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STATISTICAL IMMERSIONS BETWEEN STATISTICAL MANIFOLDS OF CONSTANT CURVATURE

CHOL RIM MIN, SONG OK CHOE, YUN HO AN

ABSTRACT. The condition for the curvature of a statistical manifold to admit a kind of standard hypersurface is given. We study the statistical hypersurfces of some types of the statistical manifolds (M, ∇, g), which enable $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.

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1. INTRODUCTION

Since Lauritzen introduced the notation of statistical manifolds in 1987 [5], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry (see, for example, [2,4,8]). In this paper we study the hypersurface of statistical manifolds.

Let M be an n-dimensional manifold, ∇ a torsion-free affine connection on M, g a Riemannian metric on M, and R a curvature tensor field on M. We denote by TM the set of vector fields on M, and by $TM^{(r,s)}$ the set of tensor fields of type (r,s) on M.

Definition 1.1. A pair (∇, g) is called a *statistical structure* on M if (M, ∇, g) is a statistical manifold, that is,∇ is a torsion-free affine connection and for all $X, Y, Z \in T(M)$, $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$.

Let ∇° be a Levi-Civita connection of g. Certainly, a pair (∇°, g) is a statistical structure, which is called a Riemannian statistical structure or a trivial statistical structure (see [3]). On the other hand, the statistical structure is also introduced from affine differential geometry which was proposed by Blasche (see [6]). Recently the relation between statistical structures and Hessian geometry has been studied (see [3,7]).

For all $\alpha \in \mathbf{R}$, a connection $\nabla^{(\alpha)}$ is defined by

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$$

Coresponding author Chol Rim Min, e-mail: mincholrim@yahoo.com.

where ∇ and ∇^* are dual connections on M. We study a statistical hypersurface of a statistical manifold (M, ∇, g) which enables $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.
In section 3, a statistical manifold (M, ∇, g), which enables $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature, is considered. In section 4, we study characteristics of statistical immersions between statistical manifolds (M, ∇, g) which enable $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbf{R}$ to admit the structure of a constant curvature.

2. Preliminaries

A statistical manifold (M, ∇, g) is said to be of constant curvature $k \in \mathbf{R}$ if

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}, \forall X, Y, Z \in TM \tag{2.1}$$

holds, where R is the curvature tensor field of ∇. A pair (∇, g) is called a Hessian structure if a statistical manifold (M, ∇, g) is of constant curvature 0.
A Riemannian metric g on a flat manifold (M, g) is called a Hessian metric if g can be locally expressed by

$$g = Dd\varphi,$$

that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j},$$

where $\{x^1, \cdots, x^n\}$ is an affine coordinate system with respect to ∇. Then (M, ∇, g) is called a Hessian manifold (see [7]).
Let (M, ∇, g) be a Hessian manifold and $K(X, Y) := \nabla_X Y - \nabla^\circ_X Y$ be the difference tensor between the Levi-Civita connection ∇° of g and ∇. A covariant differential of differential tensor K is called a Hessian curvature tensor for (∇, g). A Hessian manifold (M, ∇, g) is said to be of constant Hessian curvature $c \in \mathbf{R}$ if

$$(\nabla_X K)(Y, Z) = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}, \forall X, Y, Z \in TM$$

holds (see [7]).

Example 2.1. ([3])
Let $(H, \tilde{g})$ be the upper half space:

$$H := \left\{ y = (y^1, \cdots, y^{n+1})^T \in \mathbf{R}^{n+1} \middle| y^{n+1} > 0 \right\}, \tilde{g} := (y^{n+1})^{-2} \sum_{i=1}^{n+1} dy^i dy^i.$$

We define an affine connection $\tilde{\nabla}$ on H by the following relations:

$$\tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^{n+1}} = (y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}}, \tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij} (y^{n+1})^{-1} \frac{\partial}{\partial y^{n+1}},$$

$$\tilde{\nabla}_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^{n+1}} = \tilde{\nabla}_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^j} = 0,$$

where $i, j = 1, \cdots, n$. Then $(H, \tilde{\nabla}, \tilde{g})$ is a Hessian manifold of constant Hessian curvature 4.

