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# ON THE CIORANESCU-(HASLAM-JONES)-LANCZOS GENERALIZED DERIVATIVE

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ABSTRACT. The Cioranescu-(Haslam-Jones)-Lanczos derivative, which performs derivation through integration, is obtained from a quadrature technique with boundary values. Besides, our approach permits to generalize this derivative for higher orders, in harmony with the result of Rangarajan-Purushothaman. We apply the Lanczos derivative to Fourier series with the natural presence of the $\sigma$-factors.

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## 1. INTRODUCTION

Lanczos [1] used the Least Squares Method (LSM) of Legendre [2]-Gauss [3]-Laplace [4] to obtain an integral expression that gives the derivative of a function, that is, derivation via integration [1, 5-8]:

$$f'_L(x,\epsilon) = \frac{3}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} t f(x+t)dt, \quad \epsilon << 1 \tag{1}$$

when $\epsilon \to 0$, then (1) tends to the ordinary derivative:

$$\lim_{\epsilon\to 0} f'_L(x,\epsilon) = f'(x) \tag{2}$$

From the Taylor series:

$$f(x+t) = f(x) + f'(x)t + \frac{1}{2}f''(x)t^2 + ..., \tag{3}$$

which on substituting in (1) implies:

$$f'_L(x,\epsilon) = f'(x) + \frac{\epsilon^2}{10}f'''(x) + ..., \tag{4}$$

and thus (2) is immediate. In (4) we observe that when $\epsilon \to 0$ then is closer the equality between the two types of derivatives.

Lanczos [1, 9] calculated, for example in $x = 0$, the derivative of an empirical function tabulated in equidistant data, then he saw that with five points could fit a parabola through them, thus proposing the curve $y = a + bx + cx^2$ whose coefficients were obtained via the LSM, and in doing with b because it is clear that $y'(0) = b$. In other words, $f'_L(0,\epsilon)$ is the derivative of this parabola when the number $n$ of data tends to infinite and the separation $h$ between them reduces to zero, all this happening in the vicinity

$[-\epsilon,\epsilon],\epsilon << 1$, about $x = 0$, and the empirical function approaches to a continuous function. In applying the technique of LSM to seek $a, b, c$ that minimize the square error $\sum_{k=-n}^{n}(a + bx_k + cx_k^2 - y_k)^2$ , one of the resulting equations is:

$$a\sum x_k + b\sum x_k^2 + c\sum x_k^3 = \sum x_k y_k, \tag{5}$$

but $x_k$ are distributed symmetrically about the origin, then $\sum x_k = \sum x_k^3 = 0$ cancelling the coefficients a and c in (5), and assuming $n \to \infty$ and $h \to 0$, the relation (5) involve:

$$b\int_{-\epsilon}^{\epsilon} t^2 dt = \int_{-\epsilon}^{\epsilon} tf(t)dt \quad \therefore \quad b = \frac{3}{2\epsilon^2}\int_{-\epsilon}^{\epsilon} tf(0+t)dt,$$

and if instead of $f_L'(0,\epsilon)$ we had been interested in $f_L'(x,\epsilon)$ then would have been (1), and hence proved. It is important to note the significance of the LSM in the deduction of the Lanczos formula [10, 11]. We must emphasize that (1) was first deduced by Cioranescu [12] and Haslam-Jones [13].

In Sec. 2 we employ quadrature by differentiation [14, 15] to obtain (1), which permits to generalize the Lanczos derivative for higher orders, in compatibility with the corresponding expression constructed in [16]. The Sec. 3 exhibits the application of (1) to Fourier series, which generates the presence of the celebrated Lanczos $\sigma$-factors [1, 17-20] of great importance in the study of the Gibbs phenomenon [1, 17-19, 21-25], because by the method of the sigma smoothing the convergence of the Fourier series is increased due to a reduction of the amplitudes of the Gibbs oscillations [26-28]. We note that it is an open problem to find a geometrical meaning of the Cioranescu-(Haslam-Jones)-Lanczos derivative [8].

