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ANOTHER PROOF OF FLOOR VAN LAMOEN'S GENERALIZED THEOREM

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ABSTRACT. Another proof of Floor van Lamoen's generalized theorem will be introduced by using the algebraic form of Law of sine.

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1. INTRODUCTION

In 2000, Dutch mathematician Floor van Lamoen gave a nice theorem on *American Mathematical Monthly* [1].

Theorem 1. *If G is the centroid of triangle ABC and AG, BG, CG intersects with BC, CA, AB at A_1, B_1, C_1 respectively, then the circumcenters of six triangles GBC_1, GB_1C, GCA_1, GC_1A, GAB_1, GA_1B are concyclic.*

Theorem 1 is called Floor van Lamoen's theorem, whose proof can be found in [1], [2], [3] and [4].
In 2002, theorem 1 was expanded by A. Myakishev and then re-stated by Barry Wolk [5].

Theorem 2. *If two triangles ABC and $A_1B_1C_1$ have the same centroid and AA_1, BB_1, CC_1 are concurrent at O, then the circumcenters of six triangles OBC_1, OB_1C, OCA_1, OC_1A, OAB_1, OA_1B are concyclic.*

Theorem 2 is called Floor van Lamoen's generalized theorem, which was proven for the first time in 2003 by Darij Grinberg [6].
In this article, I will introduce a different proof of theorem 2 by using the algebraic form of Law of sine.
As in [7], the signed distances from the point A to the point B denoted by $\overline{AB}$.

2. PROOF OF THEOREM 2

Three lemmas are required.

Lemma 1. *If $\overrightarrow{\alpha}$, $\overrightarrow{\beta} \neq \overrightarrow{0}$ and $k \neq 0$ then* $\sin(k\overrightarrow{\alpha}, \overrightarrow{\beta}) = \frac{k}{|k|}\sin(\overrightarrow{\alpha}, \overrightarrow{\beta})$.

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Proof.
In this proof, the following signs ↑↑ and ↑↓ refer to the co-directionality and contra-directionality of two vectors.
There are two cases to consider.
Case 1. $k > 0$. Noting that $k\overrightarrow{\alpha} \uparrow\uparrow \overrightarrow{\alpha}$, we have

$$\sin(k\overrightarrow{\alpha},\ \overrightarrow{\beta}) = \sin((k\overrightarrow{\alpha},\ \overrightarrow{\alpha}) + (\overrightarrow{\alpha},\ \overrightarrow{\beta})) = \sin(0 + (\overrightarrow{\alpha},\ \overrightarrow{\beta})) = 1.\sin(\overrightarrow{\alpha},\ \overrightarrow{\beta}) = \frac{k}{|k|}\sin(\overrightarrow{\alpha},\ \overrightarrow{\beta}).$$

Case 2. $k < 0$. Noting that $k\overrightarrow{\alpha} \uparrow\downarrow \overrightarrow{\alpha}$, we have

$$\sin(k\overrightarrow{\alpha},\ \overrightarrow{\beta}) = \sin((k\overrightarrow{\alpha},\ \overrightarrow{\alpha}) + (\overrightarrow{\alpha},\ \overrightarrow{\beta})) = \sin(\pi + (\overrightarrow{\alpha},\ \overrightarrow{\beta})) = (-1)\,.\sin(\overrightarrow{\alpha},\ \overrightarrow{\beta}) = \frac{k}{|k|}\sin(\overrightarrow{\alpha},\ \overrightarrow{\beta}).$$

Lemma 2. *Given triangle ABC. $\overrightarrow{\alpha},\ \overrightarrow{\beta},\ \overrightarrow{\gamma}$ are the direction vectors of lines BC, CA, AB respectively. Then*
1) $\frac{\sin(\overrightarrow{AB},\overrightarrow{AC})}{\overline{BC}} = \frac{\sin(\overrightarrow{BC},\overrightarrow{BA})}{\overline{CA}} = \frac{\sin(\overrightarrow{CA},\overrightarrow{CB})}{\overline{AB}}$.
2) $\frac{\sin(\overrightarrow{\beta},\overrightarrow{\gamma})}{\overline{BC}} = \frac{\sin(\overrightarrow{\gamma},\overrightarrow{\alpha})}{\overline{CA}} = \frac{\sin(\overrightarrow{\alpha},\overrightarrow{\beta})}{\overline{AB}}$.

