it is seen that the claim of the theorem is correct, easily.

Definition 2.3. *Let the line element of H_1^2 be*

$$ds^2 = \dot{a}^2 + \sinh^2 a\dot{\theta}^2 = \varepsilon. \tag{17}$$

The curve $c : t \in R \to c(t) = (a(t), \theta(t)) \in H_1^2$ providing the equations in (2.17) is called as the time like, the light like or the space like curve providing that $\varepsilon = -1$, $\varepsilon = 0$ or $\varepsilon = 1$, respectively.

In the rest of the paper, the curve c will be assumed as a geodesic of H_1^2. To find a general equation characterizing the time like, the light like or the space like geodesics on H_1^2, we get

$$\left(\frac{da}{d\theta}\dot{\theta}\right)^2 + \sinh^2 a\dot{\theta}^2 = \varepsilon. \tag{18}$$

from (2.17). If we solve the differential equation in (2.16), we get

$$\left\{\frac{d}{da}\left(\dot{\theta}\right) + 2\coth a\dot{\theta}\right\}\dot{a} = 0 \Rightarrow \dot{\theta} = k\csc h^2 a \vee \dot{a} = 0, \tag{19}$$

and the value $\dot{\theta} = k\csc h^2 a$ put in the equation (2.18) , the general equation characterizing the time like, the light like and the space like geodesics on H_1^2 are obtained as follows:

$$\frac{da}{d\theta} = \frac{\sqrt{\varepsilon\sinh^4 a - k^2\sinh^2 a}}{k}. \tag{20}$$

Theorem 2.6. *The time like geodesics of pseudo hyperbolic 2-space H_1^2 are given by the following generalized and rectangular coordinates of H_1^2 :*

$$\sqrt{1 + k^2\csc h^2 a} + k\coth a = \cos\theta - i\sin\theta,$$

and

$$\left(x_2 - \sqrt{x_2^2 + x_3^2 + k^2} - kx_1\right)^2 + x_3^2 = 0.$$

Proof. The one parameter curve family obtained by putting $\varepsilon = -1$ in (20) defines a lot of planes. The time like geodesics of pseudo hyperbolic 2-space H_1^2 are cross-section curves between the planes and the surface H_1^2. The following curve on H_1^2 is given by an example to the time like geodesic:

$$c(t) = (t, \frac{5t^2 - 1}{4t}, \frac{3t^2 + 1}{4t}i),$$

for $k = 1$.

Theorem 2.7. *The light like geodesics on pseudo hyperbolic space H_1^2 are given by the following generalized or rectangular coordinates of H_1^2 :*

$$\csc ha - \coth a = \cos\theta + i\sin\theta, \ \ (x_1 - x_2 - 1)^2 + x_3^2 = 0.$$

Proof. The one parameter curve family obtained by putting $\varepsilon = 0$ in (20) defines two planes. The light like geodesics of pseudo hyperbolic 2-space H_1^2 are cross-section curves between the planes and the surface H_1^2. The following curve on H_1^2 is given by an example to the light like geodesic:

$$c(t) = (t, t, i).$$

Theorem 2.8. *The space like geodesics on pseudo hyperbolic 2- space H_1^2 are given by the following generalized or rectangular coordinates of H_1^2 :*

$$\frac{\sqrt{1-k^2 \csc h^2 a}}{\sqrt{1+k^2}} = \sin\theta,\ x_2^2 = k^2(x_3^2+1).$$

Proof. The one parameter curve family obtained by putting $\varepsilon = 1$ in (20) defines surfaces. The space like geodesics of pseudo hyperbolic 2-space H_1^2 are cross-section curves between these surfaces and the surface H_1^2. The following curve on H_1^2 is given by an example to the space like geodesic:

$$c(t) = (\sqrt{2}\sqrt{t^2+1}, \sqrt{t^2+1}, t)$$

for $k = 1$.

3. The Tangent Sphere Bundle with radius ε of Pseudo Hyperbolic Two Space

This section consists of some subjects as the representation by the local coordinate function of any point on $T_\varepsilon H_1^2$, the orthonormal base at any point of $T_\varepsilon H_1^2$, the covariant derivations of this orthonormal base elements, Sasaki semi Riemann metric g^S on $T_\varepsilon H_1^2$, the adapted base and adapted dual base on $T_\varepsilon H_1^2$ with respect to g^S. Furthermore, in this section contains the subjects as the connection coefficients of the Levi Civita connection of Sasaki semi Riemann manifold $\left(T_\varepsilon H_1^2, g^S\right)$, a differential equation's system, which give geodesics on $\left(T_\varepsilon H_1^2, g^S\right)$. Finally, the coefficients of the Riemann curvature tensor of $\left(T_\varepsilon H_1^2, g^S\right)$ are calculated.