## 2. LANCZOS GENERALIZED DERIVATIVE VIA A QUADRATURE METHOD.

Here we shall deduce the Lanczos derivative from a quadrature technique with boundary values. In numerical analysis there exist several algorithms to determine the area under a curve F(x), known as quadrature methods, for instance [1], the Simpson and trapezoidal rules, the non-equidistant process of Gauss [29, 30] based on the Legendre polynomials [31-33], etc., which employ the values of F(x) in different points into the given interval $[a, b]$. Lanczos [1] obtained a remarkable quadrature formula where only are needed the values of the function and its derivatives at the end points a and b:

$$\int_a^b F(t) = \sum_{k=1}^{n} \frac{(2n-k)!!}{(2n)!}\binom{n}{k}\left[F^{(k-1)}(a) - (-1)^k F^{(k-1)}(b)\right]h^k, \;\; h = b-a \tag{6}$$

The Lanczos expression is quite efficient because with only few terms it gives a result very near to the exact one, and this property is useful to solve differential equations with boundary values; for $n = 3$ the relation (6) turns out to be:

$$\int_a^b F(t)dt = \frac{h}{2}[F(a) + F(b)] + \frac{h^2}{10}[F'(a) - F'(b)] + \frac{h^3}{120}[F''(a) + F''(b)]. \tag{7}$$

In this Section is proven that (6) leads, naturally, to Lanczos derivative (1), and our treatment shows that it can be obtained without the explicit use of the LSM. In fact, (6) is applied for $F(t) = tf(x+t)$ with $a = -\epsilon, b = \epsilon, \epsilon ł 1$, so $h = 2\epsilon$ is very small and then with (7) it can be performed our analysis, therefore:

$$\frac{3}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} = \frac{9}{5}\frac{f(x+\epsilon)-f(x-\epsilon)}{2\epsilon} - \frac{2}{5}[f'(x+\epsilon)+f'(x-\epsilon)] + \frac{\epsilon}{10}[f''(x+\epsilon)-f''(x-\epsilon)],$$

consequently:

$$\lim_{\epsilon\to 0}\frac{3}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} tf(x+t)dt = \frac{9}{5}f'(x) - \frac{4}{5}f'(x) = f'(x), \tag{8}$$

in harmony with (1, 2).
If we apply (7) for $F(t) = t^2 f(x+t)$, then:

$$\int_{-\epsilon}^{\epsilon} t^2 f(x+t)dt = \frac{\epsilon^5}{15}[f''(x+\epsilon)+f''(x-\epsilon)] + \frac{\epsilon^3}{3}[f(x+\epsilon)+f(x-\epsilon)] - \\ - \frac{2\epsilon^4}{15}[f'(x+\epsilon)+f'(x-\epsilon)], \tag{9}$$

but (7) with $F(t) = f(x+t)$, gives:

$$\frac{\epsilon^3}{3}[f(x+\epsilon)+f(x-\epsilon)] - \frac{2\epsilon^4}{15}[f'(x+\epsilon)+f'(x-\epsilon)] = \\ \frac{\epsilon^2}{3}\int_{-\epsilon}^{\epsilon} f(x+t)dt - \frac{\epsilon^5}{45}[f''(x+\epsilon)+f''(x-\epsilon)],$$

therefore (9) implies:

$$\frac{1}{\epsilon^3}\int_{-\epsilon}^{\epsilon}\left(\frac{t^2}{\epsilon^2}-\frac{1}{3}\right) f(x+t)dt = \frac{2}{45}[f''(x+\epsilon)+f''(x-\epsilon)],$$

finally:

$$f''(x) = \lim_{\epsilon\to 0}\frac{15}{4\epsilon^3}\int_{-\epsilon}^{\epsilon}\left(\frac{3t^2}{\epsilon}-1\right) f(x+t)dt = \lim_{\epsilon\to 0}\frac{5!!}{2\epsilon^3}\int_{-\epsilon}^{\epsilon} P_2\left(\frac{t}{\epsilon}\right) f(x+t)dt \tag{10}$$

where $P_2(u) = \frac{1}{2}(3u^2-1)$ is a Legendre polynomial [31-33]. A similar process with $F(t) = t^3 f(x+t)$ leads to:

$$f'''(x) = \lim_{\epsilon\to 0}\frac{7!!}{2\epsilon^4}\int_{-\epsilon}^{\epsilon} P_3\left(\frac{t}{\epsilon}\right) f(x+t),\ P_3(u) = \frac{1}{2}(5u^3-3u), \tag{11}$$

thus (8,10,11) permit to write the Lanczos derivative for higher orders:

$$f_L^n(x) = \lim_{\epsilon\to 0}\frac{(2n+1)!!}{2\epsilon^{n+1}}\int_{-\epsilon}^{\epsilon} P_n\left(\frac{t}{\epsilon}\right) f(x+t)dt, n = 0,1,2,... \tag{12}$$

in according with the result of Rangarajan-Purushothaman [16]. In this manner, the method of differentiation by integration due to Cioranescu-(Haslam-Jones)-Lanczos is generalized to cover derivatives of arbitrary order. We note that Lanczos [1] uses the Legendre polynomials to deduce the quadrature formula (6), then in (12) is natural the participation of these polynomials. Washburn [8] comments the possible presence of the Legendre polynomials in an expression for $f_L^{(n)}(x)$.