Let $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical manifold and $f : M \to \tilde{M}$ be an immersion. We define g and ∇ on M by

$$g = f^*\tilde{g}, \quad g(\nabla_X Y, Z) = \tilde{g}(\tilde{\nabla}_X f_* Y, f_* Z), \quad \forall X, Y, Z \in TM.$$

Then the pair (∇, g) is a statistical structure on M, which is called the one by f from $(\tilde{\nabla}, \tilde{g})$ (see [3]).

Let (M, ∇, g) and $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ be two statistical manifolds. An immersion $f : M \to \tilde{M}$ is called a statistical immersion if (∇, g) coincides with the induced statistical structure (see [3]).

Let $f : (M, \nabla, g) \to (\tilde{M}, \tilde{\nabla}, \tilde{g})$ be a statistical immersion of codimension one, and ξ a unit normal vector field of f. Then we define $h, h^* \in TM^{(0,2)}$, $\tau, \tau^* \in TM^*$ and $A, A^* \in TM^{(1,1)}$ by the following Gauss and Weingarten formulae:

$$\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \quad \tilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi,$$

$$\tilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + h^*(X, Y)\xi, \quad \tilde{\nabla}^*_X \xi = -f_* A X + \tau(X)\xi, \quad \forall X, Y \in TM,$$

where $\tilde{\nabla}^*$ is the dual connection of $\tilde{\nabla}$ with respect to $\tilde{g}$.

In addition, we define $II \in TM^{(0,2)}$ and $S \in TM^{(1,1)}$ by using the Riemannian Gauss and Weingarten formulae:

$$\tilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + II(X, Y)\xi, \quad \tilde{\nabla}^*_X \xi = -f_* S X.$$

For more details on the Gauss, Codazzi and Ricci formulae on statistical hypersurfaces, we refer to [3].

3. The condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$

In this section we consider a condition that a statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$.

Theorem 3.1. *A statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature for any $\alpha \in \mathbf{R}$ iff there exist $\alpha_1, \alpha_2 \in \mathbf{R}(|\alpha_1| \neq |\alpha_2|)$ such that statistical manifolds $(M, \nabla^{(\alpha_1)}, g)$ and $(M, \nabla^{(\alpha_2)}, g)$ are of constant curvature.*

Proof. Necessity is obvious. We find sufficiency. Without loss of generality, we assume $\alpha_1 \neq 0$. Then since

$$\nabla^{(\alpha)} = \frac{\alpha_1 + \alpha}{2\alpha_1} \nabla^{(\alpha_1)} + \frac{\alpha_1 - \alpha}{2\alpha_1} \nabla^{(-\alpha_1)}$$

holds for all $\alpha \in \mathbf{R}$, the following relation

$$\begin{aligned} R^{(\alpha)}(X, Y)Z = & \frac{\alpha_1 + \alpha}{2\alpha_1} R^{(\alpha_1)}(X, Y)Z + \frac{\alpha_1 - \alpha}{2\alpha_1} R^{(-\alpha_1)}(X, Y)Z \\ & + \frac{\alpha_1^2 - \alpha^2}{4\alpha_1^2} [K(Y, K(Z, X)) - K(X, K(Y, Z))] \end{aligned}$$

holds, where $K(X, Y) := \nabla_X Y - \nabla^{\circ}_X Y$ is the difference tensor field of a statistical manifold.

From the relations

$$R^{(\alpha_1)}(X, Y)Z = k_1\{g(Y, Z)X - g(X, Z)Y\},$$

$$R^{(\alpha_2)}(X,Y)Z = k_2\{g(Y,Z)X - g(X,Z)Y\},$$

the relation

$$R^{(\alpha)}(X,Y)Z = \frac{k_2\alpha_1^2 - k_1\alpha_2^2 + (k_1-k_2)\alpha^2}{\alpha_1^2-\alpha_2^2}\{g(Y,Z)X - g(X,Z)Y\}$$

holds, that is, a statistical manifold $(M,\nabla^{(\alpha)},g)$ is of constant curvature $\frac{k_2\alpha_1^2-k_1\alpha_2^2+(k_1-k_2)\alpha^2}{\alpha_1^2-\alpha_2^2}$.