Definition 3.1. $T_\varepsilon H_1^2 = \underset{\forall e_1(a,\theta)\in H_1^2}{\cup} (u \in T_{e_1} H_1^2 : g(u,u) = \varepsilon)$ *is the disjoint union of the tangent vector spaces including all unit tangent vectors at every point of H_1^2. Thus, $T_\varepsilon H_1^2$ is the total space of time like, light like and space like vectors with respect to the induced metric g from standart semi Euclidean metric in E_1^3 and $T_\varepsilon H_1^2$ is called as the tangent sphere bundle with radius ε of H_1^2.*

Since H_1^2 has 2 dimensional manifold structure, $T_\varepsilon H_1^2$ should be 3 dimensional manifold structure. Let $\pi : T_\varepsilon H_1^2 \to H_1^2$ be a canonical projection map and e_2 be an element of $T_\varepsilon H_1^2$ at the point $e_1(a,\theta)$ of H_1^2. If we denote the angle between f_2 and e_2 by ω, then (a,θ,ω) can be considered as local coordinates for e_2 in $\pi^{-1}(H_1^2)$. Therefore, e_2 and e_3 have the following local expression:

$$\begin{aligned} e_2(a,\theta,\omega) &= \cos\omega f_2 + \sin\omega f_3, \\ e_3(a,\theta,\omega) &= -\sin\omega f_2 + \cos\omega f_3, \end{aligned} \tag{21}$$

where e_3 is an element of $T_\varepsilon H_1^2$ at the point $e_1(a,\theta)$ of H_1^2.

Theorem 3.1. *Let $T_\varepsilon H_1^2$ be the tangent sphere bundle with radius ε of pseudo hyperbolic 2-space H_1^2. If e_2, e_3 have been considered as the tangent vectors at a point $e_1(a,\theta)$ on H_1^2 given by the equations* (3.1) *then e_2 and e_3 are the space like unit vectors.*

Proof. The value of the unit tangent vectors e_2 and e_3 given by (3.1) under the semi Euclidean metric in E_1^3 are obtained as follows:

$$< e_2, e_2 >= \cos^2\omega < f_2, f_2 > + \sin^2\omega < f_3, f_3 >= 1,$$
$$< e_3, e_3 >= \sin^2\omega < f_2, f_2 > + \cos^2\omega < f_3, f_3 >= 1.$$

Thus, e_2 and e_3 are the space like unit vectors.

Theorem 3.2. *Let $T_\varepsilon H_1^2$ be the tangent sphere bundle with radius ε of pseudo hyperbolic 2-space and e_1, e_2, e_3 be unit-orthogonal elements of $T_\varepsilon H_1^2$. The covariant derivations of these elements are given by*

$$\begin{aligned} de_1 &= (\cos\omega da + \sinh a \sin\omega d\theta)\, e_2 + (-\sin\omega da + \sinh a \cos\omega d\theta)\, e_3, \\ de_2 &= (\cos\omega da + \sinh a \sin\omega d\theta)\, e_1 + (d\omega + \cosh a d\theta)\, e_3, \\ de_3 &= (-\sin\omega da + \sinh a \cos\omega d\theta)\, e_1 - (d\omega + \cosh a d\theta)\, e_2. \end{aligned}$$

Proof. We use the covariant derivations of e_1, e_2, e_3 in order to examine the change of the base vectors on different two points with infinitesimal distance on $T_\varepsilon H_1^2$ (i.e. (e_1, e_2, e_3) and $(e_1 + de_1, e_2 + de_2, e_3 + de_3)$). The covariant derivatives of e_1, e_2, e_3 are obtained by helping the partial derivation, easily.