## 3. Lanczos derivative applied to Fourier series

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It is very known that if the operator $\frac{d}{dx}$ acts on each term into a convergent Fourier series, then it may result a divergent series. This situation is remedied [1] applying the symmetric derivative [34] to Fourier series, which implies the existence of the important -factors. Here we show that the Lanczos derivative also leads to these factors. If on the Fourier series:

$$f(x) = 1/2a_0 + \sum_{k=1}^{\infty} k[-a_k \cos(kx) + b_k \sin(kx)], \tag{13}$$

convergent into $[-\pi, \pi]$, we apply the operator $\frac{d}{dx}$ results:

$$\frac{d}{dx} f(x) = \sum_{k=1}^{\infty} k[-a_k \cos(kx) + b_k \sin(kx)], \tag{14}$$

which it may be divergent [1, 28]. This problem was remedied by Lanczos [1] with $f'(x)$ defined as a symmetric derivative [34]:

$$f'(x) = \lim_{n\to\infty} \frac{1}{\frac{2\pi}{n}} \left[ f_n\left(x + \frac{\pi}{n}\right) - f_n\left(x - \frac{\pi}{n}\right)\right], \tag{15}$$

with the partial sums:

$$f_n(x) = g_n(x) + h_n(x),$$

$$g_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \cos(kx), \quad h_n(x) = \sum_{k=1}^{n} b_k \sin(kx) \tag{16}$$

resulting the convergent expression:

$$f'(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \sigma_k \frac{d}{dx}[a_k \cos(kx) + b_k \sin(kx)], \tag{17}$$

with the Lanczos $\sigma$-factors [1, 18]:

$$\sigma_0 = 1, \sigma_n = 0, \sigma_k = \frac{sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}}, k = 1, 2, ..., n-1. \tag{18}$$

This set of factors, for a given n, it is equivalent to a discrete sampling function. In (14, 15) we employ two types of derivatives, however, it is natural to ask if the Lanczos derivative lead to relation (17). The answer is yes, in fact:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t g_n(x+t)dt = \frac{3}{2\epsilon^3} \sum_{k=1}^{n} a_k \int_{-\epsilon}^{\epsilon} t\cos(kx+kt)dt = -3 \sum_{k=1}^{n} a_k \frac{\sin(kx)}{k^2} A_k,$$

such that $A_k(\epsilon) = \frac{1}{\epsilon^3}[\sin(k\epsilon) - k\epsilon\cos(k\epsilon)]$. Similarly:

$$\frac{3}{2\epsilon^3} \int_{-\epsilon}^{\epsilon} t h_n(x+t)dt = 3 \sum_{k=1}^{n} b_k \frac{\cos(kx)}{k^2} A_k \tag{19}$$

Therefore, the Lanczos derivative applied to partial sum (16) gives, taking $\epsilon = \frac{\pi}{n}$:

$$f'_L(x) = \lim_{n\to\infty} 3 \sum_{k=1}^{n} A_k[-a_k \sin(kx) + b_k \cos(kx)],$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{3A_k}{k^3} \frac{d}{dx}[a_k \cos(kx) + b_k \sin(kx)], \tag{20}$$

but the Bernoulli-H$\hat{o}$pital rule permits to observe the following behavior for $n1$:

$$A_k\left(\epsilon = \frac{\pi}{n}\right) \longrightarrow \frac{k^3}{3} \frac{\sin(k\epsilon)}{k\epsilon} = \frac{k^3}{3} \frac{\sin\left(\frac{k\pi}{n}\right)}{\frac{k\pi}{n}} = \frac{k^3}{3}\sigma_k,$$

and this value into (20) implies (17), q.e.d. Thus, it is proved that the symmetric and Lanczos derivatives give the same expression for the derivative of an infinite Fourier series, with the important presence of the $\sigma$-factors.

## References

[1] C. Lanczos, *Applied analysis*, Prentice-Hall, New Jersey (1956).

[2] A. M. Legendre, *Nouvelles methods pour la determination orbites des cometes*, Paris (1806).

[3] C. F. Gauss, *Theoria motus corporum coelestium in sectionibus conicis solem ambientium*, Göttingen (1809) 205-224.

[4] P. S. Laplace, *Théorie analytique des probabilities*, Paris (1812), Chap. 4.

[5] C. W. Groetsch, *Lanczos generalized derivative*, Am. Monthly 105, No. 4 (1998), 320-326.