Example 3.1. Let (M,g) be a family of normal distributions:

$$M := \left\{ p(x,\theta) \,\middle|\, p(x,\theta) = \frac{1}{\sqrt{2\pi(\theta^2)^2}}\exp\left\{-\frac{1}{2(\theta^2)^2}(x-\theta^1)^2\right\}\right\}, \quad g := 2(\theta^2)^{-2}\sum d\theta^i d\theta^i,$$

$$x\in\mathbf{R},\quad \theta^1\in\mathbf{R},\quad \theta^2>0.$$

We define an $\alpha-$ connection by the following relations:

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^1} = (-1+2\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2},\quad \nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^2} = (1+\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2},$$

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^2} = \nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^1} = 0.$$

Then the statistical manifold $(M,\nabla^{(0)},g)$ is of constant curvature $(-\frac{1}{2})$, and the statistical manifold $(M,\nabla^{(1)},g)$ is of constant curvature 0. Hence for all $\alpha\in\mathbf{R}$, the statistical manifold $(M,\nabla^{(\alpha)},g)$ is of constant curvature $\frac{\alpha^2-1}{2}$.

Example 3.2. Let (M,g) be a family of random walk distributions ([1]):

$$M := \left\{ p(x;\theta^1,\theta^2) \,\middle|\, p(x;\theta^1,\theta^2) = \sqrt{\frac{\theta^2}{2\pi x}}\exp\left\{-\frac{\theta^2 x}{2}+\frac{\theta^2}{\theta^1}-\frac{\theta^2}{2(\theta^1)^2x}\right\},\quad x,\mu,\lambda>0\right\},$$

$$g := \frac{\theta^2}{(\theta^1)^3}(d\theta^1)^2 + \frac{1}{2(\theta^2)^2}(d\theta^2)^2.$$

We define an $\alpha-$ connection by the following relations:

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^1} = \frac{-3(1+\alpha)}{2}(\theta^1)^{-1}\frac{\partial}{\partial\theta^1} + (-1+\alpha)(\theta^1)^{-3}(\theta^2)^2\frac{\partial}{\partial\theta^2},$$

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^2} = \nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^1} = \frac{(1+\alpha)}{2}(\theta^2)^{-1}\frac{\partial}{\partial\theta^1},$$

$$\nabla^{(\alpha)}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^2} = (-1+\alpha)(\theta^2)^{-1}\frac{\partial}{\partial\theta^2}.$$

Then the statistical manifold $(M,\nabla^{(0)},g)$ is of constant curvature $(-\frac{1}{2})$, and the statistical manifold $(M,\nabla^{(1)},g)$ is of constant curvature 0. Hence for all $\alpha\in\mathbf{R}$, the statistical manifold $(M,\nabla^{(\alpha)},g)$ is of constant curvature $\frac{\alpha^2-1}{2}$.

Theorem 3.1 implies the following fact.

Corollary 3.1. *If there exist $\alpha_1,\alpha_2\in\mathbf{R}(|\alpha_1|\neq|\alpha_2|)$ such that the statistical manifold $(M,\nabla^{(\alpha_1)},g)$ is of constant curvature k_1 and the statistical manifold $(M,\nabla^{(\alpha_2)},g)$ is of constant curvature k_2, and $k_1\neq k_2$, then for $\alpha\in\mathbf{R}$ satisfying that $\alpha^2 = (k_2\alpha_1^2 - k_1\alpha_2^2)/(k_2-k_1)$, the statistical manifold $(M,\nabla^{(\alpha)},g)$ is flat.*

Example 3.3. $k_1 = -1/2$, $k_2 = 0$, $\alpha_1 = 0$ and $\alpha_2 = 1$ hold in example 3.1 and example 3.2. Hence for $\alpha \in \mathbf{R}$ satisfying that $\alpha^2 = 1$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is flat.

