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CURVE RECONSTRUCTION IN LORENTZ-MINKOWSKI 3-SPACE

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ABSTRACT. In this paper, we introduce the signature curve in Lorentz-Minkowski 3-space. In addition, we propose a method for curve reconstruction by using the its signature curve. Experimental results show that the proposed method can successfully reproduce any kinds of smooth curves with a high speed and a high accuracy in Lorentz-Minkowski 3-space.

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1. INTRODUCTION

In the Euclidean plane, the signature curve $S(t)$ of a curve $\alpha(t)$ is given parametrically as $(\kappa(t), \kappa_s(t))$, where $\kappa(t)$ is the curvature and $\kappa_s(t)$ its derivative with respect to arc length s. Calabi et al[4] used signature curves for the invariant recognition of visual objects. Surazhsky and Elber[9] investigated how can one uniquely derive the curvature signature of a planar curve and more importantly, how can one reconstruct the curve from its curvature signature. Many geometers studied these questions and generalized the situation. Boutin[8] introduced the three dimensional version of the Euclidean signature curve, namely the curvature, the torsion and their derivatives with respect to arc length. Recently, Wu and Li[6] presented the algorithm that reproduction of a curve by using the signature curve in Euclidean space. The aim of this paper is carrying out this concept to the Lorentzian space. We distinguish the cases that curve is spacelike and timelike.

The three dimensional Minkowski space $\mathbb{R}^3_1$ is a real linear space with an indefinite inner product given by the matrix $G = diag(1; 1; -1)$ [1-3,10-12]. The inner product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_1$ is defined as

$$< \mathbf{u}, \mathbf{v} >= u_1v_1 + u_2v_2 - u_3v_3 \tag{1}$$

The norm of $\mathbf{u}$ defined by

$$\|\mathbf{u}\| = \sqrt{|\langle u, u\rangle|} \tag{2}$$

We say that a Lorentzian vector $\mathbf{u}$ is spacelike, lightlike or timelike if $\langle \mathbf{u}, \mathbf{u}\rangle > 0$ and $\mathbf{u} = 0$, $\langle \mathbf{u}, \mathbf{u}\rangle = 0$, $\langle \mathbf{u}, \mathbf{u}\rangle < 0$, respectively.

For any $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_1$, the Lorentz vector product of $\mathbf{u}$ and $\mathbf{v}$ is defined as follows:

$$\mathbf{u} \wedge \mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3, u_2v_1 - u_1v_2) \tag{3}$$

This yields

$$e_1 \wedge e_2 = -e_3, \qquad e_3 \wedge e_1 = e_2, \qquad e_2 \wedge e_3 = e_1 \tag{4}$$

where $\{e_1, e_2, e_3\}$ are the base of the space $\mathbb{R}^3_1$.
A curve segment $c(t) \in \mathbb{R}^3_1$, $t \in [a;b]$ is called spacelike, timelike or lightlike if its tangent vector $\frac{dc(t)}{dt}$, is spacelike, timelike or lightlike, respectively.
The Minkowski arc-length s can be calculated by

$$s = \int_a^b v(t)dt \tag{5}$$

where $v(t) = \left\| \frac{dc(t)}{dt} \right\|$. Which implies that

$$v(t) = \frac{ds}{dt} \tag{6}$$

Denote by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ the moving Frenet frame $F(t)$ along the curve $\alpha(t)$ in the Minkowski space $\mathbb{R}^3_1$. Then Frenet formulas of spacelike curve may be written as

$$\frac{d}{dt}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa v & 0 \\ -\epsilon\kappa v & 0 & \tau v \\ 0 & \tau v & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{7}$$

where $< \mathbf{n}, \mathbf{n} >= \epsilon$, $\epsilon = \pm 1$. The Minkowski curvature and torsion of spacelike curve $\alpha(t)$ are obtained by, respectively

$$\kappa = \left\| \mathbf{t}' \right\|, \tau = - < \mathbf{n}', \mathbf{b} > \tag{8}$$

Furthermore, for a timelike curve $\alpha(t)$, the following Frenet formulas are given:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa v & 0 \\ \kappa v & 0 & \tau v \\ 0 & -\tau v & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix} \tag{9}$$