Definition 3.2. *The 1-forms providing the equation $w_{ij} =< de_i, e_j >$, for $i, j \in \{1, 2, 3\}$ are called as the connection 1-forms on the cotangent space $T^*_{(e_1,e_2)} T_\varepsilon H_1^2$ where w_{ij} is given by*

$$\begin{aligned} \eta^1 &= w_{12} = -w_{21} = \cos\omega da + \sinh a \sin\omega d\theta, \\ \eta^2 &= w_{13} = -w_{31} = -\sin\omega da + \sinh a \cos\omega d\theta, \\ \eta^3 &= w_{23} = -w_{32} = d\omega + \cosh a d\theta. \end{aligned} \tag{22}$$

Theorem 3.3. *In semi Euclidean space E_1^3, the line element between infinitely close two point on $T_\varepsilon H_1^2$ is given by*

$$d\sigma^2 = (da)^2 - (d\theta)^2 - 2\cosh a d\theta d\omega - (d\omega)^2 . \tag{23}$$

Proof. In semi Euclidean space E_1^3, let $\{e_1, e_2, e_3\}$ be the orthonormal base at any point $e_2 \in \pi^{-1}(\{e_1\})$ on $T_1 H_1^2$ and $\{e_1 + de_1, e_2 + de_2, e_3 + de_3\}$ be the orthonormal base at another point to be infinitely close point to e_2. The infinitesimal length between this two point is obtained as follows:

$$\begin{aligned} d\sigma^2 &=< de_1, de_1 > - < de_2, e_3 >^2 \\ &= \eta^1 \wedge \eta^1 + \eta^2 \wedge \eta^2 - \eta^3 \wedge \eta^3 \\ &= (da)^2 - (d\theta)^2 - 2\cosh a d\theta d\omega - (d\omega)^2 . \end{aligned}$$

Definition 3.3. *$d\sigma^2 : (g^S)$ is called as a metric structure on the manifold $T_\varepsilon H_1^2$. Moreover, $\{\eta^1, \eta^2, \eta^3\}$ is called as an adapted dual base on the cotangent space $T^*_{(e_1,e_2)} T_\varepsilon H_1^2$ with respect to g^S. If the tangent vectors $\xi_i; i \in \{1, 2, 3\}$ providing the following equation:*

$$\eta^i(\xi_i) = g^S(\xi_i, \xi_i) = \varepsilon_i, \varepsilon_i = \begin{cases} 1 & for \quad i = 1, 2 \\ -1 & for \quad i = 3 \end{cases}, \tag{24}$$

$\{\xi_1, \xi_2, \xi_3\}$ is called as adapted base of the tangent space $T_{(e_1,e_2)}T_\varepsilon H_1^2$ with respect to the metric structure g^S where ξ_i $i \in \{1,2,3\}$ is defined by

$$\begin{aligned} \xi_1 &= \cos\omega \frac{\partial}{\partial a} + \frac{\sin\omega}{\sinh a}\frac{\partial}{\partial\theta} - \coth a \sin\omega \frac{\partial}{\partial\omega}, \\ \xi_2 &= -\sin\omega \frac{\partial}{\partial a} + \frac{\cos\omega}{\sinh a}\frac{\partial}{\partial\theta} - \coth a \cos\omega \frac{\partial}{\partial\omega}, \\ \xi_3 &= \frac{\partial}{\partial\omega}. \end{aligned} \tag{25}$$

Theorem 3.4. *Let $T_\varepsilon H_1^2$ be the tangent sphere bundle with radius ε of pseudo hyperbolic 2-space. If $T_{(e_1,e_2)}T_\varepsilon H_1^2$ is a tangent vector space at any point on $T_\varepsilon H_1^2$, g^S is semi Riemann metric on $T_\varepsilon H_1^2$ where g^S is defined by*

$$\begin{array}{rccc} g^S: & T_{(e_1,e_2)}T_\varepsilon H_1^2 \times T_{(e_1,e_2)}T_\varepsilon H_1^2 & \to & IR. \\ & \left(\widetilde{X}, \widetilde{Y}\right) & \to & g^S\left(\widetilde{X}, \widetilde{Y}\right) \end{array} \tag{26}$$

Proof. Let $\widetilde{X} = x^i\xi_i$, $\widetilde{Y} = y^j\xi_j$ and $\widetilde{Z} = z^k\xi_k$ for $i, j, k \in \{1,2,3\}$ be the tangent vectors at any point (e_1, e_2) of $T_\varepsilon H_1^2$ where $\{\xi_1, \xi_2, \xi_3\}$ is a orthonormal base of $T_{(e_1,e_2)}T_\varepsilon H_1^2$. For all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in T_{(e_1,e_2)}T_\varepsilon H_1^2$ and any $\alpha, \beta \in IR$, we get

$$\begin{aligned} g^S(\alpha\widetilde{X} + \beta\widetilde{Y}, \widetilde{Z}) &= g^S(\{\alpha\left[x^i\xi_i\right] + \beta\left[y^i\xi_i\right]\}, z^j\xi_j) \\ &= \alpha g^S(\widetilde{X}, \widetilde{Z}) + \beta g^S(\widetilde{Y}, \widetilde{Z}). \end{aligned}$$

