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ON THE TANGENT SPHERE BUNDLE WITH THE SASAKI SEMI RIEMANN METRIC OF A SPACE FORM

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ABSTRACT. In this paper, the Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε $T_\varepsilon S_1^2$ of the unit 2-sphere S_1^2 in semi Euclidean space E_1^3 is obtained . In addition, the connection coefficients of the Levi Civita connection on the semi Riemann manifold $(T_\varepsilon S_1^2, g^S)$ are found. Furthermore, a non-linear differential equation's system which gives geodesics of $T_\varepsilon S_1^2$ is obtained. Finally, the components of the Riemann curvature tensor of $T_\varepsilon S_1^2$ are calculated.

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1. INTRODUCTION

The geometry of the tangent bundles with a semi Riemann metric is one of well known the subjects for the scientists related to the bundle geometry. But the geometry of the tangent sphere bundles with a semi Riemann metric is a new subject.
The tangent sphere bundle of an n-dimensional manifold is defined as the disjoint union of the tangent vector space created by the unit tangent vectors at all points of this manifold. The first time was considered that the disjoint union of the tangent vector space created by the unit tangent vectors at all points of a geodesic circle of the unit 2-sphere gave a sphere and by moving this sphere along the geodesic circle was produced a torus by Klingenberg and Sasaki in [3]. Moreover, the authors studied on the torus family, which contains produced all torus along each geodesic circle of the unit 2-sphere. The authors in their study proved that T_1S^2 was a Riemann manifold with constant sectional curvature. Then they indicated that the unit vector fields which make a constant angle with geodesic circles of the unit 2-sphere produced the geodesics of T_1S^2.Furthermore, they obtained a surface $\widetilde{F}$ on T_1S^2 by defining a surface F between two parallel circles with equal distance from the equator circle of S^2 and they asserted that the surface $\widetilde{F}$ are homomorphic to a torus. They found that the surface $\widetilde{F}$ is flat. Nagy [4] calculated the components of the Riemann sectional curvature of tangent sphere bundle T_1M of a 2-dimensional Riemann manifold M. Moreover, he obtained that a curve $(x(t), y(t))$ in the unit tangent bundle had the geodesic curve if and only if the geodesic curvature of $x(t)$ with Gaussian curvature of M must have been a constant rate or the parallel displacement of the vector component $y(t)$ along the curve $x(t)$ must have drawn a helical curve. Sasaki [6] classified the geodesics on the tangent sphere bundle of the unit n-sphere S^n

and the hyperbolic n-space H^n by using the general formula of the Sasaki Riemann metric on T_1S^n and T_1H^n and taking regard of this classification, he obtained three different types geodesics on T_1S^3 and T_1H^2. Ayhan [1] obtained Sasaki Riemann metric of the tangent sphere bundle of the unit 3-sphere by using the geodesic polar coordinate of the unit 3-sphere. Furthermore, he calculated the general geodesic equations of the tangent sphere bundle of the unit 3-sphere.
The aim of this study is to examine the geometry of tangent sphere bundle with radius ε of the unit 2-sphere in 3-dimensional semi Euclidean space with index one as a space form. Firstly, the Sasaki semi Riemann metric g^S on the tangent sphere bundle with radius ε $T_\varepsilon S_1^2$ is obtained by using the parametric representation of the unit 2-sphere S_1^2. Then, the connection coefficients of the Levi Civita connection of $\left(T_\varepsilon S_1^2,\ g^S\right)$ have been calculated and then a non-linear differential equation's system which gives geodesics of $T_\varepsilon S_1^2$ has been obtained. Finally, the components of the Riemann curvature tensor of $T_\varepsilon S_1^2$ have been calculated.

2. Two Sphere in E_1^3

In this section, the parametric representation of the unit 2 sphere in semi Euclidean space, the induced semi Riemann metric on S_1^2, base vectors of the tangent vector space at any point on S_1^2, the Christoffel symbols of S_1^2, a system of differential equations, which gives geodesics of S_1^2 are re-considered by using the studies in [2] and [5].

Definition 2.1. *Let $<,>$ be non degenerate, symmetric, bilinear form in semi Euclidean space E_1^3 defined by*

$$< u,v >= -u_1v_1 + u_2v_2 + u_3v_3, \tag{1}$$

for any vectors $u,v \in E_1^3$. S_1^2 is a surface in E_1^3 given by

$$S_1^2 = \left\{u = (x_1,x_2,x_3) :< u,u >= 1, u \in E_1^3\right\}. \tag{2}$$

S_1^2 is called as unit 2 sphere in E_1^3. S_1^2 is given by the following equation:

$$-x_1^2 + x_2^2 + x_3^2 = 1, \tag{3}$$

with respect to rectangular coordinate system. The parametric representation of S_1^2 are given by

$$\begin{aligned} x_1 &= \sinh a, \\ x_2 &= \cosh a \cos\theta, \\ x_3 &= \cosh a \sin\theta, \end{aligned} \tag{4}$$

and a curve on the surface S_1^2 is described by

$$c : t \to c(t) = (a(t), \theta(t)), \tag{5}$$

where (a,θ) is called as the generalized coordinates of S_1^2.