The Minkowski curvature and torsion of timelike curve are calculated by, respectively

$$\kappa = \left\| \mathbf{t}' \right\|, \tau = < \mathbf{n}', \mathbf{b} > \tag{10}$$

2. Reproduction of a Curve from Signature of a Spacelike Curve

Let

$$\begin{aligned} \alpha : \quad & I \to \mathbb{R}^3_1 \\ & t \to \alpha(t) \quad = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \end{aligned} \tag{11}$$

where $\{0\} \subset I$, be a differentiable spacelike curve in Minkowski 3-space.
Definition 1.1. The signature curve S of spacelike curve is defined in terms of Minkowski differential invariants: $\kappa(t)$, $\tau(t)$ and their derivatives $\kappa_s(t)$, $\tau_s(t)$ with respect to the Minkowski arc-length parameter s:

$$S(t) = \{\kappa(t), \tau(t), \kappa_s(t), \tau_s(t)\} \tag{12}$$

where $\kappa_s = d\kappa/ds$ and $\tau_s = d\tau/ds$.
It is easy to see that the signature curve S is isometric invariant and fully defines the curve shape and all important geometric features. Now we introduce an algorithm for the curve reconstruction in Minkowski space $\mathbb{R}^3_1$.

From (6), it is easy to see that

$$\kappa_s(t) = \frac{d\kappa(t)}{dt}\frac{dt}{ds} = \frac{d\kappa(t)}{dt}\frac{1}{v(t)} \tag{13}$$

$$\tau_s(t) = \frac{d\tau(t)}{dt}\frac{dt}{ds} = \frac{d\tau(t)}{dt}\frac{1}{v(t)} \tag{14}$$

The aim of this paper is to formulate an algorithm to reproduce the spacelike curve $\alpha(t)$ from its signature S, with the predefined initial starting point and initial motion direction $\Theta(1)$. Under the concept of Frenet frame, the initial motion direction $\Theta(1) = \{\mathbf{t}(1), \mathbf{n}(1), \mathbf{b}(1)\}$ is represented by the three vectors of a Frenet frame.
Let $t - \Delta t$ and $t + \Delta t$ be two points in the domain of spacelike curve $\alpha(t)$. By using the central finite difference, $\frac{d\kappa(t)}{dt}$ can be calculated by

$$\frac{d\kappa(t)}{dt} = \frac{\kappa(t+\Delta t) - \kappa(t-\Delta t)}{2\Delta t} \tag{15}$$

where Δt sufficiently small.
Using (13) and (15), we can further derive the speed formula $v(t)$ from signature data in the following form:

$$v(t) = \frac{\kappa(t+\Delta t) - \kappa(t-\Delta t)}{2\Delta t \kappa_s(t)} \tag{16}$$

Let t and $t + \Delta t$ be two consecutive points. It is obvious that the derivatives of Frenet frame vectors $F(t)$ can be obtained by

$$\begin{aligned} \frac{d\mathbf{t}(t)}{dt} &= \frac{\mathbf{t}(t+\Delta t) - \mathbf{t}(t)}{\Delta t} \\ \frac{d\mathbf{n}(t)}{dt} &= \frac{\mathbf{n}(t+\Delta t) - \mathbf{n}(t)}{\Delta t} \\ \frac{d\mathbf{b}(t)}{dt} &= \frac{\mathbf{b}(t+\Delta t) - \mathbf{b}(t)}{\Delta t} \end{aligned} \tag{17}$$

From (7) and (17) we can derive the following iteration equation expressed in matrix form as

$$\begin{bmatrix} \mathbf{t}(t+\Delta t) \\ \mathbf{n}(t+\Delta t) \\ \mathbf{b}(t+\Delta t) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t\kappa(t)v(t) & 0 \\ -\epsilon\Delta t\kappa(t)v(t) & 1 & \Delta t\tau(t)v(t) \\ 0 & \Delta t\tau(t)v(t) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \\ \mathbf{b}(t) \end{bmatrix} \tag{18}$$

