STATISTICAL IMMERSIONS BETWEEN STATISTICAL MANIFOLDS OF CONSTANT CURVATURE

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ABSTRACT. The condition for the curvature of a statistical manifold to admit a kind of standard hypersurface is given. We study the statistical hypersurfaces of some types of the statistical manifolds $(M, \nabla, g)$, which enable $(M, \nabla^{(\alpha)}, g), \forall \alpha \in \mathbb{R}$ to admit the structure of a constant curvature.

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1. INTRODUCTION

Since Lauritzen introduced the notation of statistical manifolds in 1987 [5], the geometry of statistical manifolds has been developed in close relations with affine differential geometry and Hessian geometry as well as information geometry (see, for example, [2][4][8]). In this paper we study the hypersurfaces of statistical manifolds.

Let $M$ be an $n$-dimensional manifold, $\nabla$ a torsion-free affine connection on $M$, $g$ a Riemannian metric on $M$, and $R$ a curvature tensor field on $M$. We denote by $TM$ the set of vector fields on $M$, and by $TM^{(r,s)}$ the set of tensor fields of type $(r,s)$ on $M$.

**Definition 1.1.** A pair $(\nabla, g)$ is called a statistical structure on $M$ if $(M, \nabla, g)$ is a statistical manifold, that is, $\nabla$ is a torsion-free affine connection and for all $X, Y, Z \in T(M)$, $(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$.

Let $\nabla^0$ be a Levi-Civita connection of $g$. Certainly, a pair $(\nabla^0, g)$ is a statistical structure, which is called a Riemannian statistical structure or a trivial statistical structure (see [3]). On the other hand, the statistical structure is also introduced from affine differential geometry which was proposed by Blasche (see [6]). Recently the relation between statistical structures and Hessian geometry has been studied (see [3][7]).

For all $\alpha \in \mathbb{R}$, a connection $\nabla^{(\alpha)}$ is defined by

$$\nabla^{(\alpha)} = \frac{1 + \alpha}{2} \nabla + \frac{1 - \alpha}{2} \nabla^*$$

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where $\nabla$ and $\nabla^a$ are dual connections on $M$. We study a statistical hypersurface of a statistical manifold $(M, \nabla^a, g)$ which enables $(M, \nabla^{(a)}, g), \forall \alpha \in \mathbb{R}$ to admit the structure of a constant curvature.

In section 3, a statistical manifold $(M, \nabla, g)$, which enables $(M, \nabla^{(a)}, g), \forall \alpha \in \mathbb{R}$ to admit the structure of a constant curvature, is considered. In section 4, we study characteristics of statistical immersions between statistical manifolds $(M, \nabla, g)$ which enable $(M, \nabla^{(a)}, g), \forall \alpha \in \mathbb{R}$ to admit the structure of a constant curvature.

2. Preliminaries

A statistical manifold $(M, \nabla, g)$ is said to be of constant curvature $k \in \mathbb{R}$ if

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}, \forall X, Y, Z \in TM$$

(2.1)

holds, where $R$ is the curvature tensor field of $\nabla$. A pair $(\nabla, g)$ is called a Hessian structure if a statistical manifold $(M, \nabla, g)$ is of constant curvature 0.

A Riemannian metric $g$ on a flat manifold $(M, g)$ is called a Hessian metric if $g$ can be locally expressed by

$$g = Dd\varphi,$$

that is,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j},$$

where $\{x^1, \cdots, x^n\}$ is an affine coordinate system with respect to $\nabla$. Then $(M, \nabla, g)$ is called a Hessian manifold (see [7]).

Let $(M, \nabla, g)$ be a Hessian manifold and $K(X, Y) := \nabla_X Y - \nabla^X Y$ be the difference tensor between the Levi-Civita connection $\nabla^X$ of $g$ and $\nabla$. A covariant differential of differential tensor $K$ is called a Hessian curvature tensor for $(\nabla, g)$. A Hessian manifold $(M, \nabla, g)$ is said to be of constant Hessian curvature $c \in \mathbb{R}$ if

$$(\nabla K)(Y, Z) = -\frac{c}{2}\{g(X, Y)Z + g(X, Z)Y\}, \forall X, Y, Z \in TM$$

holds (see [7]).