Similarly we get $g^S(\widetilde{X}, \alpha\widetilde{Y} + \beta\widetilde{Z}) = \alpha g^S(\widetilde{X}, \widetilde{Y}) + \beta g^S(\widetilde{X}, \widetilde{Z})$. Thus g^S is bilinear transformation. Since the follow equality is hold

$$g^S(\widetilde{X}, \widetilde{Y}) = g^S(x^i\xi_i, y^j\xi_j) = y^i x^i \varepsilon_i = g^S(\widetilde{Y}, \widetilde{X}).$$

g^S must be symmetric map. Finally, g^S is a non degenerate map because g^S provides

$$g^S(\widetilde{X}, \widetilde{Y}) = 0 \Longleftrightarrow \widetilde{Y} = 0 \quad \text{for all } \widetilde{X} \in T_{e_1}H_1^2.$$

Since g^S is non degenerate, symmetric, bilinear form, g^S is a semi Riemann metric on the tangent sphere bundle with radius ε $T_\varepsilon H_1^2$. g^S is called as the Sasaki semi Riemann metric on $T_\varepsilon H_1^2$. Moreover $\left(T_\varepsilon H_1^2, g^S\right)$ is also called as the Sasaki semi Riemann manifold.

Theorem 3.5. *Let $T_\varepsilon H_1^2$ be the tangent sphere bundle with radius ε of pseudo hyperbolic 2-space and $\{\xi_1, \xi_2, \xi_3\}$ be a orthonormal base of $T_{(e_1,e_2)}T_\varepsilon H_1^2$ with respect to Sasaki semi Riemann metric g^S. Then ξ_1, ξ_2 are the space like unit vectors, ξ_3 is a the time like unit vector and $\frac{1}{\sqrt{2}}\{\xi_1 + \xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_2 + \xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_1 - \xi_2\}$ are the light like vectors.*

Proof. The image of the unit tangent vectors ξ_1 and ξ_2, ξ_3 given by (3.5) under the Sasaki semi Riemann metric g^S are

$$\begin{aligned} g^S(\xi_1,\xi_1) &= \cos^2\omega g^S(\frac{\partial}{\partial a},\frac{\partial}{\partial a}) - \frac{\sin^2\omega}{\sinh^2 a} g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) + \\ &- \frac{\sin^2\omega}{\sinh^2 a}\cosh a g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\omega}) + \coth^2 a \sin^2\omega g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} g^S(\xi_2,\xi_2) &= \sin^2\omega g^S(\frac{\partial}{\partial a},\frac{\partial}{\partial a}) - \frac{\cos^2\omega}{\sinh^2 a} g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) \\ &- \frac{\cos^2\omega}{\sinh^2 a}\cosh a g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\omega}) + \coth^2 a \cos^2\omega g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) \\ &= 1, \\ g^S(\xi_3,\xi_3) &= g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) = -1. \end{aligned}$$

As a consequence $g^S(\xi_3,\xi_3) = -1$ and $g^S(\xi_1,\xi_1) = g^S(\xi_2,\xi_2) = 1$, ξ_3 is a the time like unit vectors and ξ_1, ξ_2 are the space like unit vectors with respect to g^S. Furthermore, it is seen that $\frac{1}{\sqrt{2}}\{\xi_1+\xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_2+\xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_1-\xi_2\}$ are the light like vectors with respect to g^S, easily.
Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε of pseudo hyperbolic 2-space has the following matrix representation:

$$g_{\alpha\beta} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -\cosh a \\ 0 & -\cosh a & -1 \end{pmatrix} \text{ for } \alpha,\beta \in \{1,2,3\}. \tag{27}$$

The inverse matrix of $g_{\alpha\beta}$ is given by

$$g^{\beta\alpha} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & \csc h^2 a & -\csc ha \coth a \\ 0 & -\csc ha\coth a & \csc h^2 a \end{pmatrix}. \tag{28}$$

Theorem 3.6. *Let $(T_\varepsilon H_1^2, g^S)$ be Sasaki semi Riemann manifold. Let ∇ be Levi Civita connection of $(T_\varepsilon H_1^2, g^S)$ and $\Gamma^\gamma_{\alpha\beta}; \alpha,\beta,\gamma \in \{1,2,3\}$ be coefficients of the Christoffel symbols with related to ∇. Then the non-zero the Christoffel symbols of $(T_\varepsilon H_1^2, g^S)$ are given by*

$$\begin{array}{ll} \Gamma^1_{23} = \frac{1}{2}\sinh a, & \\ \Gamma^2_{12} = \frac{1}{2}\coth a, & \Gamma^2_{13} = -\frac{1}{2}\csc ha, \\ \Gamma^3_{12} = -\frac{1}{2}\csc ha, & \Gamma^3_{13} = \frac{1}{2}\coth a, \end{array} \tag{29}$$

where $\Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha}$ for all $\alpha,\beta,\gamma \in \{1,2,3\}$.

