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PRINCIPAL DIRECTION CURVES OF BIHARMONIC CURVES IN THE HEISENBERG GROUP

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ABSTRACT. In this paper, we study principal-direction curve of biharmonic curves in the Heisenberg group Heis^3. Finally, we illustrate principal-direction curve of biharmonic curves in the Heisenberg group Heis^3.

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1. INTRODUCTION

The local theory of space curves are mainly developed by the Frenet–Serret theorem which expresses the derivative of a geometrically chosen basis of R^3 by the aid of itself is proved. Then it is observed that by the solution of some of special ordinary differential equations, further classical topics, for instance spherical curves, Bertrand curves, involutes and evolutes are investigated. One of the mentioned works is spherical images of a regular curve in the Euclidean space. It is a well known concept in the local differential geometry of curves. Such curves are obtained in terms of the Frenet–Serret vector fields, [4].
In this paper, we study principal-direction curve of biharmonic curves in the Heisenberg group Heis^3. Finally, we illustrate principal-direction curve of biharmonic curves in the Heisenberg group Heis^3.

2. BACKGROUND ON THE HEISENBERG GROUP HEIS^3

Heisenberg group Heis^3 can be seen as the space $\mathbb{R}^3$ endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y})$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group. The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1=\frac{\partial}{\partial x},\ \ \mathbf{e}_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ \ \mathbf{e}_3=\frac{\partial}{\partial z}, \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_1,\mathbf{e}_2]=\mathbf{e}_3,\ \ [\mathbf{e}_2,\mathbf{e}_3]=[\mathbf{e}_3,\mathbf{e}_1]=0$$

with

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=g(\mathbf{e}_3,\mathbf{e}_3)=1.$$

Let $\gamma : I \longrightarrow Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:
$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N},\\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T}+\tau\mathbf{B},\\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}g(\mathbf{T},\mathbf{T}) &= 1,\ g(\mathbf{N},\mathbf{N})=1,\ g(\mathbf{B},\mathbf{B})=1,\\ g(\mathbf{T},\mathbf{N}) &= g(\mathbf{T},\mathbf{B})=g(\mathbf{N},\mathbf{B})=0.\end{aligned}$$

Theorem 2.1. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic curve with non-zero natural curvatures. Then, the parametric equations of γ are*

$$\begin{aligned}x(s) &= \cos\mathcal{C}s+\mathcal{B}_3,\\ y(s) &= \frac{1}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s+\mathcal{B}_2]+\mathcal{B}_4,\\ z(s) &= \frac{1}{\mathcal{B}_1^2}\sin\mathcal{C}\cos\mathcal{C}\cos[\mathcal{B}_1 s+\mathcal{B}_2]+\frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos\mathcal{C}\sin[\mathcal{B}_1 s+\mathcal{B}_2]\\ &\quad+\frac{\mathcal{B}_3}{\mathcal{B}_1}\sin\mathcal{C}\sin[\mathcal{B}_1 s+\mathcal{B}_2]-\frac{1}{\mathcal{B}_1}\sin\mathcal{C}\cos[\mathcal{B}_1 s+\mathcal{B}_2]+\mathcal{B}_5,\end{aligned}$$

where $\mathcal{B}_1,\mathcal{B}_2,\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5$ are constants of integration.

3. Principal-Direction Curves of Biharmonic Curves in the Heisenberg Group Heis^3

Recall that a smooth curve in Heis^3 is a smooth map $\gamma : I \to Heis^3$, where I is an interval in $\mathbb{R}$. For any $a \in I$, the tangent vector of γ at the point $\gamma(a)$ is

$$\gamma'(a)=\frac{d\gamma}{ds}(a)=d\gamma_a\left(\frac{d}{ds}\right),$$

where $\frac{d}{ds}$ is the standard coordinate tangent vector.

Definition 3.1. *Let X be a smooth vector field on $Heis^3$. We say that a smooth curve $\gamma : I \to Heis^3$ is an integral curve of X if for any $s \in I$,*

$$\gamma'(s) = X_{\gamma(s)}.$$

Then,

$$X_{\gamma(s)} = u(s)\,\mathbf{T}(s) + v(s)\,\mathbf{N}(s) + w(s)\,\mathbf{B}(s), \tag{3.1}$$

Then, an integral curve $\bar{\gamma}(s)$ of $X_{\gamma(s)}$ defined on I is a unit speed curve in Heis^3.

Remark 3.2. *A principal-direction curve is an integral curve of $X_{\gamma(s)} = \mathbf{N}(s)$.*

Theorem 3.3. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic curve. Then, the equations of principal-direction curve of γ are*

$$\begin{aligned}
\bar{x}(s) &= \frac{1}{4\mathcal{B}_1\kappa}\sin^2\mathcal{C}(-\cos[2\mathcal{B}_2]\cos[2\mathcal{B}_1 s] + \sin[2\mathcal{B}_2]\sin[2\mathcal{B}_1 s]),\\
\bar{y}(s) &= -\frac{1}{\mathcal{B}_1\kappa}\sin\mathcal{C}(\mathcal{B}_1 + \cos\mathcal{C})(-\cos[\mathcal{B}_2]\cos[\mathcal{B}_1 s] + \sin[\mathcal{B}_2]\sin[\mathcal{B}_1 s]), 3.2 \qquad (1)\\
\bar{z}(s) &= -\frac{1}{24\mathcal{B}_1^2\kappa^2}\sin^3\mathcal{C}(\mathcal{B}_1 + \cos\mathcal{C})(-3\cos[\mathcal{B}_1 s + \mathcal{B}_2] + \cos 3[\mathcal{B}_1 s + \mathcal{B}_2])\\
&\quad + \frac{1}{\kappa}\sin\mathcal{C}(\cos[\mathcal{B}_2]\sin[\mathcal{B}_1 s] + \cos[\mathcal{B}_2]\sin[\mathcal{B}_1 s]),
\end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$ are constants of integration.