Example 2.1. ([3])

Let $(H, \bar{g})$ be the upper half space:

$$H := \left\{ y = (y^1, \cdots, y^{n+1})^T \in \mathbb{R}^{n+1} \middle| y^{n+1} > 0 \right\}, \bar{g} := (y^{n+1})^{-2}\sum_{i=1}^{n+1} dy^i dy^i.$$

We define an affine connection $\nabla$ on $H$ by the following relations:

$$\nabla_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^{n+1}} = (y^{n+1})^{-1}\frac{\partial}{\partial y^{n+1}}, \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = 2\delta_{ij}(y^{n+1})^{-1}\frac{\partial}{\partial y^{n+1}},$$

$$\nabla_{\frac{\partial}{\partial y^{n+1}}} \frac{\partial}{\partial y^i} = \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^{n+1}} = 0,$$

where $i, j = 1, \cdots, n$. Then $(H, \nabla, \bar{g})$ is a Hessian manifold of constant Hessian curvature 4.
Let \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) be a statistical manifold and \(f : M \to \tilde{M}\) be an immersion. We define \(g\) and \(\nabla\) on \(M\) by

\[
g = f^* \tilde{g}, \quad g(\nabla_X Y, Z) = \tilde{g}(\nabla_X f Y, f_* Z), \quad \forall X, Y, Z \in TM.
\]

Then the pair \((\nabla, g)\) is a statistical structure on \(M\), which is called the one by \(f\) from \((\tilde{\nabla}, \tilde{g})\) (see [3]).

Let \((M, \nabla, g)\) and \((\tilde{M}, \tilde{\nabla}, \tilde{g})\) be two statistical manifolds. An immersion \(f : M \to \tilde{M}\) is called a statistical immersion if \((\tilde{\nabla}, \tilde{g})\) coincides with the induced statistical structure (see [3]).

Let \(f : (M, \nabla, g) \to (\tilde{M}, \tilde{\nabla}, \tilde{g})\) be a statistical immersion of codimension one, and \(\xi\) a unit normal vector field of \(f\). Then we define \(h, h^* \in TM^{(0,2)}, \tau, \tau^* \in TM^*\) and \(A, A^* \in TM^{(1,1)}\) by the following Gauss and Weingarten formulæ:

\[
\tilde{\nabla}_X f_* Y = f_* \nabla_X Y + h(X, Y)\xi, \quad \tilde{\nabla}_X \xi = -f_* A^* X + \tau^*(X)\xi,
\]

\[
\tilde{\nabla}^*_X f_* Y = f_* \nabla^*_X Y + h^*(X, Y)\xi, \quad \tilde{\nabla}^*_X \xi = -f_* A X + \tau(X)\xi, \quad \forall X, Y \in TM,
\]

where \(\tilde{\nabla}^*\) is the dual connection of \(\tilde{\nabla}\) with respect to \(\tilde{g}\).

In addition, we define \(II \in TM^{(0,2)}\) and \(S \in TM^{(1,1)}\) by using the Riemannian Gauss and Weingarten formulæ:

\[
\tilde{\nabla}_X^* f_* Y = f_* \nabla^*_X Y + II(X, Y)\xi, \quad \tilde{\nabla}^*_X \xi = -f_* SX.
\]

For more details on the Gauss, Codazzi and Ricci formulæ on statistical hypersurfaces, we refer to [3].

### 3. The Condition That a Statistical Manifold \((M, \nabla^{(\alpha)}, g)\) is of Constant Curvature for Any \(\alpha \in \mathbb{R}\)

In this section we consider a condition that a statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature for any \(\alpha \in \mathbb{R}\).

**Theorem 3.1.** A statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature for any \(\alpha \in \mathbb{R}\) iff there exist \(\alpha_1, \alpha_2 \in \mathbb{R}(|\alpha_1| \neq |\alpha_2|)\) such that statistical manifolds \((M, \nabla^{(\alpha_1)}, g)\) and \((M, \nabla^{(\alpha_2)}, g)\) are of constant curvature.