In order to find the arc length parameter of any curve on S_1^2 for $t_0 \le t \le t_1$, it is used the covariant derivations of x_1, x_2, x_3 as follow:

$$\begin{aligned} dx_1 &= \cosh a da, \\ dx_2 &= \sinh a \cos\theta da - \cosh a \sin\theta d\theta, \\ dx_3 &= \sinh a \sin\theta da + \cosh a \cos\theta d\theta. \end{aligned} \tag{6}$$

Definition 2.2. *In semi Euclidean space E_1^3, the arc length between different two point with infinitesimal distance on the surface S_1^2 (i.e. (x_1, x_2, x_3) and $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$) is calculated with the following equations:*

$$\begin{aligned} ds^2 &= < (dx_1, dx_2, dx_3), (dx_1, dx_2, dx_3 > \\ &= -(dx_1)^2 + (dx_2)^2 + (dx_3)^2 . \end{aligned} \tag{7}$$

By using the (6), we get

$$ds^2 = -(da)^2 + \cosh^2 a \, (d\theta)^2 \tag{8}$$

and also the matrix representation of this equation

$$g_{ik} : \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2 a \end{pmatrix}, i, k \in \{1, 2\} \tag{9}$$

where g_{ik} is called as the components of the induced metric on S_1^2 from E_1^3. The inverse of g_{ik} has the following matrix representation:

$$g^{kj} : \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{\cosh^2 a} \end{pmatrix}. \tag{10}$$

Assuming that $e_1(a, \theta)$ is any point on S_1^2 given by

$$e_1(a, \theta) = (\sinh a, \cosh a \cos \theta, \cosh a \sin \theta) \tag{11}$$

with respect to standard orthonormal base of E_1^3. Since a curve on the surface S_1^2 is described by $c : t \to c(t) = (a(t), \theta(t))$, the unit tangent vector on $a-$curves and $\theta-$curves passing through the point $e_1(a, \theta)$ must be expressed by

$$f_2 = \frac{\partial}{\partial a} \quad \text{and} \quad f_3 = \frac{1}{\cosh a} \frac{\partial}{\partial \theta}. \tag{12}$$

The unit tangent vectors f_2 and f_3 has the following local expression:

$$\begin{aligned} f_2(a, \theta) &= (\cosh a, \sinh a \cos \theta, \sinh a \sin \theta), \\ f_3(a, \theta) &= (0, -\sin \theta, \cos \theta), \end{aligned} \tag{13}$$

with respect to standard orthonormal base of E_1^3. Thus $\{e_1, f_2, f_3\}$ is another orthonormal base of E_1^3.

Theorem 2.1. *Let S_1^2 be the unit 2-sphere in E_1^3. If $T_{e_1}S_1^2$ is a tangent vector space at any point $e_1(a, \theta)$ on S_1^2, g is a semi Riemann metric on S_1^2 defined by*

$$\begin{aligned} g : \quad T_{e_1}S_1^2 \times T_{e_1}S_1^2 &\to \quad IR \\ (X, Y) \quad &\to \quad g(X, Y) . \end{aligned} \tag{14}$$

Proof. *Let $X = af_2 + bf_3$, $Y = cf_2 + df_3$ and $Z = pf_2 + qf_3$ be the tangent vectors at any point on S_1^2 where $\{f_2,\ f_3\}$ is a orthonormal frame on $T_{e_1}S_1^2$. For all X, Y, $Z \in T_{e_1}S_1^2$ and α, $\beta \in IR$, we get*

$$\begin{aligned} g(\alpha X + \beta Y, Z) &= g(\alpha [af_2 + bf_3] + \beta [cf_2 + df_3], [pf_2 + qf_3]) \\ &= \alpha g(X, Z) + \beta g(Y, Z). \end{aligned}$$

Similarly we get $g(X, \alpha Y + \beta Z) = \alpha g(X, Y) + \beta g(X, Z)$. Thus g is a bilinear transformation. Furthermore g is a symmetric map for

$$\begin{aligned} g(X, Y) &= g(af_2 + bf_3, cf_2 + df_3) \\ &= g(Y, X). \end{aligned}$$

Finally, g is a non degenerate map such that

$$g(X,Y)=0 \Longleftrightarrow Y=0 \quad \text{for } \forall X \in T_{e_1}S_1^2.$$

Since g is a non degenerate, symmetric, bilinear form, g is a semi Riemann metric on the surface S_1^2*.*

Theorem 2.2. *Let S_1^2 be the unit 2-sphere in E_1^3. Let f_2, f_3 be the unit tangent vectors at any point on S_1^2 given by the equations (12) and (13). Then, f_2 is a time like vector and f_3 is a space like vector.*

Proof. *Since the value of the unit tangent vectors f_2 and f_3 given by (13) under the semi Euclidean metric $<,>$ in E_1^3 have the following expression:*

$$\begin{aligned} < \quad & f_2, f_2 >= -\cosh^2 a + \sinh^2 a \cos^2\theta + \sinh^2 a \sin^2\theta = -1, \\ < \quad & f_3, f_3 >= \sin^2\theta + \cos^2\theta = 1, \end{aligned}$$

f_2 and f_3 must be time like and space like unit tangent vector, respectively. If we consider the unit tangent vectors f_2 and f_3 given by (12), we must use the induced metric on S_1^2 from E_1^3 given by (9). As a consequence of this fact, we get

$$\begin{aligned} g(f_2,f_2) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2 a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -1, \\ g(f_3,f_3) &= \begin{pmatrix} 0 & \frac{1}{\cosh a} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & \cosh^2 a \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\cosh a} \end{pmatrix} = 1. \end{aligned}$$

Thus, f_2 is a time like vector and f_3 is a space like vector.