Proof.
Without the loss of generality, assume that ΔABC has a positive direction and $\overrightarrow{\alpha},\ \overrightarrow{\beta},\ \overrightarrow{\gamma}$ are unit vectors.
According to Law of sine and lemma 1, we have

$$\frac{BC}{CA} = \frac{\sin\widehat{BAC}}{\sin\widehat{CBA}} = \frac{\sin(\overrightarrow{AB},\ \overrightarrow{AC})}{\sin(\overrightarrow{BC},\ \overrightarrow{BA})} = \frac{\sin(\overline{AB}.\overrightarrow{\gamma},\ \overline{AC}.\overrightarrow{\beta})}{\sin(\overline{BC}.\overrightarrow{\alpha},\overline{BA}.\overrightarrow{\gamma})} = \frac{\frac{\overline{AB}}{AB}.\frac{\overline{AC}}{AC}\sin(\overrightarrow{\gamma},\ \overrightarrow{\beta})}{\frac{\overline{BC}}{BC}.\frac{\overline{BA}}{BA}\sin(\overrightarrow{\alpha},\ \overrightarrow{\gamma})}.$$

Therefore, $\frac{\sin(\overrightarrow{AB},\overrightarrow{AC})}{\overline{BC}} = \frac{\sin(\overrightarrow{BC},\overrightarrow{BA})}{\overline{CA}}$ and $\frac{\sin(\overrightarrow{\beta},\overrightarrow{\gamma})}{\overline{BC}} = \frac{\sin(\overrightarrow{\gamma},\overrightarrow{\alpha})}{\overline{CA}}$.
Likewise, $\frac{\sin(\overrightarrow{BC},\overrightarrow{BA})}{\overline{CA}} = \frac{\sin(\overrightarrow{CA},\overrightarrow{CB})}{\overline{AB}}$ and $\frac{\sin(\overrightarrow{\gamma},\overrightarrow{\alpha})}{\overline{CA}} = \frac{\sin(\overrightarrow{\alpha},\overrightarrow{\beta})}{\overline{AB}}$.
In short,
1) $\frac{\sin(\overrightarrow{AB},\overrightarrow{AC})}{\overline{BC}} = \frac{\sin(\overrightarrow{BC},\overrightarrow{BA})}{\overline{CA}} = \frac{\sin(\overrightarrow{CA},\overrightarrow{CB})}{\overline{AB}}$.
2) $\frac{\sin(\overrightarrow{\beta},\overrightarrow{\gamma})}{\overline{BC}} = \frac{\sin(\overrightarrow{\gamma},\overrightarrow{\alpha})}{\overline{CA}} = \frac{\sin(\overrightarrow{\alpha},\overrightarrow{\beta})}{\overline{AB}}$.
Lemma 2 is actually the algebraic form of Law of sine.

Lemma 3. *Given triangle ABC and the following pairs of points (A_1, A_2), (B_1, B_2), (C_1, C_2) belonging to lines BC, CA, AB respectively. If the following groups of four points (B_1, B_2, C_1, C_2), (C_1, C_2, A_1, A_2), (A_1, A_2, B_1, B_2) are concyclic each, then six points A_1, A_2, B_1, B_2, C_1, C_2 lie on the same circle.*

Proof of lemma 3 is very simple and hence, not presented here.
Return to the proof of theorem 2.
Proof.[Proof of theorem 2]Let A_b, A_c, B_c, B_a, C_a, C_b be the circumcenters of triangles OBC_1, OB_1C, OCA_1, OC_1A, OAB_1, OA_1B, respectively and X, Y, Z be the intersections of the following pairs of lines (A_bC_b, B_cA_c), (B_cA_c, C_aB_a), (C_aB_a, A_bC_b) respectively. Let H, K be the projections of X on A_cC_a, A_bB_a respectively and let M, M_1, N, N_1 be the midpoints of OB, OB_1, OC, OC_1 respectively.

Direct the lines $AA_1, BB_1, CC_1, YZ, ZX, XY$ respectively by unit vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}, \overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}$.
Evidently, $B_cC_b \parallel YZ$; $XH \parallel BB_1$; $XK \parallel CC_1$. Therefore, without the loss of generality, direct the lines B_cC_b, XH, XK respectively by vectors $\overrightarrow{x}, \overrightarrow{b}, \overrightarrow{c}$.
Clearly, YZ, ZX, XY are perpendicular to AA_1, BB_1, CC_1 respectively. Therefore, without the loss of generality, assume that $(\overrightarrow{x}, \overrightarrow{a}) \equiv (\overrightarrow{y}, \overrightarrow{b}) \equiv (\overrightarrow{z}, \overrightarrow{c}) \equiv \frac{\pi}{2} \pmod{2\pi}$.
Since $B_cC_b \parallel YZ$, according to lemma 2, $\frac{\overline{XB_c}}{\overline{XC_b}} = \frac{\sin(\overrightarrow{x}, \overrightarrow{y})}{\sin(\overrightarrow{x}, \overrightarrow{z})}$ (1).
Noting that $\overline{XK} = \overline{NN_1}$; $\overline{XH} = \overline{MM_1}$; $A_bK \parallel XY$; $XK \parallel CC_1$; $A_cH \parallel XZ$; $XH \parallel BB_1$, according to lemma 2, noting that $\sin(\overrightarrow{z}, \overrightarrow{c}) = \sin(\overrightarrow{y}, \overrightarrow{b}) = 1$, we have

$$\frac{\overline{XA_b}}{\overline{XA_c}} = \frac{\overline{XA_b}}{\overline{XK}} \cdot \frac{\overline{NN_1}}{\overline{MM_1}} \cdot \frac{\overline{XH}}{\overline{XA_c}} = \frac{\sin(\overrightarrow{z}, \overrightarrow{c})}{\sin(\overrightarrow{z}, \overrightarrow{y})} \cdot \frac{\frac{1}{2}\overline{CC_1}}{\frac{1}{2}\overline{BB_1}} \cdot \frac{\sin(\overrightarrow{y}, \overrightarrow{z})}{\sin(\overrightarrow{y}, \overrightarrow{b})} = -\frac{\overline{CC_1}}{\overline{BB_1}}.$$