Proof. On the Sasaki semi Riemann manifold $(T_\varepsilon H_1^2, g^S)$ there is a unique connection ∇ such that ∇ is torsion free and compatible with semi Riemann metric g^S. This connection is called as Levi Civita connection and characterized by the Kozsul formula:

$$2g^S\left(\nabla_{\partial_a}\partial_\theta,\partial_\omega\right)=\partial_a g^S\left(\partial_\theta,\partial_\omega\right)+\partial_\theta g^S\left(\partial_\omega,\partial_a\right)-\partial_\omega g^S\left(\partial_a,\partial_\theta\right)+$$
$$-g^S\left(\left[\partial_a,\partial_\theta\right],\partial_\omega\right)+g^S\left(\left[\partial_\theta,\partial_\omega\right],\partial_a\right)+g^S\left(\left[\partial_\omega,\partial_a\right],\partial_\theta\right),$$

where $\partial_a=\frac{\partial}{\partial a}=\partial_1,\partial_\theta=\frac{\partial}{\partial\theta}=\partial_2$ and $\partial_\omega=\frac{\partial}{\partial\omega}=\partial_3$. Since ∇ is symmetric, $\left[\partial_a,\partial_\theta\right],\left[\partial_\theta,\partial_\omega\right],\left[\partial_\omega,\partial_a\right]$ must be zero. If we get $\nabla_{\partial_1}\partial_2=\Gamma^1_{12}\partial_1+\Gamma^2_{12}\partial_2+\Gamma^3_{12}\partial_3$, from Kozsul formula, Christoffel symbols are obtained as follows:

$$\Gamma^1_{12}=\frac{1}{2}g^{1k}\left(\partial_1 g_{k2}+\partial_2 g_{1k}-\partial_k g_{12}\right)=0,$$
$$\Gamma^2_{12}=\frac{1}{2}g^{2k}\left(\partial_1 g_{k2}+\partial_2 g_{1k}-\partial_k g_{12}\right)=\frac{1}{2}\coth a,$$
$$\Gamma^3_{12}=\frac{1}{2}g^{3k}\left(\partial_1 g_{k2}+\partial_2 g_{1k}-\partial_k g_{12}\right)=-\frac{1}{2}\csc ha,$$

where $k\in\{1,2,3\}$. Other Christoffel symbols can be obtained by using the similar method.

Theorem 3.7. *Let $\left(T_\varepsilon H^2_1,g^S\right)$ be Sasaki semi Riemann manifold and $c:t\in R\to c(t)=(a(t),\theta(t),\omega(t))$ be a curve on the tangent sphere bundle with radius ε $T_\varepsilon H^2_1$. c is geodesic if and only if the following second order differential equation's system must be provided:*

$$\begin{aligned}\ddot{a}+\sinh a\dot{\theta}\dot{\omega}&=0,\\ \ddot{\theta}+\coth a\dot{a}\dot{\theta}-\csc ha\dot{a}\dot{\omega}&=0,\\ \ddot{\omega}-\csc ha\dot{a}\dot{\theta}+\cot ha\dot{a}\dot{\omega}&=0.\end{aligned}\tag{30}$$

Proof. $c(t)=(a(t),\theta(t),\omega(t))$ is geodesic if and only if $\nabla_{\dot{c}}\dot{c}$ must be zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega$, $\nabla_{\dot{c}}\dot{c}$ is equal to

$$\nabla_{\dot{c}}\dot{c}=\nabla_{\dot{a}\partial_a}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega\right)+\nabla_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega\right)+$$
$$+\nabla_{\dot{\omega}\partial_\omega}\left(\dot{a}\partial_a+\dot{\theta}\partial_\theta+\dot{\omega}\partial_\omega\right).$$