Theorem 3.2. *If the Hessian manifold (M, ∇, g) is of constant Hessian curvature, then for all $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature.*

Proof. If the Hessian manifold (M, ∇, g) is of constant Hessian curvature, then for all $X, Y, Z \in TM$,

$$(\nabla K)(Y, Z; X) = -\frac{c}{2}\{g(X,Y)Z + g(X,Z)Y\}, c \in \mathbf{R}$$

holds. On the other hand, the curvature tensor R° of Levi-Civita connection ∇° is written by

$$\begin{aligned} R^\circ(X,Y)Z = R(X,Y)Z &- (\nabla K)(Y,Z;X) + (\nabla K)(Z,X;Y) \\ &+ K(X, K(Y,Z)) - K(Y, K(Z,X)), \end{aligned}$$

where R is the curvature tensor of ∇ and $K(X,Y) = \nabla_X Y - \nabla^\circ_X Y$ is difference tensor. Then

$$\begin{aligned} &(\nabla K)(Y,Z;X) - (\nabla K)(Z,X;Y) \\ &\quad = 2\{K(X, K(Y,Z)) - K(Y, K(Z,X))\} + \frac{1}{2}\{R(X,Y)Z - R^*(X,Y)Z\} \end{aligned}$$

implies

$$R^\circ(X,Y)Z = -\frac{c}{4}\{g(Y,Z)X - g(X,Z)Y\},$$

where R^* is curvature tensor of dual connection ∇^*, that is, the statistical manifold (M, ∇°, g) is of constant curvature. On the other hand, the statistical manifold (M, ∇, g) is flat, that is, constant curvature 0. Therefore we finish the proof of theorem by applying Theorem 3.1.

Hitherto we found some conditions that for any $\alpha \in \mathbf{R}$, the statistical manifold $(M, \nabla^{(\alpha)}, g)$ is of constant curvature.

4. The hypersurfaces of statistical manifolds of constant curvature

We consider statistical hypersurfaces of some type of statistical manifolds, which enable for any $\alpha \in \mathbf{R}$ a statistical manifold $(M, \nabla^{(\alpha)}, g)$ to be of constant curvature.

Theorem 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k, $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a statistical manifold of constant curvature $\tilde{k}$ with a Riemannian manifold of constant curvature $\overset{\circ}{\tilde{k}}$ $(\neq \tilde{k})$ $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, and $f : M \to \tilde{M}$ a statistical immersion of codimension one. Then $f : M \to \tilde{M}$ is equiaffine, that is, τ^* vanishes.*

Proof. If $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold of constant curvature $\tilde{k}$ with a Riemannian manifold of constant curvature $\overset{\circ}{\tilde{k}}$ $(\neq \tilde{k})$ $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, the following equation

$$\begin{aligned} &(\tilde{\nabla}_X \tilde{K})(f_*Y, f_*Z) - (\tilde{\nabla}_Y \tilde{K})(f_*X, f_*Z) \\ &\qquad = 2\{\tilde{R}(f_*X, f_*Y)f_*Z - \tilde{R}^\circ(f_*X, f_*Y)f_*Z\} \\ &\qquad = 2(\tilde{k} - \overset{\circ}{\tilde{k}})\{\tilde{g}(f_*Y, f_*Z)f_*X - \tilde{g}(f_*X, f_*Z)f_*Y\} \end{aligned} \tag{4.1}$$

holds by Eq.(2.2) and Eq.(2.3) in [3]. By above equation and equation Eq.(3.6) in [3], we have

$$\begin{aligned}
-2(\tilde{k}-\overset{\circ}{\tilde{k}})\{g(Y,Z)X-g(X,Z)Y\} &= (\nabla_X K)(Y,Z)-(\nabla_Y K)(X,Z)\\
&\quad -b(Y,Z)A^*X+b(X,Z)A^*Y+h(X,Z)B^*Y-h(Y,Z)B^*X\\
0 &= (\nabla_X b)(Y,Z)-(\nabla_Y b)(X,Z)+\tau^*(X)b(Y,Z)-\tau^*(Y)b(X,Z)\\
&\quad -\tau^*(Y)h(X,Z)+\tau^*(X)h(Y,Z)\\
0 &= -\tau^*(Y)A^*X+-\tau^*(X)A^*Y-(\nabla_X B^*)Y+(\nabla_Y B^*)X+\tau^*(X)B^*Y-\tau^*(Y)B^*X\\
0 &= -h(X,B^*Y)+h(Y,B^*X)+(\nabla_X\tau^*)(Y)-(\nabla_Y\tau^*)(X)+b(Y,A^*X)-b(X,A^*Y).
\end{aligned} \tag{4.2}$$