Since the Frenet vectors are unit vectors, we need to normalize the vectors, then we have

$$\begin{aligned} \widetilde{\mathbf{t}}(t+\Delta t) &= \frac{\mathbf{t}(t+\Delta t)}{\|\mathbf{t}(t+\Delta t)\|} \\ \widetilde{\mathbf{n}}(t+\Delta t) &= \frac{\mathbf{n}(t+\Delta t)}{\|\mathbf{n}(t+\Delta t)\|} \\ \widetilde{\mathbf{b}}(t+\Delta t) &= \frac{\mathbf{b}(t+\Delta t)}{\|\mathbf{b}(t+\Delta t)\|} \end{aligned} \tag{19}$$

The reproduced Frenet frame $F(t + \Delta t)$ at the point $t + \Delta t$ from $F(t)$ denoted by

$$F(t+\Delta t) = \left\{\widetilde{\mathbf{t}}(t+\Delta t), \widetilde{\mathbf{n}}(t+\Delta t), \widetilde{\mathbf{b}}(t+\Delta t)\right\} \tag{20}$$

Now start from this initial motion direction $\left\{\widetilde{\mathbf{t}}(1),\widetilde{\mathbf{n}}(1),\widetilde{\mathbf{b}}(1)\right\}$ and the given signature S of curve $\alpha(t)$, iterate equations (15), (18) and (19), we can obtain the Frenet frame $F(t+\Delta t)$ for all the points of curve $\alpha(t)$.
Using again finite difference, the derivative of $\alpha(t)$ is obtained by

$$\alpha'(t) = \frac{\alpha(t+\Delta t) - \alpha(t)}{\Delta t} \tag{21}$$

Substituting $\alpha'(t) = v(t)\mathbf{t}(t)$ into (21) gives

$$\alpha(t+\Delta t) = \alpha(t) + \Delta t v(t)\mathbf{t}(t) \tag{22}$$

Assume that we are given the initial starting point and initial motion direction $\Theta(1)$ then using (22), we can reproduce all the curve points for spacelike curve $\alpha(t)$ that has the signature of S.
In addition, using (18) implies that

$$< \mathbf{t}(t+\Delta t), \mathbf{t}(t+\Delta t) >= 1 + \epsilon \Delta t^2 \kappa^2 v^2 \tag{23}$$

When Δt goes to zero, the reproduced curve $\alpha(t+\Delta t)$ from original spacelike curve $\alpha(t)$ is also spacelike.

3. Reproduction of a Curve from Signature of a Timelike Curve

Let

$$\begin{aligned} \alpha: \quad & I \rightarrow \mathbb{R}^3_1 \\ & t \rightarrow \alpha(t) \quad = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \end{aligned} \tag{24}$$

where $\{0\} \subset I$, be a differentiable timelike curve in Minkowski 3-space.
From (9) and (17) we can derive the following iteration equation to calculate $F(t+\Delta t)$ from $F(t)$ expressed in matrix form as

$$\begin{bmatrix} \mathbf{t}(t+\Delta t) \\ \mathbf{n}(t+\Delta t) \\ \mathbf{b}(t+\Delta t) \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \kappa(t) v(t) & 0 \\ \Delta t \kappa(t) v(t) & 1 & \Delta t \tau(t) v(t) \\ 0 & -\Delta t \tau(t) v(t) & 1 \end{bmatrix} \begin{bmatrix} \mathbf{t}(t) \\ \mathbf{n}(t) \\ \mathbf{b}(t) \end{bmatrix} \tag{25}$$

Hence

$$< \mathbf{t}(t+\Delta t), \mathbf{t}(t+\Delta t) >= -1 + \Delta t^2 \kappa^2 v^2 \tag{26}$$

Since $\Delta t \rightarrow 0$, the reproduced curve $\alpha(t+\Delta t)$ from timelike curve $\alpha(t)$ is also timelike.
From now on, by applying the same procedure as that used for spacelike curve. We are able to reproduce all the timelike curve points for timelike curve $\alpha(t)$ that has the signature of S by using (25) and (22).