[6] J. Shen, *On the Lanczos generalized derivative*, Am. Monthly 106, No. 8 (1999), 766-768.

[7] D. L. Hicks and L. M. Liebrock, *Lanczos generalized derivative: Insights and applications*, Applied Maths. Comp. 112, No. 1 (2006) 63-73.

[8] L. Washburn, *The Lanczos derivative*, Senior Project Archive, Dept. of Maths., Whitman College, USA (2006).

[9] J. López-Bonilla, A. Iturri-Hinojosa and A. Bucur, *Lanczos derivative for a discontinuous function*, The Aligarh Bull. Maths. 29, No. 1 (2010) 35-39.

[10] N. Burch, P. E. Fishback and R. Gordon, *The least square property of the Lanczos derivative*, Maths. Magazine 78, No. 5 (2005) 368-378.

[11] R. Gordon, *A least squares approach to differentiation*, Real Analysis Exchange 35, No. 1 (2009) 205-228.

[12] N. Cioranescu, *La generalization de la premiére formule de la moyenne*, Enseign. Math. 37 (1938) 292-302.

[13] U. S. Haslam-Jones, *On a generalized derivative*, Quart. J. Math. Oxford Ser. (2) 4 (1953) 190-197.

[14] W. Squire, *Some applications of quadrature by differentiation*, J. Soc. Indust. Appl. Maths. 9, No. 1 (1961) 94-108.

[15] J. López-Bonilla, J. Rivera R. and S. Vidal B., *Lanczos derivative via a quadrature method*, Int. J. Pure Appl. Sci. Technol. 1, No. 2 (2010) 100-103.

[16] S. K. Rangarajan and S. P. Purushothaman, *Lanczos generalized derivative for higher orders*, J. Comp. Appl. Maths. 177, No. 2 (2005) 461-465.

[17] C. Lanczos, *Discourse on Fourier series*, Oliver and Boyd, London (1966).

[18] C. Lanczos, *Linear differential operators*, Dover, New York (1997).

[19] A. J. Jerri, *Lanczos-like $\sigma$-factors for reducing the Gibbs phenomenon in general orthogonal expansions and other representations*, J. Comp. Analysis Appls. 2 (2000) 1-17.

[20] R. Cruz S., J. López-Bonilla and R. López-Vázquez, *Lanczos derivative applied to Fourier series*, J. Sci. Res. (India) 57 (2013) 165-167.

[21] F. Ustina, *Henry Wilbraham and Gibbs phenomenon in 1848*, Historia Mathematica 1 (1974) 83-84.

[22] E. Hewitt and R. E. Hewitt, *The Gibbs-Wilbraham phenomenon: An episode in Fourier analysis*, Arch. Hist. Exact Sci. 21 (1979) 129-160.

[23] D. Gottlieb and Chi-Wang Shu, *On the Gibbs phenomenon and its resolution*, SIAM Rev. 39, No. 4 (1997) 644-668.

[24] T. H. Fay and P. H. Kloppers, *The Gibbs phenomenon*, Int. J. Math. Educ. Sci. Technol. 32, No. 1 (2001) 73-89.

[25] A. J. Jerri, *The Gibbs phenomenon*, Sampling Pub., New York (2007).

[26] H. Wilbraham, *On a certain periodic function*, Cambridge and Dublin Math. J. 3 (1848) 198-201.

[27] A. Michelson, *Fouriers series*, Nature 58 (1898) 544-545.

[28] J. W. Gibbs, *Fourier series*, Nature 59 (1898-1899) 200 and 606.

[29] C. F. Gauss, *Methodus nova integralium valores per approximationen inveniendi*, (1814).
[30] J. H. Caltenco, J. López-Bonilla and J. Rivera R., *Gaussian quadrature via Hermite and Lagrange interpolations*, J. Sci. Res. (India) 55 (2011) 177-180.
[31] A. M. Legendre, *Recherches sur lattraction des spheroids homogénes*, Mémoires de Mathématiques et Physique, Acad. Roy. Sci. Paris, 10 (1785) 411-435.
[32] C. Lanczos, *Legendre versus Chebyshev polynomials*, in Topics in Numerical Analysis [Proc. Roy. Irish Acad. Conf. on Numerical Analysis, 1972], Ed. John J.H. Miller, Academic Press, London (1973) 191-201.
[33] V. Barrera, J. López-Bonilla and J. Sosa, *Multiple root finder algorithm for Legendre and Chebyshev polynomials via Newton method*, Annales Mathematicae et Informaticae 33 (2006) 3-13.
[34] C. E. Aull, *The first symmetric derivative*, Am. Math. Monthly 74, No. 6 (1967) 708-711.

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