**Proof.** Necessity is obvious. We find sufficiency. Without loss of generality, we assume \(\alpha_1 \neq 0\). Then since

\[
\nabla^{(\alpha)} = \frac{\alpha_1 + \alpha}{2\alpha_1} \nabla^{(\alpha_1)} + \frac{\alpha_1 - \alpha}{2\alpha_1} \nabla^{(-\alpha_1)}
\]

holds for all \(\alpha \in \mathbb{R}\), the following relation

\[
R^{(\alpha)}(X, Y)Z = \frac{\alpha_1 + \alpha}{2\alpha_1} R^{(\alpha_1)}(X, Y)Z + \frac{\alpha_1 - \alpha}{2\alpha_1} R^{(-\alpha_1)}(X, Y)Z + \frac{\alpha_1^2 - \alpha^2}{4\alpha_1^2} [K(Y, K(Z, X)) - K(X, K(Y, Z))]
\]

holds, where \(K(X, Y) := \nabla_X Y - \nabla^0_X Y\) is the difference tensor field of a statistical manifold.

From the relations

\[
R^{(\alpha_1)}(X, Y)Z = k_1 \{g(Y, Z)X - g(X, Z)Y\},
\]

where

\[
k_1 = \frac{\alpha_1}{\alpha_1^2 - \alpha^2}.
\]
Then the statistical manifold holds, that is, a statistical manifold \( g \) is of constant curvature \( \frac{k_2\alpha_1^2 - k_1\alpha_2^2 + (k_1 - k_2)\alpha^2}{\alpha_1^2 - \alpha_2^2} \).

**Example 3.1.** Let \((M, \nabla^{(\alpha)}, g)\) be a family of normal distributions:

\[
M := \left\{ p(x, \theta) \left| p(x, \theta) = \frac{1}{\sqrt{2\pi(\theta^2)}} \exp \left\{ -\frac{1}{2(\theta^2)} (x - \theta^1)^2 \right\} \right. \right\}, \quad g := 2(\theta^2)^{-2} \sum d\theta^i d\theta^i,
\]

\[x \in \mathbb{R}, \quad \theta^1 \in \mathbb{R}, \quad \theta^2 > 0.\]

We define an \(\alpha\)-connection the following relations:

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = (-1 + 2\alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^1}, \quad \nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = (1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = \nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = 0.
\]

Then the statistical manifold \((M, \nabla^{(0)}, g)\) is of constant curvature \(-\frac{1}{2}\), and the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature 0. Hence for all \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature \(\frac{\alpha^2 - 1}{2}\).

**Example 3.2.** Let \((M, g)\) be a family of random walk distributions (11):

\[
M := \left\{ p(x; \theta^1, \theta^2) \left| p(x; \theta^1, \theta^2) = \sqrt{\frac{\theta^2}{2\pi x}} \exp \left\{ -\frac{\theta^2 x}{2} + \frac{\theta^2}{2(\theta^1)^2 x} \right\} \right. \right\}, \quad x, \mu, \lambda > 0 \}
\]

\[g := \frac{\theta^2}{(\theta^1)^3} (d\theta^1)^2 + \frac{1}{2(\theta^2)^2} (d\theta^2)^2.\]

We define an \(\alpha\)-connection the following relations:

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = -\frac{3(1 + \alpha)}{2} (\theta^1)^{-1} \frac{\partial}{\partial \theta^1} + (-1 + \alpha)(\theta^1)^{-3}(\theta^2)^2 \frac{\partial}{\partial \theta^2},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = \nabla^{(\alpha)} \frac{\partial}{\partial \theta^1} = \frac{(1 + \alpha)}{2} (\theta^2)^{-1} \frac{\partial}{\partial \theta^1},
\]

\[
\nabla^{(\alpha)} \frac{\partial}{\partial \theta^2} = (-1 + \alpha)(\theta^2)^{-1} \frac{\partial}{\partial \theta^2}.
\]

Then the statistical manifold \((M, \nabla^{(0)}, g)\) is of constant curvature \(-\frac{1}{2}\), and the statistical manifold \((M, \nabla^{(1)}, g)\) is of constant curvature 0. Hence for all \(\alpha \in \mathbb{R}\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is of constant curvature \(\frac{\alpha^2 - 1}{2}\).

Theorem 3.1.1 implies the following fact.