Theorem 2.3. *Let S_1^2 be the unit 2-sphere in E_1^3 and $\{e_1, f_2, f_3\}$ is a orthonormal base in E_1^3. The covariant derivations of these unit-orthogonal vectors are given by*

$$\begin{aligned} de_1 &= da f_2 + \cosh a d\theta f_3, \\ df_2 &= da e_1 + \sinh a d\theta f_3, \\ df_3 &= -\cosh a d\theta e_1 + \sinh a d\theta f_2. \end{aligned}$$

Proof. *We use the covariant derivations of orthonormal vectors e_1, f_2, f_3 in order to examine the change of the frames on different two points with infinitesimal distance on S_1^2 (i.e. (e_1, f_2, f_3) and $(e_1+de_1, f_2+df_2, f_3+df_3)$). The covariant derivatives of these vectors are calculated by using the partial derivation as follow:*

$$\begin{aligned} de_1 &= \frac{\partial e_1}{\partial a} da + \frac{\partial e_1}{\partial \theta} d\theta = da f_2 + \cosh a d\theta f_3, \\ df_2 &= \frac{\partial f_2}{\partial a} da + \frac{\partial f_2}{\partial \theta} d\theta = da e_1 + \sinh a d\theta f_3, \\ df_3 &= \frac{\partial f_3}{\partial a} da + \frac{\partial f_3}{\partial \theta} d\theta = -\cosh a d\theta e_1 + \sinh a d\theta f_2. \end{aligned}$$

Theorem 2.4. *Let (S_1^2, g) be a semi Riemann manifold. Let D be Levi Civita connection of (S_1^2, g) and $\phi_{ij}^k; i,j,k \in \{1,2\}$ be Christoffel symbols with respect to the semi Riemann metric g. Then the non-zero the Christoffel symbols of (S_1^2, g) have the following components:*

$$\phi_{22}^1 = \sinh a \cosh a, \quad \phi_{12}^2 = \tanh a,$$

where $\phi_{ij}^k = \phi_{ji}^k$ for all $i,j,k \in \{1,2\}$.

Proof. *Since D is Levi Civita connection of the semi Riemann manifold $\left(S_1^2, g\right)$, D is torsion free and compatible with g and D is characterized by the Kozsul formula:*

$$\begin{aligned} 2g\left(D_{\partial_a}\partial_\theta, \partial_\theta\right) &= \partial_a g\left(\partial_\theta, \partial_\theta\right) + \partial_\theta g\left(\partial_\theta, \partial_a\right) - \partial_\theta g\left(\partial_a, \partial_\theta\right) + \\ &\quad -g\left(\left[\partial_a, \partial_\theta\right], \partial_\theta\right) + g\left(\left[\partial_\theta, \partial_\theta\right], \partial_a\right) + g\left(\left[\partial_\theta, \partial_a\right], \partial_\theta\right) \end{aligned}$$

where $\partial_a = \frac{\partial}{\partial a} = \partial_1$, and $\partial_\theta = \frac{\partial}{\partial \theta} = \partial_2$. Since D is symmetric, $[\partial_a, \partial_\theta]$ must be zero. If we get $D_{\partial_a}\partial_\theta = \phi_{12}^1\partial_a + \phi_{12}^2\partial_\theta$, from Kozsul formula, Christoffel symbols are obtained by

$$\begin{aligned} \phi_{12}^1 &= \frac{1}{2}g^{1m}\left(\partial_1 g_{m2} + \partial_2 g_{2m} - \partial_m g_{12}\right) = 0, \\ \phi_{12}^2 &= \frac{1}{2}g^{2m}\left(\partial_1 g_{m2} + \partial_2 g_{2m} - \partial_m g_{12}\right) = \tanh a \end{aligned}$$

where $m \in \{1, 2\}$. The other Christoffel symbols can be obtained by using the similar method.

Theorem 2.5. *Let $\left(S_1^2, g\right)$ be semi Riemann manifold and $c : t \in R \to c(t) = (a(t), \theta(t)) \in S_1^2$ be a curve on S_1^2. c is a geodesic if and only if the following differential equation's system has been provided:*

$$\ddot{a} + \sinh a \cosh a\dot{\theta}^2 = 0, \tag{15}$$

$$\ddot{\theta} + 2\tanh a\dot{a}\dot{\theta} = 0. \tag{16}$$

Proof. *$c(t) = (a(t), \theta(t))$ is geodesic if and only if $D_{\dot{c}}\dot{c}$ must be zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a + \dot{\theta}\partial_\theta$, $D_{\dot{c}}\dot{c}$ is equal to $D_{\dot{a}\partial_a}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta\right) + D_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta\right)$. For $D_{\dot{c}}\dot{c} = 0$*

$$D_{\dot{c}}\dot{c} = \left(\ddot{a} + \sinh a \cosh a\dot{\theta}^2\right)\partial_a + \left(\ddot{\theta} + 2\tanh a\dot{a}\dot{\theta}\right)\partial_\theta$$

it is seen easily that the claim of the theorem is correct.

Definition 2.3. *The first fundamental form of S_1^2 is given by*

$$ds^2 = -\dot{a}^2 + \cosh a\dot{\theta}^2 = \varepsilon. \tag{17}$$

A curve $c : t \in R \to c(t) = (a(t), \theta(t)) \in S_1^2$ which provides the equations in (17) is called as a time like curve or a light like curve or a space like curve for $\varepsilon = -1$, $\varepsilon = 0$ or $\varepsilon = 1$, respectively.