4. Examples

Example 1: Assume that spacelike curve $\alpha(t)$ is parametrized by

$$\alpha(t) = (\sqrt{2}\cos(t), \sqrt{2}\sin(t), t) \tag{27}$$

From (5) we have $v(t) = 1$, which implies that $\alpha(t)$ is parameterized by its arc length.
We have the Frenet frame as

$$\begin{aligned} \mathbf{t} &= (-\sqrt{2}\sin t, \sqrt{2}\cos t, 1) \\ \mathbf{n} &= (-\cos t, -\sin t, 0) \\ \mathbf{b} &= (\sin t, -\cos t, -\sqrt{2}) \end{aligned} \tag{28}$$

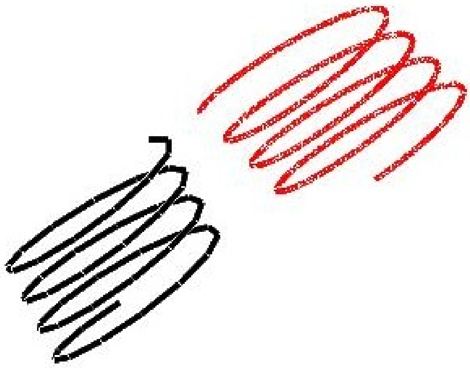

FIGURE 1. The red curve is the reproduced curve, that was produced from the signature of original black curve.

where $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ are spacelike, spacelike and timelike, repectively.
The curvature and torsion of the curve are obtained by

$$\kappa = \sqrt{2}, \quad \tau = -1 \tag{29}$$

In this example, we can simply set $\Theta(1) = \{\mathbf{t}(1), \mathbf{n}(1), \mathbf{b}(1)\}$ the initial motion direction in terms of the Frenet vector as

$$\begin{aligned} \mathbf{t}(1) &= (1,0,0) \\ \mathbf{n}(1) &= (0,1,0) \\ \mathbf{b}(1) &= (0,0,1) \end{aligned} \tag{30}$$

The initial starting point is chosen as

$$\alpha(1) = (1,2,3) \tag{31}$$

We reproduced the spacelike curve with points $t_i \in (1,17)$ with $\Delta t = t_{i+1} - t_i$, for $\Delta t = 0.01$. The reproduced curve(red) and original curve(black) are illustrated in Figures 1.
Example 2: Assume that timelike curve $\alpha(t)$ is parametrized by

$$\alpha(t) = (t, \sqrt{2}\cosh(t), \sqrt{2}\sinh(t)) \tag{32}$$

The Frenet frame can be obtained by

$$\begin{aligned} \mathbf{t} &= (1, \sqrt{2}\sinh t, \sqrt{2}\cosh t,) \\ \mathbf{n} &= (0, \cosh t, \sinh t) \\ \mathbf{b} &= (-\sqrt{2}, -\sinh t, -\cosh t) \end{aligned} \tag{33}$$

where $\mathbf{t}, \mathbf{n}$ and $\mathbf{b}$ vectors are timelike, spacelike and spacelike, repectively.
The curvature and torsion of the curve are obtained by

$$\kappa = \sqrt{2}, \quad \tau = 1 \tag{34}$$

In this example, it is convenient to choose the initial motion direction $\Theta(1)$ the same as Frenet frame of original curve then we have

$$\begin{aligned}\mathbf{t}(1) &= (1, \sqrt{2}\sinh 1, \sqrt{2}\cosh 1)\\ \mathbf{n}(1) &= (0, \cosh 1, \sinh 1)\\ \mathbf{b}(1) &= (-\sqrt{2}, -\sinh 1, -\cosh 1)\end{aligned} \quad (35)$$

The initial starting point $\alpha(1) = (1, \sqrt{2}\cosh(1), \sqrt{2}\sinh(1))$. We computed the signature with points $t_i \in (1, 5)$ with $\Delta t = t_{i+1} - t_i$, for $\Delta t = 0.0002$. The reproduced curve(red) and original curve(dotted black) are illustrated in Figures 2.

FIGURE 2. Reproduced curve(red) and original curve(black dotted) with the same signature, initial starting point and initial motion direction.

Note that since we have chosen as $\Theta(1)$ and $\alpha(1)$ the same as original curve, the reproduced curve and original curve have the same graphics shown in Figures 2. Moreover, in example 1, the signatures of curves just are the same.

5. CONCLUSION

The aim of the study is to present the performance of the curve reproduction in Minkowski 3-space. Examples shows that the method of curve reproduction used in this study can be used effectively in Minkowski space as well as in Euclidean space. We can easily reproduce the curve since the method relies on the intrinsic properties of the curve (curvature and torsion), and is simple, local.

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