**Corollary 3.1.** If there exist \(\alpha_1, \alpha_2 \in \mathbb{R} (|\alpha_1| \neq |\alpha_2|)\) such that the statistical manifold \((M, \nabla^{(\alpha_1)}, g)\) is of constant curvature \(k_1\) and the statistical manifold \((M, \nabla^{(\alpha_2)}, g)\) is of constant curvature \(k_2\), and \(k_1 \neq k_2\), then for \(\alpha \in \mathbb{R}\) satisfying that \(\alpha^2 = (k_2\alpha_1^2 - k_1\alpha_2^2)/(k_2 - k_1)\), the statistical manifold \((M, \nabla^{(\alpha)}, g)\) is flat.
Example 3.3. $k_1 = -1/2$, $k_2 = 0$, $a_1 = 0$ and $a_2 = 1$ hold in example 3.1 and example 3.2. Hence for $a \in \mathbb{R}$ satisfying that $a^2 = 1$, the statistical manifold $(M, \nabla^{(a)}, g)$ is flat.

Theorem 3.2. If the Hessian manifold $(M, \nabla, g)$ is of constant Hessian curvature, then for all $a \in \mathbb{R}$, the statistical manifold $(M, \nabla^{(a)}, g)$ is of constant curvature.

Proof. If the Hessian manifold $(M, \nabla, g)$ is of constant Hessian curvature, then for all $X, Y, Z \in TM$,

$$(\nabla X)(Y, Z; X) = -\frac{c}{2} \{g(X, Y)Z + g(X, Z)Y\}, c \in \mathbb{R}$$

holds. On the other hand, the curvature tensor $R^\circ$ of Levi-Civita connection $\nabla^\circ$ is written by

$$R^\circ(X, Y)Z = R(X, Y)Z - (\nabla X)(Y, Z; X) + (\nabla X)(Z, X; Y)$$

$$+ K(X, K(Y, Z)) - K(Y, K(Z, X)),$$

where $R$ is the curvature tensor of $\nabla$ and $K(X, Y) = \nabla X Y - \nabla^\circ X Y$ is difference tensor. Then

$$(\nabla K)(Y, Z; X) - (\nabla K)(Z, X; Y)$$

$$= 2\{K(X, K(Y, Z)) - K(Y, K(Z, X))\} + \frac{1}{2} \{R(X, Y)Z - R^\circ(X, Y)Z\}$$

implies

$$R^\circ(X, Y)Z = -\frac{c}{4} \{g(Y, Z)X - g(X, Z)Y\},$$

where $R^\circ$ is curvature tensor of dual connection $\nabla^\circ$, that is, the statistical manifold $(M, \nabla^\circ, g)$ is of constant curvature. On the other hand, the statistical manifold $(M, \nabla, g)$ is flat, that is, constant curvature 0. Therefore we finish the proof of theorem by applying Theorem 3.1.

Hitherto we found some conditions that for any $a \in \mathbb{R}$, the statistical manifold $(M, \nabla^{(a)}, g)$ is of constant curvature.

4. THE HYPERSURFACES OF STATISTICAL MANIFOLDS OF CONSTANT CURVATURE

We consider statistical hypersurfaces of some type of statistical manifolds, which enable for any $a \in \mathbb{R}$ a statistical manifold $(M, \nabla^{(a)}, g)$ to be of constant curvature.

Theorem 4.1. Let $(M, \nabla, g)$ be a trivial statistical manifold of constant curvature $k$, $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ a statistical manifold of constant curvature $\tilde{k}$ with a Riemannian manifold of constant curvature $\tilde{k}$ $(\neq \tilde{k}) (\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, and $f : M \to \tilde{M}$ a statistical immersion of codimension one. Then $f : M \to \tilde{M}$ is equiaffine, that is, $\tau^* \tilde{k}$ vanishes.

Proof. If $(\tilde{M}, \tilde{\nabla}, \tilde{g})$ is a statistical manifold of constant curvature $\tilde{k}$ with a Riemannian manifold of constant curvature $\tilde{k}$ $(\neq \tilde{k}) (\tilde{M}, \tilde{\nabla}^\circ, \tilde{g})$, the following equation

$$(\tilde{\nabla}_X \tilde{k})(f, Y, f, Z) - (\tilde{\nabla}_Y \tilde{k})(f, X, f, Z)$$

$$= 2\{\tilde{R}(f, X, f, Y)f, Z - \tilde{R}^\circ(f, X, f, Y)f, Z\}$$

$$= 2(\tilde{k} - \tilde{k})\{\tilde{g}(f, Y, f, Z)f, X - \tilde{g}(f, X, f, Z)f, Y\}$$