In the rest of the paper, the curve c will be assumed as a geodesic of S_1^2. Now let find a general equation characterizing time like, light like or space like geodesics on S_1^2. From (17), we get

$$-\left(\frac{da}{d\theta}\dot{\theta}\right)^2 + \cosh a\dot{\theta}^2 = \varepsilon. \tag{18}$$

If we solve the differential equation in (16)

$$\left\{\frac{d}{da}\left(\dot{\theta}\right) + 2\tanh a\dot{\theta}\right\}\dot{a} = 0 \Rightarrow \dot{\theta} = k \sec h^2 a \vee \dot{a} = 0. \tag{19}$$

and the value $\dot{\theta} = k \sec h^2 a$ put in the equation (18) , the general equation characterizing time like, light like, space like geodesics on S_1^2 is obtained as follows:

$$\frac{da}{d\theta} = \frac{\sqrt{k^2\cosh^2 a - \varepsilon\cosh^4 a}}{k}. \tag{20}$$

Theorem 2.6. *Time like geodesics on the surface S_1^2 are given by the following parametric or rectangular coordinates of S_1^2 :*

$$\sin\theta = \frac{k}{\sqrt{k^2+1}}\tanh a,\ x_3 = \frac{k}{\sqrt{k^2+1}}x_1.$$

Proof. The one parameter curves family, defines a plane, is obtained by putting $\varepsilon = -1$ in (20). Time like geodesics of S_1^2 are cross-section curves between one parameter curves family and S_1^2.

Theorem 2.7. *The light like geodesics on the surface S_1^2 are given by the following parametric or rectangular coordinates of S_1^2 :*

$$\tan\left(\frac{\theta}{2} - \frac{\pi}{4}\right) = \tanh\left(\frac{a}{2}\right),\ x_1x_3 = -2x_2\sqrt{1+x_1^2}.$$

Proof. *The one parameter curves family, defines a surfaces, is obtained by putting $\varepsilon = 0$ in (20). Light like geodesics of S_1^2 are cross-section curves between one parameter curves family and S_1^2.*

Theorem 2.8. *Space like geodesics on the surface S_1^2 are given by the following parametric or rectangular coordinates of S_1^2 :*

$$\sin\theta = \frac{k}{\sqrt{k^2-1}}\tanh a,\ x_3 = \frac{k}{\sqrt{k^2-1}}x_1.$$

Proof. *The one parameter curves family, defines a plane, is obtained by putting $\varepsilon = 1$ in (20). Space like geodesics of S_1^2 are cross-section curve between one parameter curves family and S_1^2.*

3. The Tangent Sphere Bundle with radius ε of the Unit 2-Sphere in E_1^3

In this section, we studied on some subjects as the expression in terms of the local coordinate function of any point on $T_\varepsilon S_1^2$, the orthonormal frame at any point on $T_\varepsilon S_1^2$, the covariant derivations of this orthonormal base elements, Sasaki semi Riemann metric g^S on $T_\varepsilon S_1^2$, the adapted base and dual base vectors on $T_\varepsilon S_1^2$ with respect to g^S. Furthermore,we obtained the connection coefficients of the Levi Civita connection of Sasaki semi Riemann manifold $\left(T_\varepsilon S_1^2, g^S\right)$, a differential equation's system which give geodesics on the Sasaki semi Riemann manifold. Finally, we calculated the components of the Riemann curvature tensor of $\left(T_\varepsilon S_1^2, g^S\right)$.

Definition 3.1. $T_\varepsilon S_1^2 = \bigcup_{\forall e_1(a,\theta)\in S_1^2}\{u \in T_{e_1}S_1^2 : g(u,u) = \varepsilon\}$ *is the disjoint union of the tangent vector spaces including all unit tangent vectors at every point of S_1^2. Thus, $T_\varepsilon S_1^2$ is the total space of time like, lightlike and space like vectors with respect to the induced metric g from standart semi Euclidean metric in E_1^3 and $T_\varepsilon S_1^2$ is called as the tangent sphere bundle with radius ε of S_1^2.*

Let $\pi : T_\varepsilon S_1^2 \to S_1^2$ be a canonical projection map and e_2 be an element of $T_\varepsilon S_1^2$ at the point $e_1(a,\theta)$ of S_1^2. If we denote the angle between f_2 and e_2 by ω, then (a,θ,ω) can be considered as local coordinates for e_2 in $\pi^{-1}(S_1^2)$. Therefore, e_2 and e_3 has the local expression

$$\begin{aligned} e_2(a,\theta,\omega) &= \cosh\omega f_2 + \sinh\omega f_3, \\ e_3(a,\theta,\omega) &= \sinh\omega f_2 + \cosh\omega f_3, \end{aligned} \tag{21}$$

where e_3 is an element of $T_\varepsilon S_1^2$ at the point $e_1(a,\theta)$ of S_1^2.

Theorem 3.1. *Let $T_\varepsilon S_1^2$ be the tangent sphere bundle with radius ε of S_1^2 in E_1^3 and e_2, e_3 be the tangent vectors at a point $e_1(a,\theta)$ on S_1^2 given by the equations (3.1), then e_2 is the unit time like vector and e_3 is the unit space like vectors.*
Proof. *The value of the tangent vectors e_2 and e_3 given by (3.1) under the semi Euclidean metric in E_1^3 are*

$$< e_2, e_2 >= \cosh^2\omega < f_2, f_2 > + \sinh^2\omega < f_3, f_3 >= -1,$$
$$< e_3, e_3 >= \sinh^2\omega < f_2, f_2 > + \cosh^2\omega < f_3, f_3 >= 1.$$

Thus, e_2 is the unit time like vector and e_3 is the unit space like vector.