Therefore we get

$$\begin{aligned}\nabla_{\dot{c}}\dot{c}=&\ddot{a}\partial_a+\dot{a}\dot{\theta}\left(\frac{1}{2}\coth a\partial_\theta-\frac{1}{2}\csc ha\partial_\omega\right)\\&+\dot{a}\dot{\omega}\left(-\frac{1}{2}\csc ha\partial_\theta+\frac{1}{2}\coth a\partial_\omega\right)+\ddot{\theta}\partial_\theta+\\&+\dot{a}\dot{\theta}\left(\frac{1}{2}\coth a\partial_\theta-\frac{1}{2}\csc ha\right)\partial_\omega+\dot{\theta}\dot{\omega}\sinh a\partial_a+\\&+\dot{a}\dot{\omega}\left(-\frac{1}{2}\csc ha\partial_\theta+\frac{1}{2}\coth a\partial_\omega\right)+\ddot{\omega}\partial_\omega.\end{aligned}$$

If we organize $\nabla_{\dot{c}}\dot{c}$,

$$\nabla_{\dot{c}}\dot{c}=\left(\ddot{a}+\sinh a\dot{\theta}\dot{\omega}\right)\partial_a+\left(\ddot{\theta}+\coth a\dot{a}\dot{\theta}-\csc ha\dot{a}\dot{\omega}\right)\partial_\theta$$
$$+\left(\ddot{\omega}-\csc ha\dot{a}\dot{\theta}+\cot ha\dot{a}\dot{\omega}\right)\partial_\omega.$$

it can be seen that the claim of the theorem is true.

Theorem 3.8. *The non-zero components of the Riemann curvature tensor of the semi Riemann manifold* $\left(T_\varepsilon H_1^2, g^S\right)$ *are given by*

$$R^1_{321} = -\tfrac{1}{4}\cosh a \quad R^1_{231} = -\tfrac{1}{4}\cosh a, \quad R^1_{331} = -\tfrac{1}{4}, \quad R^1_{212} = \tfrac{1}{4},$$
$$R^2_{232} = -\tfrac{1}{4}\cosh a, \quad R^2_{332} = -\tfrac{1}{4}, \quad R^2_{112} = \tfrac{1}{4}, \quad R^2_{323} = \tfrac{1}{4}, \quad R^2_{332} = -\tfrac{1}{4},$$
$$R^3_{232} = \tfrac{1}{4} \quad R^3_{323} = -\tfrac{1}{4}\cosh a, \quad R^3_{113} = \tfrac{1}{4}, \quad R^3_{223} = -\tfrac{1}{4}, \quad R^3_{121} = 0,$$

where $R^\mu_{\alpha\beta\gamma} = -R^\mu_{\alpha\gamma\beta}$ *for* $\alpha, \beta, \gamma \in \{1,2,3\}$.

Proof. Let $\Gamma^\gamma_{\alpha\beta}$, $\alpha, \beta, \gamma \in \{1,2,3\}$ be the Christoffel symbols of the semi Riemann manifold $\left(T_\varepsilon H_1^2, g^S\right)$ and $R^\mu_{\alpha\beta\gamma}$, $\alpha, \beta, \gamma \in \{1,2,3\}$ be the components of the Riemann curvature tensor. By using the known formula of the Riemann curvature tensor

$$R^\mu_{\alpha\beta\gamma} = \partial_\beta \Gamma^\mu_{\alpha\gamma} - \partial_\gamma \Gamma^\mu_{\alpha\beta} + \Gamma^\mu_{\delta\beta}\Gamma^\delta_{\alpha\gamma} - \Gamma^\mu_{\delta\gamma}\Gamma^\delta_{\alpha\beta},$$

and the Christoffel symbols of $\left(T_\varepsilon H_1^2, g^S\right)$ in (3.9), it is seen that the claim of the theorem is correct, easily.

4. MAIN RESULT

In this section, the obtained data in second and third section are summarized. Furthermore, two theorem with related to the relations between geodesics of H_1^2 and $T_\varepsilon H_1^2$ are given. Finaly, the particular examples of the time like, the light like and the space like geodesics on the surface H_1^2 are given and the relation between these geodesics and geodesics of $T_\varepsilon H_1^2$ are given.