By $K=0$, $B^*=A^*-S$ and Gauss equation $(3.3)_1$ in [3], from Eq.$(4.2)_1$, we have

$$\begin{aligned}
-2(\tilde{k}-\overset{\circ}{\tilde{k}})\{g(Y,Z)X-g(X,Z)Y\} &= -b(Y,Z)A^*X+b(X,Z)A^*Y\\
&\quad +h(X,Z)A^*Y-h(X,Z)SY-h(Y,Z)A^*X+h(Y,Z)SX\\
&= -b(Y,Z)A^*X+b(X,Z)A^*Y-h(X,Z)SY+h(Y,Z)SX+\tilde{R}(X,Y)Z-R(X,Y)Z.
\end{aligned}$$

By $b=h-II$, $B^*=A^*-S$ and Riemannian Gauss equation $(3.5)_1$ in [3], we have

$$\begin{aligned}
&-2(\tilde{k}-\overset{\circ}{\tilde{k}})\{g(Y,Z)X-g(X,Z)Y\}\\
&= -h(Y,Z)A^*X+II(Y,Z)A^*X+h(X,Z)A^*Y-II(X,Z)A^*Y\\
&\qquad -h(X,Z)SY+h(Y,Z)SX+\tilde{R}(X,Y)Z-R(X,Y)Z\\
&= -h(Y,Z)B^*X+h(X,Z)B^*Y+II(Y,Z)B^*X+II(Y,Z)SX\\
&\qquad -II(X,Z)B^*Y-II(X,Z)SY+\tilde{R}(X,Y)Z-R(X,Y)Z\\
&= -b(Y,Z)B^*X+b(X,Z)B^*Y+R^\circ(X,Y)Z-\tilde{R}^\circ(X,Y)Z+\tilde{R}(X,Y)Z-R(X,Y)Z.
\end{aligned}$$

Since (M,∇,g) is Riemannian manifold, clearly $R^\circ(X,Y)Z=R(X,Y)Z$. Hence we have

$$0=(\tilde{k}-\overset{\circ}{\tilde{k}})\{g(Y,Z)X-g(X,Z)Y\}-b(Y,Z)B^*X+b(X,Z)B^*Y.$$

And since $b(Y,Z)=g(BY,Z)$, $b(X,Z)=g(BX,Z)$, from above equation we have

$$0=(\tilde{k}-\overset{\circ}{\tilde{k}})\{g(Y,Z)X-g(X,Z)Y\}-g(BY,Z)B^*X+g(BX,Z)B^*Y. \tag{4.3}$$

From Eq.$(4.2)_3$, $B^*=A^*-S$ and Codazzi equation on A we get

$$\begin{aligned}
0 &= -\tau^*(Y)A^*X+\tau^*(X)A^*Y-(\nabla_X A^*)Y+(\nabla_X S)Y+(\nabla_Y A^*)X-(\nabla_Y S)X\\
&\quad +\tau^*(X)B^*Y-\tau^*(Y)B^*X\\
&= (\nabla_X S)Y-(\nabla_Y S)X+\tau^*(X)B^*Y-\tau^*(Y)B^*X
\end{aligned}$$

and by $\nabla=\nabla^\circ$ and Codazzi equation on S, we also get

$$0=\tau^*(X)B^*Y-\tau^*(Y)B^*X. \tag{4.4}$$

From Eq.$(4.2)_4$, $B^*=A^*-S$ and Ricci equation we have

$$b(X,B^*Y)-b(Y,B^*X)=0,$$

and since $b(X,B^*Y)=g(BX,B^*Y)$ and $b(Y,B^*X)=g(BY,B^*X)$, we have

$$g(BX,B^*Y)-g(BY,B^*X)=0.$$

Since $g(BX, B^*Y) = g(B^*Y, BX) = b^*(BX, Y) = g(B^*BX, Y)$, we have

$$0 = -g([B, B^*]X, Y). \tag{4.5}$$

From Eq.(4.5), B and B^* are simultaneously diagonalizable.
In the case that B^* is of the form $\lambda^* I$, we see easily that τ^* vanishes from Eq.(4.4) if $\lambda^* \neq 0$ and $\overset{\circ}{\tilde{k}} = \tilde{k}$ from Eq.(4.3) otherwise. In the case that B^* is not of the form $\lambda^* I$, there are λ_1^*, λ_2^* with $\lambda_1^* \neq \lambda_2^*$ such that $B^* X_j = \lambda_j^* X_j$, where $g(X_i, X_j) = \delta_{ij},\ i, j = 1, 2$. Besides there are λ_1, λ_2 such that $BX_j = \lambda_j X_j$. Eq.(4.3) implies that