Theorem 3.2. *Let $T_\varepsilon S_1^2$ be the tangent sphere bundle with radius ε of the unit 2-sphere in E_1^3 and e_1, e_2, e_3 be unit-orthogonal elements of $T_\varepsilon S_1^2$. The covariant derivations of these elements are given by*

$$\begin{aligned} de_1 &= (\cosh\omega da - \sinh\omega\cosh a d\theta)\, e_2 + (-\sinh\omega da + \cosh\omega\cosh a d\theta)\, e_3, \\ de_2 &= (\cosh\omega da - \sinh\omega\cosh a d\theta)\, e_1 + (d\omega + \sinh a d\theta)\, e_3, \\ de_3 &= (\sinh\omega da - \cosh\omega\cosh a d\theta)\, e_1 + (d\omega + \sinh a d\theta)\, e_2. \end{aligned}$$

Proof. *We can use the covariant derivations of e_1, e_2, e_3 in order to examine the change of the frames on different two points with infinitesimal distance on $T_\varepsilon S_1^2$ (i.e. (e_1, e_2, e_3) and $(e_1 + de_1, e_2 + de_2, e_3 + de_3)$). The covariant derivatives of e_1, e_2, e_3 are obtained by helping the partial derivation, easily.*

Definition 3.2. *The 1-forms providing the equation $w_{ij} =< de_i, e_j >$, for $i, j \in \{1,2,3\}$ are called as the connection 1-forms on the cotangent space $T^*_{(e_1,e_2)}T_\varepsilon S_1^2$ where w_{ij} is given by*

$$\begin{aligned} \eta^1 &= w_{12} = -w_{21} = -\cosh\omega da + \sinh\omega\cosh a d\theta, \\ \eta^2 &= w_{13} = -w_{31} = -\sinh\omega da + \cosh\omega\cosh a d\theta, \\ \eta^3 &= w_{23} = -w_{32} = d\omega + \sinh a d\theta. \end{aligned} \tag{22}$$

Theorem 3.3. *The line element between infinitely close two point on $T_\varepsilon S_1^2$ is given by*

$$d\sigma^2 = -(da)^2 + \cosh 2a\,(d\theta)^2 + 2\sinh a d\theta d\omega + (d\omega)^2. \tag{23}$$

Proof. *Let $\{e_1, e_2, e_3\}$ be the orthonormal frame at any point $e_2 \in \pi^{-1}(\{e_1\})$ on $T_\varepsilon S_1^2$ and $\{e_1 + de_1, e_2 + de_2, e_3 + de_3\}$ be the orthonormal frame at another point to be infinitely close point to e_2. The infinitesimal length between this two point is obtained as follows:*

$$\begin{aligned} d\sigma^2 &= < de_1, de_1 > + < de_2, e_3 >^2 \\ &= -\eta^1 \wedge \eta^1 + \eta^2 \wedge \eta^2 + \eta^3 \wedge \eta^3 \\ &= -(da)^2 + \cosh 2a\,(d\theta)^2 + 2\sinh a d\theta d\omega + (d\omega)^2. \end{aligned}$$

Definition 3.3. *$d\sigma^2 = g^S$ is called as a metric structure on the manifold $T_\varepsilon S_1^2$. Moreover, $\{\eta^1, \eta^2, \eta^3\}$ is called as an adapted basis 1-forms for the cotangent space $T^*_{(e_1,e_2)}T_\varepsilon S_1^2$ with respect to g^S. The tangent vectors $\xi_i; i \in \{1,2,3\}$ providing the equation*

$$\eta^i(\xi_i) = g^S(\xi_i, \xi_i) = \varepsilon_i, \varepsilon_i = \begin{cases} 1 & for \quad i = 2,3 \\ -1 & for \quad i = 1 \end{cases} \tag{24}$$

are called as adapted basis vectors of the tangent space $T_{(e_1,e_2)}T_\varepsilon S_1^2$ with respect to the metric structure g^S where ξ_i is defined by

$$\begin{aligned} \xi_1 &= \cosh\omega\frac{\partial}{\partial a}+\frac{\sinh\omega}{\cosh a}\frac{\partial}{\partial\theta}-\tanh a\sinh\omega\frac{\partial}{\partial\omega}, \\ \xi_2 &= \sinh\omega\frac{\partial}{\partial a}+\frac{\cosh\omega}{\cosh a}\frac{\partial}{\partial\theta}-\tanh a\cosh\omega\frac{\partial}{\partial\omega}, \\ \xi_3 &= \frac{\partial}{\partial\omega}. \end{aligned} \tag{25}$$

Theorem 3.4. *Let $T_\varepsilon S_1^2$ be the tangent sphere bundle with radius ε of the unit 2-sphere in E_1^3. If $T_{(e_1,e_2)}T_\varepsilon S_1^2$ is a tangent vector space at any point on $T_\varepsilon S_1^2$, g^S is semi Riemann metric on $T_\varepsilon S_1^2$ where g^S is defined by*

$$\begin{array}{rcl} g^S: \; T_{(e_1,e_2)}T_\varepsilon S_1^2\times T_{(e_1,e_2)}T_\varepsilon S_1^2 & \to & IR \\ (X,Y) & \to & g(X,Y) \end{array} \tag{26}$$