In the second section, we obtained a differential equation's system which gives geodesic of the surface H_1^2 as follows:

$$\begin{aligned} \ddot{a} - \sinh a \cosh a \dot{\theta}^2 &= 0, \\ \ddot{\theta} + 2\coth a \dot{a}\dot{\theta} &= 0, \end{aligned}$$

and the general equation characterizing the time like, the light like and the space like geodesics on H_1^2 are obtained as follows:

$$\frac{da}{d\theta} = \frac{\sqrt{\varepsilon \sinh^4 a - k^2 \sinh^2 a}}{k}.$$

Furthermore, the time like geodesic equations are cross-section curves of the pseudo hyperbolic space H_1^2 with the following surfaces given by generalized coordinates (a, θ) and cartesian coordinates (x_1, x_2, x_3), respectively as follows:

$$\sqrt{1 + k^2 \csc h^2 a} + k \coth a = \cos\theta - i \sin\theta\,,$$

and

$$\left(x_2 - \sqrt{x_2^2 + x_3^2 + k^2} - k x_1\right)^2 + x_3^2 = 0.$$

The following curve on H_1^2 can be given by an example to the time like geodesic:

$$c(t) = (t, \frac{5t^2-1}{4t}, \frac{3t^2+1}{4t}i),$$

for $k = 1$.
The light like geodesic equations are cross-section curves of the pseudo hyperbolic space H_1^2 with the following surfaces given by generalized coordinates (a, θ) and cartesian coordinates (x_1, x_2, x_3), respectively as follows:

$$\csc ha - \coth a = \cos\theta + i\sin\theta, \ \ (x_1 - x_2 - 1)^2 + x_3^2 = 0.$$

The following curve on H_1^2 can be given by an example to the light like geodesic:

$$c(t) = (t, t, i).$$

The space like geodesic equations are found with respect to generalized coordinates (a, θ) and cartesian coordinates (x_1, x_2, x_3), respectively as follows:

$$\frac{\sqrt{1 - k^2 \csc h^2 a}}{\sqrt{1+k^2}} = \sin\theta, \ x_2^2 = k^2(x_3^2 + 1).$$

The following curve on H_1^2 can be given by an example to the space like geodesic:

$$c(t) = (\sqrt{2}\sqrt{t^2+1}, \sqrt{t^2+1}, t),$$

for $k = 1$.
In the third section, we calculated the line element on the tangent sphere bundle with radius ε $T_\varepsilon H_1^2$ of the pseudo hyperbolic 2-space H_1^2 with respect to the induced coordinates (a, θ, ω) as follows:

$$d\sigma^2 = (da)^2 - (d\theta)^2 - 2\cosh a d\theta d\omega - (d\omega)^2,$$

and we found out the connection coefficients of the Levi Civita connection of the semi Riemann manifold $\left(T_\varepsilon H_1^2, g^S\right)$ as follows:

$$\begin{array}{ll} \Gamma_{23}^1 = \frac{1}{2}\sinh a, & \\ \Gamma_{12}^2 = \frac{1}{2}\coth a, & \Gamma_{13}^2 = -\frac{1}{2}\csc ha, \\ \Gamma_{12}^3 = -\frac{1}{2}\csc ha, & \Gamma_{13}^3 = \frac{1}{2}\coth a. \end{array}$$

Furthermore, we calculated the general geodesic equations of the semi Riemann manifold $\left(T_\varepsilon H_1^2, g^S\right)$ as follows:

$$\begin{aligned} \ddot{a} + \sinh a\dot{\theta}\dot{\omega} &= 0, \\ \ddot{\theta} + \coth a\dot{a}\dot{\theta} - \csc ha\dot{a}\dot{\omega} &= 0, \\ \ddot{\omega} - \csc ha\dot{a}\dot{\theta} + \cot ha\dot{a}\dot{\omega} &= 0. \end{aligned}$$

If we consider with together two differential equation's systems which give geodesics on the surface H_1^2 and its tangent sphere bundle with radius ε $T_\varepsilon H_1^2$ we can obtain the following two theorem:

Theorem 4.1. *Let (a, θ) is generalized coordinates of H_1^2 and (a, θ, ω) is the local coordinates of $T_\varepsilon H_1^2$. The surface H_1^2 is totally geodesic sub-manifold of the tangent sphere bundle with radius ε $T_\varepsilon H_1^2$ if and only if $\dot{\omega}$ is equal to $-\cosh a\dot{\theta}$.*

Proof. If we put $-\cosh a\dot{\theta}$ instead of $\dot{\omega}$ in the differential equations system given by (29) we can get the following the differential equations system:

$$\ddot{a} - \sinh a \cosh a \left(\dot{\theta}\right)^2 = 0,$$
$$\ddot{\theta} + 2\coth a\dot{a}\dot{\theta} = 0.$$
$$\dot{\omega} + \cosh a\dot{\theta} = 0$$