(4.1)
holds by Eq.(2.2) and Eq.(2.3) in [3]. By above equation and equation Eq.(3.6) in [3], we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = (\nabla_X K)(Y, Z) - (\nabla_Y K)(X, Z) \]

\[ -b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)B^*Y - h(Y, Z)B^*X \]

\[ 0 = (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z) + \tau^*(X)b(Y, Z) - \tau^*(Y)b(X, Z) \]

\[ \tilde{\tau}^*(Y)b(X, Z) - \tau^*(X)b(Y, Z) \]

\[ 0 = -\tau^*(X)A^*X + \tau^*(Y)A^*X - (\nabla_X B^*)Y + (\nabla_Y B^*)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

By $K = 0$, $B^* = A^* - S$ and Gauss equation (3.3) in [3], from Eq.(4.2), we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = -b(Y, Z)A^*X + b(X, Z)A^*Y \]

\[ + h(X, Z)A^*Y - h(X, Z)SY - h(Y, Z)A^*X + h(Y, Z)SX \]

\[ = -b(Y, Z)A^*X + b(X, Z)A^*Y - h(X, Z)SY + h(Y, Z)SX + \tilde{R}(X, Y)Z - R(X, Y)Z. \]

By $b = h - II$, $B^* = A^* - S$ and Riemannian Gauss equation (3.5) in [3], we have

\[ -2(\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} = -b(Y, Z)A^*X + b(X, Z)A^*Y \]

\[ + h(X, Z)A^*Y - h(X, Z)SY - h(Y, Z)A^*X + h(Y, Z)SX \]

\[ = -b(Y, Z)B^*X + h(X, Z)B^*Y + II(Y, Z)B^*X + II(Y, Z)SX \]

\[ - II(Y, Z)B^*Y - II(X, Z)SY + \tilde{R}(X, Y)Z - R(X, Y)Z \]

\[ = -b(Y, Z)B^*X + b(X, Z)B^*Y + R^*(X, Y)Z - R^*(X, Y)Z + \tilde{R}(X, Y)Z - R(X, Y)Z. \]

Since $(M, \nabla, g)$ is Riemannian manifold, clearly $R^*(X, Y)Z = R(X, Y)Z$. Hence we have

\[ 0 = (\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} - b(Y, Z)B^*X + b(X, Z)B^*Y. \]

And since $b(Y, Z) = g(BY, Z)$, $b(X, Z) = g(BX, Z)$, from above equation we have

\[ 0 = (\tilde{k} - \tilde{k})\{g(Y, Z)X - g(X, Z)Y\} - g(BY, Z)B^*X + g(BX, Z)B^*Y. \]

(4.3)

From Eq.(4.2), $B^* = A^* - S$ and Codazzi equation on $A$ we get

\[ 0 = -\tau^*(Y)A^*X + \tau^*(X)A^*Y - (\nabla_X A^*)Y + (\nabla_Y A^*)X - (\nabla_Y S)X \]

\[ + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

\[ = (\nabla_X S)Y - (\nabla_Y S)X + \tau^*(X)B^*Y - \tau^*(Y)B^*X \]

and by $\nabla = \nabla^*$ and Codazzi equation on $S$, we also get

\[ 0 = \tau^*(X)B^*Y - \tau^*(Y)B^*X. \]

(4.4)

From Eq.(4.2), $B^* = A^* - S$ and Ricci equation we have

\[ b(X, B^*Y) - b(Y, B^*X) = 0, \]

and since $b(X, B^*Y) = g(BX, B^*Y)$ and $b(Y, B^*X) = g(BY, B^*X)$, we have

\[ g(BX, B^*Y) - g(BY, B^*X) = 0. \]
Since \( g(BX, B^*Y) = g(B^*Y, BX) = b^*(BX, Y) = g(B^*BX, Y) \), we have
\[
0 = -g([B, B^*]X, Y).
\] (4.5)

From Eq.(4.5), \( B \) and \( B^* \) are simultaneously diagonalizable.