Proof. *Let $\widetilde{X}=x^i\xi_i$, $\widetilde{Y}=y^j\xi_j$ and $\widetilde{Z}=z^k\xi_k$ for $i,j,k\in\{1,2,3\}$ be the tangent vectors at any point on $T_\varepsilon S_1^2$ where $\{\xi_1,\xi_2,\xi_3\}$ is orthonormal adapted base of $T_{(e_1,e_2)}T_\varepsilon S_1^2$. For all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in T_{(e_1,e_2)}T_\varepsilon S_1^2$ and $\alpha,\beta\in IR$, we get*

$$\begin{aligned} g^S(\alpha\widetilde{X}+\beta\widetilde{Y},\widetilde{Z}) &= g^S(\{\alpha\left[x^i\xi_i\right]+\beta\left[y^i\xi_i\right]\},z^j\xi_j) \\ &= \alpha g^S(\widetilde{X},\widetilde{Z})+\beta g^S(\widetilde{Y},\widetilde{Z}). \end{aligned}$$

Similarly we get $g^S(\widetilde{X},\alpha\widetilde{Y}+\beta\widetilde{Z})=\alpha g^S(\widetilde{X},\widetilde{Y})+\beta g^S(\widetilde{X},\widetilde{Z})$. Thus g^S is bilinear transformation. Since the follow equality is hold

$$g^S(\widetilde{X},\widetilde{Y}) = g^S(x^i\xi_i,y^j\xi_j)=y^ix^i\varepsilon_i=g^S(\widetilde{Y},\widetilde{X}).$$

g^S must be symmetric map. Finally, g^S is a non degenerate map because g^S provides

$$g^S(\widetilde{X},\widetilde{Y})=0\Longleftrightarrow\widetilde{Y}=0 \quad \textit{for}\ \forall\widetilde{X}\in T_{e_1}S_1^2.$$

Since g^S is non degenerate, symmetric, bilinear form with constant index, g^S is semi Riemann metric on $T_\varepsilon S_1^2$. g^S is called as Sasaki semi Riemann metric. Moreover $\left(T_\varepsilon S_1^2,g^S\right)$ is also called as Sasaki semi Riemann manifold.

Theorem 3.5. *Let $T_\varepsilon S_1^2$ be tangent sphere bundle with radius ε of the unit 2-sphere in 3 dimensional semi Euclidean space with index one and $\{\xi_1,\xi_2,\xi_3\}$ be orthonormal frame of $T_{(e_1,e_2)}T_\varepsilon S_1^2$ with respect to Sasaki semi Riemann metric g^S. Then ξ_2,ξ_3 are space like vectors, ξ_1 is a time like vector and $\frac{1}{\sqrt{2}}\{\xi_1+\xi_2\}$, $\frac{1}{\sqrt{2}}\{\xi_1+\xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_2-\xi_3\}$ are light like vectors.*

Proof. *The values of the tangent vectors ξ_1 and ξ_2, ξ_3 given by (3.5) under the Sasaki semi Riemann metric g^S are calculated as follows:*

$$\begin{aligned} g^S(\xi_1,\xi_1) &= \cosh^2\omega g^S(\frac{\partial}{\partial a},\frac{\partial}{\partial a})+\frac{\sinh^2\omega}{\cosh^2 a}g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) \\ &\quad -\frac{\sinh^2\omega}{\cosh^2 a}\sinh a g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\omega})+\tanh^2 a\sinh^2\omega g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) \\ &= -1, \end{aligned}$$

$$\begin{aligned} g^S(\xi_2,\xi_2) &= \sinh^2\omega g^S(\frac{\partial}{\partial a},\frac{\partial}{\partial a}) + \frac{\cosh^2\omega}{\cosh^2 a} g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\theta}) \\ &\quad -\frac{\cosh^2\omega}{\cosh^2 a}\sinh a g^S(\frac{\partial}{\partial\theta},\frac{\partial}{\partial\omega}) + \tanh^2 a\cosh^2\omega g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) \\ &= 1, \\ g^S(\xi_3,\xi_3) &= g^S(\frac{\partial}{\partial\omega},\frac{\partial}{\partial\omega}) = 1. \end{aligned}$$

As a consequence $g^S(\xi_1,\xi_1) = -1$ *and* $g^S(\xi_2,\xi_2) = g^S(\xi_3,\xi_3) = 1$, ξ_1 *is a time like vectors and* ξ_2, ξ_3 *are space like vectors with respect to* g^S. *Furthermore, it is seen easily that* $\frac{1}{\sqrt{2}}\{\xi_1+\xi_2\}$, $\frac{1}{\sqrt{2}}\{\xi_1+\xi_3\}$, $\frac{1}{\sqrt{2}}\{\xi_2-\xi_3\}$ *are light like vectors with respect to* g^S.