The solution curves of the above differential equations system give the horizontal geodesics of $T_\varepsilon H_1^2$, which are obtained by parallel translations of the unit vectors passing through geodesics given by (15) and (16) on the surface H_1^2. Since lifted curves with parallel vector field of each geodesic of the surface H_1^2 are also a geodesics of $T_\varepsilon H_1^2$. If we put $-cosha\dot{\theta}$ the instead of $\dot{\omega}$ in the Sasaki Riemann metric on $T_\varepsilon H_1^2$, we obtain the following equation:

$$\begin{aligned} d\sigma^2 &= (da)^2 - (d\theta)^2 + 2\cosh a\,(d\theta)^2 - \cosh^2 a\,(d\theta)^2 \\ &= (da)^2 + \sinh^2 a\,(d\theta)^2 \end{aligned}$$

Thus, we see that the time like, the light like, and the space like geodesics of the pseudo hyperbolic 2-space H_1^2 is the time like, the light like, and the space like geodesics of the tangent sphere bundle $T_\varepsilon H_1^2$. The surface H_1^2 is also submanifold of $T_\varepsilon H_1^2$ (see [7]), the surface H_1^2 is totally geodesic submanifold of $T_\varepsilon H_1^2$.

Theorem 4.2. *The horizontal lifting operation from the surface H_1^2 to $T_\varepsilon H_1^2$ preserves the causal characters of geodesics.*

Proof. Assuming that $C : t \to C(t) = (a(t), \theta(t), \omega(t))$ is a horizontal geodesic curve and $c : t \to c(t) = (a(t), \theta(t))$ is natural projection to the surface H_1^2 with $\pi \circ C = c$ where $\pi : T_\varepsilon H_1^2 \to H_1^2$ is a canonical projection. Since $g^S(X^H, X^H) = g(X, X)$ for $X^H = \dot{C}(t)$ and $X = \dot{c}(t)$ When a geodesic on the surface H_1^2 is the time like or the space like or the light like geodesic, the horizontal lifted to $T_\varepsilon H_1^2$ of this geodesic must be respectively the time like or the space like or the light like geodesic. Thus, horizontal lifting operation from the surface H_1^2 to $T_\varepsilon H_1^2$ preserves the causal characters of geodesics.
In the third section, we get the non-zero components of the Riemann curvature tensor of the semi Riemann manifold $\left(T_\varepsilon H_1^2, g^S\right)$ as follows:

$$R^1_{321} = -\tfrac{1}{4}\cosh a \quad R^1_{231} = -\tfrac{1}{4}\cosh a, \quad R^1_{331} = -\tfrac{1}{4}, \quad R^1_{212} = \tfrac{1}{4},$$
$$R^2_{232} = -\tfrac{1}{4}\cosh a, \quad R^2_{332} = -\tfrac{1}{4}, \quad R^2_{112} = \tfrac{1}{4}, \quad R^2_{323} = \tfrac{1}{4}, \quad R^2_{332} = -\tfrac{1}{4},$$
$$R^3_{232} = \tfrac{1}{4} \quad R^3_{323} = -\tfrac{1}{4}\cosh a, \quad R^3_{113} = \tfrac{1}{4}, \quad R^3_{223} = -\tfrac{1}{4}, \quad R^3_{121} = 0.$$

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References

[1] Ayhan, I., *Geodesics on The Tangent Sphere Bundle of 3-Sphere*, International Electronic Journal of Geometry, 6(2),(2013), 100-109.

[2] Ayhan I.,*On The Tangent Sphere Bundle with The Sasaki semi Riemann Metric of a Space Form,* GJARCMG, **3(1)** (2014), 25–35.

[3] Free, P., *Introduction to General Relativity*, http://personalpages.to.infn.it/~fre/PPT/virgolect.ppt.3, 2003.

[4] Kilingenberg, W., and Sasaki, S., *On the Tangent Sphere Bundle of a 2-Sphere*, Tohuku Math. Journ. 27(1975), 49–56.

[5] Nagy, P., T., *On the Tangent Sphere Bundle of a Riemann 2- manifold*, Tohuku Math. Journ. 29(1977), 203-208.

[6] O'neill, B., *Semi-Riemnn Geometry with Applications to Relativity*, Acedemic Press, New york, 1997.

[7] Sasaki, S., *On the Differential Geometry of Tangent Bundle of Riemann Manifolds II,* Tohuku Math. Journ. 14(1962), 146-155.

[8] Sasaki, S., *Geodesics on the Tangent Sphere Bundles over Space Forms*, Journ. Für die reine und angewandte math. 288(1976), 106-120.

Department of Mathematics Education,
PAMUKKALE university,
DENIZLI, 20070, TURKEY.
E-mail address: iyusufayhan@gmail.com