$$\begin{aligned}
&(\tilde{k} - \overset{\circ}{\tilde{k}})\{g(X_j, Z)X_i - g(X_i, Z)X_j\} + \lambda_j\lambda_i^* g(X_j, Z)X_i - \lambda_i\lambda_j^* g(X_i, Z)X_j \\
&= (\tilde{k} - \overset{\circ}{\tilde{k}} + \lambda_j\lambda_i^*)g(X_j, Z)X_i - (\tilde{k} - \overset{\circ}{\tilde{k}} + \lambda_i\lambda_j^*)g(X_i, Z)X_j = 0
\end{aligned}$$

and hence $\tilde{k} - \overset{\circ}{\tilde{k}} + \lambda_j\lambda_i^* = \tilde{k} - \overset{\circ}{\tilde{k}} + \lambda_i\lambda_j^* = 0$, which means that

$$\lambda_j\lambda_i^* = \lambda_i\lambda_j^* = -(\tilde{k} - \overset{\circ}{\tilde{k}}) \neq 0.$$

By Eq.(4.4) we have $\lambda_2^*\tau^*(X_1)X_2 - \lambda_1^*\tau^*(X_2)X_1 = 0$, which implies that τ^* vanishes.
Example 4.1. Suppose $\tilde{M}$ be $\mathbf{R}^3$. We define Riemannian metric and an Affine connection by the following relations:

$$\begin{gathered}
\tilde{g} = a\sum d\theta^i d\theta^i, \\
\tilde{\nabla}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^1} = \tilde{b}\frac{\partial}{\partial\theta^1}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^2} = \frac{\tilde{b}}{2}\frac{\partial}{\partial\theta^1}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}}\frac{\partial}{\partial\theta^3} = \frac{\tilde{b}}{2}\frac{\partial}{\partial\theta^1}, \\
\tilde{\nabla}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^2} = \tilde{\nabla}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^1} = \frac{\tilde{b}}{2}\frac{\partial}{\partial\theta^2}, \tilde{\nabla}_{\frac{\partial}{\partial\theta^1}}\frac{\partial}{\partial\theta^3} = \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}}\frac{\partial}{\partial\theta^1} = \frac{\tilde{b}}{2}\frac{\partial}{\partial\theta^2}, \\
\tilde{\nabla}_{\frac{\partial}{\partial\theta^2}}\frac{\partial}{\partial\theta^3} = \tilde{\nabla}_{\frac{\partial}{\partial\theta^3}}\frac{\partial}{\partial\theta^2} = 0.
\end{gathered}$$

Then $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold of constant curvature $-\frac{\tilde{b}^2}{4a}$ with a trivial statistical manifold of constant curvature 0 $(\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$. Suppose M be $\mathbf{R}^2$, and (∇, g) an induced statistical structure from $(\tilde{\nabla}, \tilde{g})$ by an immersion $f : (x, y)(\in \mathbf{R}^2) \mapsto (0, x, y)$. We remark that (M, ∇, g) is a trivial statistical manifold of constant curvature 0.
Theorem 3.2 and Theorem 4.1 imply the following fact.

Corollary 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k, $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature $\tilde{c}$, and $f : M \to \tilde{M}$ a statistical immersion of codimension one. Then $f : M \to \tilde{M}$ is equiaffine, that is, τ^* vanishes.*

We consider a shape operator of statistical immersion of a trivial statistical manifold of constant curvature into a Hessian manifold of constant Hessian curvature.