The Sasaki semi Riemann metric g^S on $T_\varepsilon S_1^2$ has the following matrix representation:

$$g_{\alpha\beta} : \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cosh 2a & \sinh a \\ 0 & \sinh a & 1 \end{pmatrix} \text{ for } \alpha,\beta \in \{1,2,3\} \tag{27}$$

The inverse matrix of $g_{\alpha\beta}$ is

$$g^{\beta\alpha} : \begin{pmatrix} -1 & 0 & 0 \\ 0 & \sec h^2 a & -\sec ha\tanh a \\ 0 & -\sec ha\tanh a & 1+\tanh^2 a \end{pmatrix}. \tag{28}$$

Theorem 3.6. *Let* $(T_\varepsilon S_1^2, g^S)$ *be Sasaki semi Riemann manifold. Let* ∇ *be Levi Civita connection of* $(T_\varepsilon S_1^2, g^S)$ *and* $\Gamma_{\alpha\beta}^\gamma; \alpha,\beta,\gamma \in \{1,2,3\}$ *be coefficients of the Christoffel symbols. Then the non-zero the Christoffel symbols of* $(T_\varepsilon S_1^2, g^S)$ *are given by*

$$\begin{array}{rr} \Gamma_{22}^1 = \sinh 2a, & \Gamma_{23}^1 = \frac{1}{2}\cosh a, \\ \Gamma_{12}^2 = \frac{3}{2}\tanh a, & \Gamma_{13}^2 = \frac{1}{2}\sec ha, \\ \Gamma_{12}^3 = \frac{3}{2}\tanh a\sinh a + \frac{1}{2}\cosh a, & \Gamma_{13}^3 = -\frac{1}{2}\tanh a, \end{array} \tag{29}$$

where $\Gamma_{\alpha\beta}^\gamma = \Gamma_{\beta\alpha}^\gamma$ *for all* $\alpha,\beta,\gamma \in \{1,2,3\}$.

Proof. *On the Sasaki semi Riemann manifold* $(T_\varepsilon S_1^2, g^S)$ *there is a unique connection* ∇ *such that* ∇ *is torsion free and compatible with semi Riemann metric* g^S. *This connection is Levi Civita connection and characterized by the Kozsul formula:*

$$\begin{aligned} 2g^S(\nabla_{\partial_a}\partial_\theta,\partial_\omega) &= \partial_a g^S(\partial_\theta,\partial_\omega) + \partial_\theta g^S(\partial_\omega,\partial_a) - \partial_\omega g^S(\partial_a,\partial_\theta) + \\ &\quad -g^S([\partial_a,\partial_\theta],\partial_\omega) + g^S([\partial_\theta,\partial_\omega],\partial_a) + g^S([\partial_\omega,\partial_a],\partial_\theta) \end{aligned}$$

where $\partial_1 = \partial_a = \frac{\partial}{\partial a}, \partial_2 = \partial_\theta = \frac{\partial}{\partial\theta}$ *and* $\partial_3 = \partial_\omega = \frac{\partial}{\partial\omega}$. *Since* ∇ *is symmetric,* $[\partial_a,\partial_\theta], [\partial_\theta,\partial_\omega], [\partial_\omega,\partial_a]$ *must be zero. If we get* $\nabla_{\partial_a}\partial_\theta = \Gamma_{12}^1\partial_a + \Gamma_{12}^2\partial_\theta + \Gamma_{12}^3\partial_\omega$, *from Kozsul formula, it is obtained the following Christoffel symbols:*

$$\begin{aligned} \Gamma_{12}^1 &= \frac{1}{2}g^{1k}(\partial_1 g_{k2} + \partial_2 g_{1k} - \partial_k g_{12}) = 0, \\ \Gamma_{12}^2 &= \frac{1}{2}g^{2k}(\partial_1 g_{k2} + \partial_2 g_{1k} - \partial_k g_{12}) = \frac{3}{2}\tanh a, \\ \Gamma_{12}^3 &= \frac{1}{2}g^{3k}(\partial_1 g_{k2} + \partial_2 g_{1k} - \partial_k g_{12}) = \frac{3}{2}\tanh a\sinh a + \frac{1}{2}\cosh a, \end{aligned}$$

where $k \in \{1,2,3\}$. Other Christoffel symbols can be obtained by using the similar method.

Theorem 3.7. *Let $\left(T_\varepsilon S_1^2, g^S\right)$ be Sasaki semi Riemann manifold and $c : t \in R \to c(t) = (a(t), \theta(t), \omega(t))$ be a curve on the tangent sphere bundle $T_\varepsilon S_1^2$. c is geodesic if and only if the following second order differential equation's system must be provided:*

$$\begin{aligned} \ddot{a} + \sinh 2a\dot{\theta}^2 + \cosh a\dot{\theta}\dot{\omega} &= 0, \\ \ddot{\theta} + 3\tanh a\dot{a}\dot{\theta} + \sec ha\dot{a}\dot{\omega} &= 0, \\ \ddot{\omega} + (\cosh a - 3\tanh a \sinh a)\,\dot{a}\dot{\theta} + \tanh ha\dot{a}\dot{\omega} &= 0. \end{aligned} \tag{30}$$

Proof. *$c(t) = (a(t), \theta(t), \omega(t))$ is geodesic if and only if $\nabla_{\dot{c}}\dot{c}$ must be zero. Since $\dot{c}$ is equal to $\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega$, $\nabla_{\dot{c}}\dot{c}$ is equal to*

$$\nabla_{\dot{a}\partial_a}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + \nabla_{\dot{\theta}\partial_\theta}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right) + \nabla_{\dot{\omega}\partial_\omega}\left(\dot{a}\partial_a + \dot{\theta}\partial_\theta + \dot{\omega}\partial_\omega\right).$$

If we organize $\nabla_{\dot{c}}\dot{c}$,

$$\begin{aligned} \nabla_{\dot{c}}\dot{c} = &\left(\ddot{a} + \sinh 2a\dot{\theta}^2 + \cosh a\dot{\theta}\dot{\omega}\right)\partial_a \\ &+ \left(\ddot{\theta} + 3\tanh a\dot{a}\dot{\theta} + \sec ha\dot{a}\dot{\omega}\right)\partial_\theta \\ &+ \left(\ddot{\omega} + (\cosh a - 3\tanh a\sinh a)\,\dot{a}\dot{\theta} + \tanh ha\dot{a}\dot{\omega}\right)\partial_\omega. \end{aligned}$$

it can be seen that the claim of the theorem is true.