In the case that \( B^* \) is of the form \( \lambda^* I \), we see easily that \( \tau^* \) vanishes from Eq.(4.4) if \( \lambda^* \neq 0 \) and \( \tilde{k} = k \) from Eq.(4.3) otherwise. In the case that \( B^* \) is not of the form \( \lambda^* I \), there are \( \lambda_1^*, \lambda_2^* \) with \( \lambda_1^* \neq \lambda_2^* \) such that \( B^*X_j = \lambda_j^*X_j \), where \( g(X_i, X_j) = \delta_{ij}, \ i, j = 1, 2 \). Besides there are \( \lambda_1, \lambda_2 \) such that \( BX_j = \lambda_j X_j \). Eq.(4.3) implies that
\[
(k - \tilde{k})\{g(X_j, Z)X_i - g(X_i, Z)X_j + \lambda_i \lambda_j^* g(X_j, Z)X_i - \lambda_i \lambda_j g(X_i, Z)X_j \}
= (k - \tilde{k} + \lambda_j^* \lambda_i)g(X_j, Z)X_i - (k - \tilde{k} + \lambda_i \lambda_j^*)g(X_i, Z)X_j = 0
\]
and hence \( k - \tilde{k} + \lambda_j^* \lambda_i = \tilde{k} - \tilde{k} + \lambda_i \lambda_j^* = 0 \), which means that
\[
\lambda_j \lambda_i^* = \lambda_i \lambda_j^* = - (k - \tilde{k}) \neq 0.
\]
By Eq.(4.4) we have \( \lambda_j^* \tau^* (X_1)X_2 - \lambda_i \tau^* (X_2)X_1 = 0 \), which implies that \( \tau^* \) vanishes.

**Example 4.1.** Suppose \( \tilde{M} \) be \( \mathbb{R}^3 \). We define Riemannian metric and an Affine connection by the following relations:
\[
\tilde{g} = a \sum d\theta^i d\theta^i,
\]
\[
\tilde{\nabla} \frac{\partial}{\partial \theta^i} \tilde{\nabla} \frac{\partial}{\partial \theta^l} = \tilde{b} \frac{\partial}{\partial \theta^l}, \quad \tilde{\nabla} \frac{\partial}{\partial \theta^j} \tilde{\nabla} \frac{\partial}{\partial \theta^l} = \tilde{b} \frac{\partial}{\partial \theta^l}, \quad \tilde{\nabla} \frac{\partial}{\partial \theta^i} \tilde{\nabla} \frac{\partial}{\partial \theta^j} = \tilde{b} \frac{\partial}{\partial \theta^j}, \quad \tilde{\nabla} \frac{\partial}{\partial \theta^i} \tilde{\nabla} \frac{\partial}{\partial \theta^j} = \tilde{b} \frac{\partial}{\partial \theta^j},
\]
\[
\tilde{\nabla} \frac{\partial}{\partial \theta^i} \tilde{\nabla} \frac{\partial}{\partial \theta^j} = \tilde{b} \frac{\partial}{\partial \theta^j} = 0.
\]

Then \( (\tilde{M}, \tilde{\nabla}, \tilde{g}) \) is a statistical manifold of constant curvature \( \frac{\tilde{b}^2}{4\tilde{a}} \) with a trivial statistical manifold of constant curvature \( 0 \) \( (\tilde{M}, \tilde{\nabla}^0, \tilde{g}) \). Suppose \( M \) be \( \mathbb{R}^2 \), and \( (\nabla, g) \) an induced statistical structure from \( (\tilde{\nabla}, \tilde{g}) \) by an immersion \( f : (x, y)(\in \mathbb{R}^2) \mapsto (0, x, y) \). We remark that \( (M, \nabla, g) \) is a trivial statistical manifold of constant curvature 0.

Theorem 3.2 and Theorem 4.1 imply the following fact.

**Corollary 4.1.** Let \( (M, \nabla, g) \) be a trivial statistical manifold of constant curvature \( k \), \( (\tilde{M}, \tilde{\nabla}, \tilde{g}) \) a Hessian manifold of constant Hessian curvature \( \tilde{c} \), and \( f : M \to \tilde{M} \) a statistical immersion of codimension one. Then \( f : M \to \tilde{M} \) is equiaffine, that is, \( \tau^* \) vanishes.

We consider a shape operator of statistical immersion of a trivial statistical manifold of constant curvature into a Hessian manifold of constant Hessian curvature.