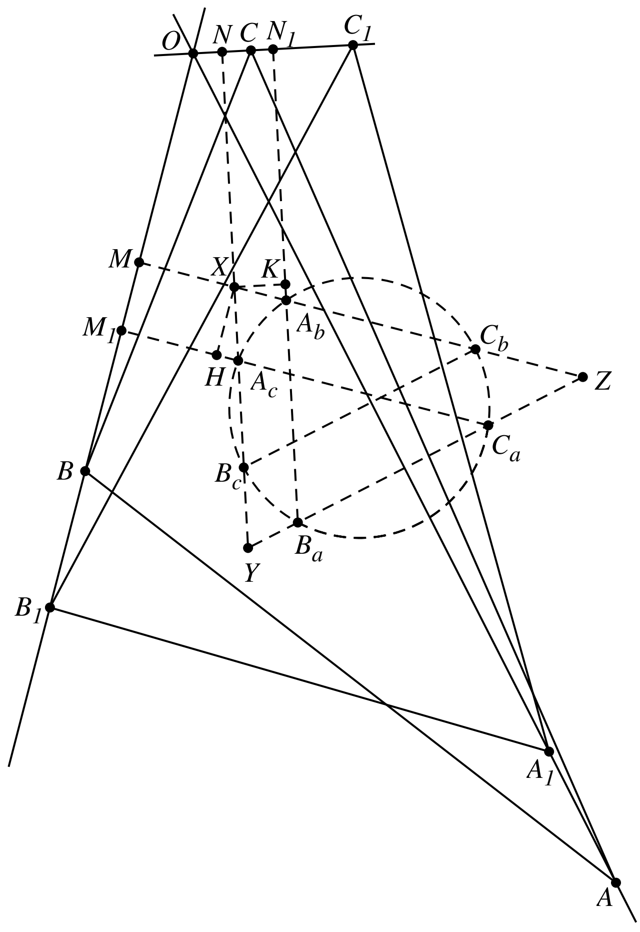

Figure 1.

Because triangles ABC and $A_1B_1C_1$ have the same centroid,

$$\overrightarrow{0} = \overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{AA_1} + \overline{BB_1}\,\overrightarrow{b} + \overline{CC_1}\,\overrightarrow{c}.$$

From this, noting that $\overrightarrow{x} \perp \overrightarrow{AA_1}$, we can deduce that

$0 = \overrightarrow{x}.\overrightarrow{0} = \overrightarrow{x}.\overrightarrow{AA_1} + \overline{BB_1}\cos(\overrightarrow{x}, \overrightarrow{b}) + \overline{CC_1}\cos(\overrightarrow{x}, \overrightarrow{c}) = \overline{BB_1}\cos(\overrightarrow{x}, \overrightarrow{b}) + \overline{CC_1}\cos(\overrightarrow{x}, \overrightarrow{c})$.

Therefore, $-\frac{\overline{CC_1}}{\overline{BB_1}} = \frac{\cos(\overrightarrow{x}, \overrightarrow{b})}{\cos(\overrightarrow{x}, \overrightarrow{c})} = \frac{\cos\left((\overrightarrow{x}, \overrightarrow{y}) + (\overrightarrow{y}, \overrightarrow{b})\right)}{\cos\left((\overrightarrow{x}, \overrightarrow{z}) + (\overrightarrow{z}, \overrightarrow{c})\right)} = \frac{\cos\left((\overrightarrow{x}, \overrightarrow{y}) + \frac{\pi}{2}\right)}{\cos\left((\overrightarrow{x}, \overrightarrow{z}) + \frac{\pi}{2}\right)} = \frac{\sin(\overrightarrow{x}, \overrightarrow{y})}{\sin(\overrightarrow{x}, \overrightarrow{z})}$.

Hence, $\frac{\overline{XA_b}}{\overline{XA_c}} = \frac{\sin(\overrightarrow{x}, \overrightarrow{y})}{\sin(\overrightarrow{x}, \overrightarrow{z})}$ (2).

From (1) and (2), we can deduce that $\frac{\overline{XB_c}}{\overline{XC_b}} = \frac{\overline{XA_b}}{\overline{XA_c}}$.

In other words, $\overline{XA_b}.\overline{XC_b} = \overline{XA_c}.\overline{XB_c}$.

This means that four points A_b, A_c, B_c, C_b are concyclic.

Similarly, the following groups of four points (B_c, B_a, C_a, A_c) and (C_a, C_b, A_b, A_c) are concyclic as well.

In short, by lemma 3, A_b, A_c, B_c, B_a, C_a, C_b belong to the same circle.

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