Lemma 4.1. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k, $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature $\tilde{c}$, and $f : M \to \tilde{M}$ a statistical immersion of codimension one. Then the following holds:*

$$A^* = k\nu\tilde{c}^{-1}I, B^* = -\frac{1}{2}\nu I, h = \tilde{c}\nu^{-1}g, A = \tilde{c}\nu^{-1}I, B = [2\tilde{c}^2 - (2k + \tilde{c})\nu^2](2\nu\tilde{c})^{-1}I.$$

Proof. Combining Eq.(2.3) and Eq.(3.6) in [3] with Eq.(2.1), we have

$$\begin{aligned}
\frac{\tilde{c}}{2}\{g(Y,Z)X-g(X,Z)Y\} &= 2(k-\overset{\circ}{k})\{g(Y,Z)X-g(X,Z)Y\}\\
&\quad -b(Y,Z)A^*X+b(X,Z)A^*Y+h(X,Z)B^*Y-h(Y,Z)B^*X\\
0 &= h(X,K(Y,Z))-h(Y,K(X,Z))+(\nabla_X b)(Y,Z)-(\nabla_Y b)(X,Z)\\
&\quad +\tau^*(X)b(Y,Z)-\tau^*(Y)b(X,Z)-\tau^*(Y)h(X,Z)+\tau^*(X)h(Y,Z) \qquad (4.6)\\
0 &= K(Y,A^*X)-K(X,A^*Y)-\tau^*(Y)A^*X+\tau^*(X)A^*Y\\
&\quad -(\nabla_X B^*)Y+(\nabla_Y B^*)X+\tau^*(X)B^*Y-\tau^*(Y)B^*X\\
0 &= -h(X,B^*Y)+h(Y,B^*X)+(\nabla_X\tau^*)(Y)-(\nabla_Y\tau^*)(X)+b(Y,A^*X)-b(X,A^*Y)
\end{aligned}$$

Taking the trace of $(4.6)_1$ with respect to X, we have

$$-\tilde{c}g(Y,Z) = -trA^*b(Y,Z)+h(B^*Z,Y)+h(B^*Y,Z)$$

and taking the trace of $(4.6)_1$ with respect to Y, we have

$$-\frac{\tilde{c}}{2}(n+1)g(X,Z) = -b(A^*X,Z)+h(X,B^*Z)+trB^*h(X,Z).$$

Using the above equation and Eq.$(4.6)_4$, we have

$$\begin{aligned}
-\frac{\tilde{c}}{2}(n+2)g(X,Y) &= -b(A^*X,Y)+h(X,B^*Y)+trB^*h(Y,Y).\\
&\quad -h(X,Y)\nu-h(X,B^*Y)+(\nabla_X\tau^*)Y+b(Y,A^*X)\\
&= trB^*h(X,Y)-h(X,Y)\nu+(\nabla_x\tau^*)Y
\end{aligned}$$

and since from Corollary 4.1 $\tau^*=0$ holds, we have

$$(\nu-trB^*)h(X,Y)=\frac{\tilde{c}}{2}(n+2)g(X,Y).$$

Hence we have

$$h=\frac{\tilde{c}}{2}(n+2)(\nu-trB^*)^{-1}g. \qquad (4.7)$$

If $\tilde{c}\neq 0$ holds, h is non-degenerated.
Since $\tilde{\nabla}$ is flat in Gaussian equation in [3], we obtain

$$k\{g(Y,Z)X-g(X,Z)Y\}=h(Y,Z)A^*X-h(X,Z)A^*Y$$

and taking the trace of above equation with respect to X, we have

$$k(n-1)g(Y,Z)=trA^*h(Y,Z)-h(A^*Y,Z)=h((trA^*I-A^*)Y,Z).$$

Since the above equation and Eq.(4.7) imply that

$$k(n-1)I=\frac{\tilde{c}}{2}(n+2)(\nu-trB^*)^{-1}(trA^*I-A^*),$$

there is $a\in\mathbf{R}$ such that $A^*=aI$ and $trA^*=an$. Therefore the above equation implies that

$$k(n-1)I=\frac{\tilde{c}}{2}(n+2)(\nu-trB^*)^{-1}(na-a)I$$

and thus since

$$2k(\nu-trB^*)=\tilde{c}(n+2)a,$$

we have

$$A^*=2k(\nu-trB^*)[\tilde{c}(n+2)]^{-1}I. \qquad (4.8)$$

If $k \neq 0$ holds,then since A^* is non-degenerated, by Eq.(4.8) we have

$$B^* = -\frac{\nu}{2}I,\ trB^* = -\frac{n\nu}{2}$$

and

$$A^* = \frac{2k(\nu + \frac{n\nu}{2})}{\tilde{c}(n+2)}I = \frac{k\nu}{\tilde{c}}I, h = \frac{\tilde{c}}{2}(n+2)(\nu + \frac{n\nu}{2})^{-1}g = \frac{\tilde{c}}{\nu}g.$$