**Lemma 4.1.** Let \( (M, \nabla, g) \) be a trivial statistical manifold of constant curvature \( k \), \( (\tilde{M}, \tilde{\nabla}, \tilde{g}) \) a Hessian manifold of constant Hessian curvature \( \tilde{c} \), and \( f : M \to \tilde{M} \) a statistical immersion of codimension one. Then the following holds:
\[
A^* = kv\tilde{c}^{-1}I, B^* = -\frac{1}{2}v^2I, h = \tilde{c}v^{-1}g, A = \tilde{c}v^{-1}I, B = [2v^2 - (2k + \tilde{c})v^2](2v\tilde{c})^{-1}I.
\]
Proof. Combining Eq.(2.3) and Eq.(3.6) in [3] with Eq.(2.1), we have
\[
\frac{\tilde{\varepsilon}}{2} \{g(Y, Z)X - g(X, Z)Y\} = 2(k - \tilde{\nu})(g(Y, Z)X - g(X, Z)Y)
\]
\[
- b(Y, Z)A^*X + b(X, Z)A^*Y + h(X, Z)B^*Y - h(Y, Z)B^*X
\]
\[
0 = h(X, K(Y, Z)) - h(Y, K(X, Z)) + (\nabla_X b)(Y, Z) - (\nabla_Y b)(X, Z)
\]
\[
+ \tau^*(X) b(Y, Z) - \tau^*(Y) b(X, Z) - \tau^*(Y) h(X, Z) + \tau^*(X) h(Y, Z)
\]
\[
4.6
\]
\[
0 = K(Y, A^*X) - K(X, A^*Y) - \tau^*(Y) A^*X + \tau^*(X) A^*Y
\]
\[
- (\nabla_X B^*) Y + (\nabla_Y B^*) X + \tau^*(X) B^* Y - \tau^*(Y) B^* X
\]
\[
0 = -h(X, B^*Y) + h(Y, B^*X) + (\nabla_X \tau^*)(Y) - (\nabla_Y \tau^*)(X) + b(Y, A^*X) - b(X, A^*Y)
\]
Taking the trace of (4.6)_1 with respect to X, we have
\[
-\tilde{\varepsilon} g(Y, Z) = -tr A^* b(Y, Z) + h(B^*Z, Y) + h(B^*Y, Z)
\]
and taking the trace of (4.6)_1 with respect to Y, we have
\[
-\frac{\tilde{\varepsilon}}{2}(n + 1) g(X, Z) = -b(A^*X, Z) + h(X, B^*Z) + trB^*h(X, Z).
\]
Using the above equation and Eq.(4.6)_1, we have
\[
-\frac{\tilde{\varepsilon}}{2}(n + 2) g(X, Y) = -b(A^*X, Y) + h(X, B^*Y) + trB^*h(Y, Y).
\]
\[
- h(X, Y) v - h(X, B^*Y) + (\nabla_X \tau^*) Y + b(Y, A^*X)
\]
\[
= trB^*h(X, Y) - h(X, Y) v + (\nabla_X \tau^*) Y
\]
and since from Corollary [4.1] \(\tau^* = 0\) holds, we have
\[
(v - trB^*)h(X, Y) = \frac{\tilde{\varepsilon}}{2}(n + 2) g(X, Y).
\]
Hence we have
\[
h = \frac{\tilde{\varepsilon}}{2}(n + 2) (v - trB^*)^{-1} g.
\]
If \(\tilde{\varepsilon} \neq 0\) holds, \(h\) is non-degenerated.
Since \(\nabla\) is flat in Gaussian equation in [3], we obtain
\[
k\{g(Y, Z)X - g(X, Z)Y\} = h(Y, Z)A^*X - h(X, Z)A^*Y
\]
and taking the trace of above equation with respect to \(X\), we have
\[
k(n - 1) g(Y, Z) = trA^*h(Y, Z) - h(A^*Y, Z) = h((trA^*I - A^*) Y, Z).
\]
Since the above equation and Eq.(4.7) imply that
\[
k(n - 1) I = \frac{\tilde{\varepsilon}}{2}(n + 2)(v - trB^*)^{-1}(trA^*I - A^*),
\]
there is \(a \in \mathbb{R}\) such that \(A^* = aI\) and \(trA^* = an\). Therefore the above equation implies that
\[
k(n - 1) I = \frac{\tilde{\varepsilon}}{2}(n + 2)(v - trB^*)^{-1}(na - a) I
\]
and thus since
\[
2k(v - trB^*) = \tilde{\varepsilon}(n + 2)a,
\]
we have
\[
A^* = 2k(v - trB^*)[\tilde{\varepsilon}(n + 2)]^{-1} I.
\]
(4.8)
If \( k \neq 0 \) holds, then since \( A^* \) is non-degenerated, by Eq.(4.8) we have
\[
B^* = -\frac{v}{2} I, \quad \text{tr}B^* = -\frac{nv}{2}
\]
and
\[
A^* = \frac{2k(v + \frac{nv}{2})}{c(n + 2)} I = \frac{k\nu}{c} I, \quad h = \frac{c}{2} (n + 2)(v + \frac{nv}{2})^{-1} g = \frac{c}{v} g.
\]