Theorem 3.8. *Let $\left(T_\varepsilon S_1^2, g^S\right)$ be Sasaki semi Riemann manifold. The non-zero components of the Riemann curvature tensor of $\left(T_\varepsilon S_1^2, g^S\right)$ are given by*

$$R_{112}^2 = \tfrac{7}{4},\quad R_{121}^3 = 2\sinh a,\quad R_{131}^3 = \tfrac{1}{4},\quad R_{212}^1 = \tfrac{3}{2}\cosh^2 a + \tfrac{1}{4},$$

$$R_{232}^2 = \tfrac{1}{4}\sinh a,\quad R_{223}^3 = \tfrac{1}{4}\sin h^2 a,\quad R_{321}^1 = \tfrac{1}{4}\sinh a,\quad R_{231}^1 = \tfrac{1}{4}\sinh a,$$

$$R_{332}^2 = \tfrac{1}{4},\quad R_{323}^3 = \tfrac{1}{4}\sinh a,\quad R_{331}^1 = \tfrac{1}{4},$$

where $R_{\alpha\beta\gamma}^\mu = R_{\alpha\gamma\beta}^\mu$ for $\alpha, \beta, \gamma \in \{1,2,3\}$.

Proof. *Let $\Gamma_{\alpha\beta}^\gamma$ be the Christoffel symbol of the semi Riemann manifold $\left(T_\varepsilon S_1^2, g^S\right)$ and $R_{\alpha\beta\gamma}^\mu$ be the components of the Riemann curvature tensor. By using the known formula of the Riemann curvature tensor*

$$R_{\alpha\beta\gamma}^\mu = \partial_\beta\Gamma_{\alpha\gamma}^\mu - \partial_\gamma\Gamma_{\alpha\beta}^\mu + \Gamma_{\delta\beta}^\mu\Gamma_{\alpha\gamma}^\delta - \Gamma_{\delta\gamma}^\mu\Gamma_{\alpha\beta}^\delta,$$

and the Christoffel symbols of $\left(T_\varepsilon S_1^2, g^S\right)$ in (3.9), it can be seen that the claim of the theorem is correct.

4. MAIN RESULT

In this paper, we calculated the line element on the tangent sphere bundle with radius ε $T_\varepsilon S_1^2$of the unit 2-sphere S_1^2 in 3-dimensional semi Euclidean space with index one E_1^3 with respect to the induced coordinates (a, θ, ω) as follows:

$$d\sigma^2 = -(da)^2 + \cosh 2a\,(d\theta)^2 + 2\sinh a d\theta d\omega + (d\omega)^2,$$

and we found out the connection coefficients of the Levi Civita connection of the semi Riemann manifold $\left(T_\varepsilon S_1^2, g^S\right)$ out as follows:

$$\begin{array}{cc} \Gamma_{22}^1 = \sinh 2a, & \Gamma_{23}^1 = \frac{1}{2}\cosh a, \\ \Gamma_{12}^2 = \frac{3}{2}\tanh a, & \Gamma_{13}^2 = \frac{1}{2}\sec ha, \\ \Gamma_{12}^3 = \frac{3}{2}\tanh a \sinh a + \frac{1}{2}\cosh a, & \Gamma_{13}^3 = -\frac{1}{2}\tanh a. \end{array}$$

Furthermore, we calculated the general geodesic equations of the semi Riemann manifold $\left(T_\varepsilon S_1^2, g^S\right)$ as follows:

$$\begin{aligned} \ddot{a} + \sinh 2a\dot{\theta}^2 + \cosh a\dot{\theta}\dot{\omega} &= 0, \\ \ddot{\theta} + 3\tanh a\dot{a}\dot{\theta} + \sec ha\dot{a}\dot{\omega} &= 0, \\ \ddot{\omega} + (\cosh a - 3\tanh a \sinh a)\,\dot{a}\dot{\theta} + \tanh ha\dot{a}\dot{\omega} &= 0. \end{aligned}$$

Finally, we get the non-zero components of the Riemann curvature tensor of the semi Riemann manifold $\left(T_\varepsilon S_1^2, g^S\right)$ as follows:

$$R_{112}^2 = \frac{7}{4}, \quad R_{121}^3 = 2\sinh a, \quad R_{131}^3 = \frac{1}{4}, \quad R_{212}^1 = \frac{3}{2}\cosh^2 a + \frac{1}{4},$$
$$R_{232}^2 = \frac{1}{4}\sinh a, \quad R_{223}^3 = \frac{1}{4}\sin h^2 a, \quad R_{321}^1 = \frac{1}{4}\sinh a, \quad R_{231}^1 = \frac{1}{4}\sinh a,$$
$$R_{332}^2 = \frac{1}{4}, \quad R_{323}^3 = \frac{1}{4}\sinh a, \quad R_{331}^1 = \frac{1}{4}.$$

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