Since $h(X,Y) = g(AX,Y)$, we have $A = \frac{\tilde{c}}{\nu}I$ and

$$B = B^* + (A - A^*) = -\frac{\nu}{2}I + (\frac{\tilde{c}}{\nu} - \frac{k\nu}{\tilde{c}})I = \frac{-\nu^2\tilde{c} + 2\tilde{c}^2 - 2k\nu^2}{2\nu\tilde{c}}I = \frac{2\tilde{c}^2 - (2k+\tilde{c})\nu^2}{2\nu\tilde{c}}I.$$

Theorem 4.2. *Let (M, ∇, g) be a trivial statistical manifold of constant curvature k, $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a Hessian manifold of constant Hessian curvature $\tilde{c}$. If there is a statistical immersion of codimension one $f : M \to \tilde{M}$, $2k + \tilde{c}$ is of non-negative. Moreover, when $\tilde{c}$ is positive, the Riemannian shape operator of $f : M \to \tilde{M}$ is given by $S = \pm\frac{1}{2}\sqrt{2k+\tilde{c}}I$.*

Proof. By Lemma 4.1 and Eq.(4.2), we have

$$\begin{aligned}&\frac{\tilde{c}}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{2\tilde{c}^2 - (2k+\tilde{c})\nu^2}{2\nu\tilde{c}}(-\frac{\nu}{2})\{g(Y,Z)X - g(X,Z)Y\}\\ &= \left[\frac{\tilde{c}}{4} - \frac{2\tilde{c}^2 - (2k+\tilde{c})\nu^2}{4\tilde{c}}\right]\{g(Y,Z)X - g(X,Z)Y\} = 0\end{aligned}$$

and thus conclude that

$$\frac{\tilde{c}}{4} - \frac{2\tilde{c}^2 - (2k+\tilde{c})\nu^2}{4\tilde{c}} = 0.$$

Since $\tilde{c}^2 = (2k+\tilde{c})\nu^2$, we have $2k + \tilde{c} \geq 0$ and

$$\nu = \pm\frac{|\tilde{c}|}{\sqrt{2k+\tilde{c}}}.$$

Thus the Riemannian shape operator S is given by

$$S = A^* - B^* = (\frac{k\nu}{\tilde{c}} + \frac{\nu}{2})I = \frac{2k+\tilde{c}}{2\tilde{c}}(\pm\frac{|\tilde{c}|}{\sqrt{2k+\tilde{c}}})I = \pm\frac{|\tilde{c}|}{2\tilde{c}}\sqrt{2k+\tilde{c}}I.$$

When $\tilde{c}$ is positive, we have $S = \pm\frac{1}{2}\sqrt{2k+\tilde{c}}I$.

Example 4.2. Let $(H, \tilde{\nabla}, \tilde{g})$ be the $(n+1)-$dimensional upper half Hessian space of constant Hessian curvature 4 as in Example 2.1. For a constant $y_0 > 0$, write the following immersion by f:

$$(y^1, \cdots, y^n)^T (\in \mathbf{R}^n) \mapsto (y^1, \cdots, y^n, y_0)^T \in H.$$

Let (∇, g) be the statistical structure on $\mathbf{R}^n$ induced by f from $(\tilde{\nabla}, \tilde{g})$. Then $(\mathbf{R}^n, \nabla, g)$ is a trivial statistical manifold of constant curvature 0 and f is a statistical immersion of a trivial statistical manifold of constant curvature into Hessian manifold of constant Hessian curvature.

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Faculty of Mathematics,
KIM IL SUNG university,
Democratic People's Republic of Korea.
E-mail address: mincholrim@yahoo.com
E-mail address: choesongok@yahoo.com
E-mail address: anyunho@yahoo.com