Since \( h(X, Y) = g(AX, Y) \), we have \( A = \frac{c}{v} I \) and
\[
B = B^* + (A - A^*) = -\frac{v}{2} I + (\frac{c}{v} - \frac{k\nu}{c}) I = -\frac{v^2c + 2c^2 - 2k\nu^2}{2vc} I = \frac{2c^2 - (2k + c)\nu^2}{2vc} I.
\]

**Theorem 4.2.** Let \( (M, \nabla, g) \) be a trivial statistical manifold of constant curvature \( k \), \( (\bar{M}, \bar{\nabla}, \bar{g}) \) a Hessian manifold of constant Hessian curvature \( \bar{c} \). If there is a statistical immersion of codimension one \( f : M \to \bar{M} \), \( 2k + \bar{c} \) is of non-negative. Moreover, when \( \bar{c} \) is positive, the Riemannian shape operator of \( f : M \to \bar{M} \) is given by \( S = \pm \frac{1}{2} \sqrt{2k + \bar{c}} I \).

**Proof.** By Lemma 4.1 and Eq.(4.2), we have
\[
\bar{c} \frac{1}{2} \left\{ g(Y, Z)X - g(X, Z)Y \right\} + \frac{2c^2 - (2k + \bar{c})\nu^2}{2vc} \left( -\frac{v}{2} \right) \left\{ g(Y, Z)X - g(X, Z)Y \right\}
\]
\[
= \left[ \frac{c}{4} - \frac{2c^2 - (2k + \bar{c})\nu^2}{4c} \right] \left\{ g(Y, Z)X - g(X, Z)Y \right\} = 0
\]
and thus conclude that
\[
\frac{c}{4} - \frac{2c^2 - (2k + \bar{c})\nu^2}{4c} = 0.
\]

Since \( \bar{c} = (2k + \bar{c})\nu^2 \), we have \( 2k + \bar{c} \geq 0 \) and
\[
\nu = \pm \frac{|\bar{c}|}{\sqrt{2k + \bar{c}}}.
\]
Thus the Riemannian shape operator \( S \) is given by
\[
S = A^* - B^* = \left( \frac{k\nu}{c} + \frac{v}{2} \right) I = \frac{2k + \bar{c}}{2c} \left( \pm \frac{|\bar{c}|}{\sqrt{2k + \bar{c}}} \right) I = \pm \frac{|\bar{c}|}{2c} \sqrt{2k + \bar{c}} I.
\]
When \( \bar{c} \) is positive, we have \( S = \pm \frac{1}{2} \sqrt{2k + \bar{c}} I \).

**Example 4.2.** Let \( (H, \nabla, \bar{g}) \) be the \((n + 1)\)-dimensional upper half Hessian space of constant Hessian curvature 4 as in Example 2.1. For a constant \( y_0 > 0 \), write the following immersion by \( f: \)
\[
(y^1, \cdots, y^n)^T (\in \mathbb{R}^n) \mapsto (y^1, \cdots, y^n, y_0)^T \in H.
\]
Let \( (\nabla, g) \) be the statistical structure on \( \mathbb{R}^n \) induced by \( f \) from \( (\nabla, \bar{g}) \). Then \( (\mathbb{R}^n, \nabla, g) \) is a trivial statistical manifold of constant curvature 0 and \( f \) is a statistical immersion of a trivial statistical manifold of constant curvature into Hessian manifold of constant Hessian curvature.

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Statistical immersions between statistical manifolds of constant curvature

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