Proof. By using the Frenet frame, we obtain

$$\begin{aligned}
\frac{d\gamma}{ds} &= \frac{1}{\kappa}\sin^2\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_1 - \frac{1}{\kappa}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C})\mathbf{e}_2\\
&\quad + \frac{1}{\kappa}\mathcal{B}_1\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\,\mathbf{e}_3.
\end{aligned}$$

Since, we express

$$\begin{aligned}
\frac{d\gamma}{ds} &= (\frac{1}{\kappa}\sin^2\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]\sin[\mathcal{B}_1 s + \mathcal{B}_2],\\
&\quad -\frac{1}{\kappa}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C}),\\
&\quad -x\frac{1}{\kappa}\sin\mathcal{C}\sin[\mathcal{B}_1 s + \mathcal{B}_2](\mathcal{B}_1 + \cos\mathcal{C}) + \frac{1}{\kappa}\mathcal{B}_1\sin\mathcal{C}\cos[\mathcal{B}_1 s + \mathcal{B}_2]).
\end{aligned}$$

So, by integratiating of the above formula, we get (3.2). This concludes the proof of theorem.

Theorem 3.4. *Let $\gamma : I \longrightarrow Heis^3$ be a unit speed biharmonic curve. Then, the position vector of principal-direction curve of γ is*

$$\begin{aligned}\bar{\gamma}(s) = & \frac{1}{4\mathcal{B}_1\kappa}\sin^2\mathcal{C}(-\cos[2\mathcal{B}_2]\cos[2\mathcal{B}_1 s]+\sin[2\mathcal{B}_2]\sin[2\mathcal{B}_1 s])\mathbf{e}_1 \\ & -\frac{1}{\mathcal{B}_1\kappa}\sin\mathcal{C}(\mathcal{B}_1+\cos\mathcal{C})(-\cos[\mathcal{B}_2]\cos[\mathcal{B}_1 s]+\sin[\mathcal{B}_2]\sin[\mathcal{B}_1 s])\mathbf{e}_2 \\ & +(-\frac{1}{24\mathcal{B}_1^2\kappa^2}\sin^3\mathcal{C}(\mathcal{B}_1+\cos\mathcal{C})(-3\cos[\mathcal{B}_1 s+\mathcal{B}_2]+\cos 3[\mathcal{B}_1 s+\mathcal{B}_2]) \\ & +\frac{1}{\kappa}\sin\mathcal{C}(\cos[\mathcal{B}_2]\sin[\mathcal{B}_1 s]+\cos[\mathcal{B}_2]\sin[\mathcal{B}_1 s]) \\ & +\frac{1}{4\mathcal{B}_1^2\kappa^2}\sin^3\mathcal{C}(-\cos[2\mathcal{B}_2]\cos[2\mathcal{B}_1 s]+\sin[2\mathcal{B}_2]\sin[2\mathcal{B}_1 s])(\mathcal{B}_1 \\ & +\cos\mathcal{C})(-\cos[\mathcal{B}_2]\cos[\mathcal{B}_1 s]+\sin[\mathcal{B}_2]\sin[\mathcal{B}_1 s]))\mathbf{e}_3,\end{aligned}$$

where $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5$ are constants of integration.

4. Some pictures

In this section we draw some pictures about γ and $\bar{\gamma}$:

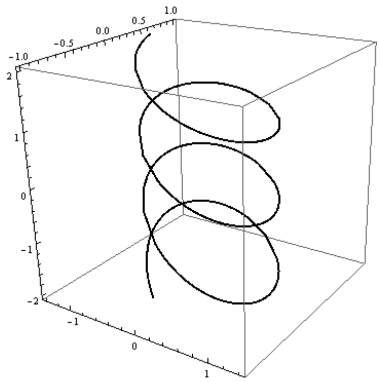

Fig.1: *A unit speed biharmonic curve.*

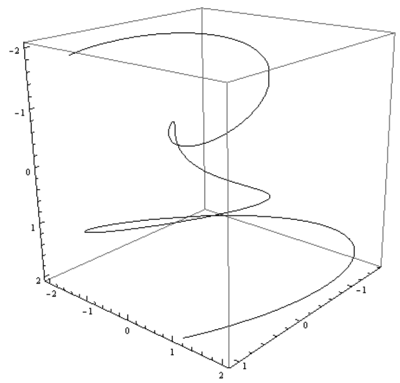

Fig.2: *Principal-direction curve* of *a unit speed biharmonic curve.*

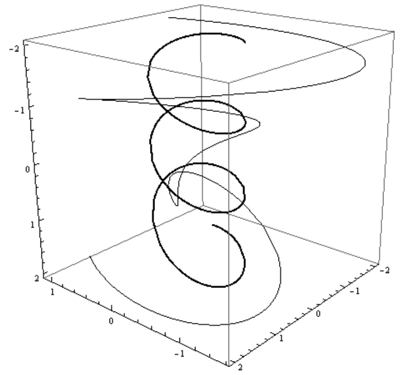

Fig.3: Using Mathematica both principal-direction curve and its mate.

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