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AFFINE OSSERMAN CONNECTIONS WHICH ARE LOCALLY SYMMETRICS

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ABSTRACT. This paper deals with affine Osserman connections. We give examples of affine Osserman connections which are locally symmetric but not flat on 3-dimensional manifolds.

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1. INTRODUCTION

Let (M, ∇) be an m-dimensional affine manifold, i.e., ∇ is a torsion free connection on the tangent bundle. Let $\mathcal{R}^{\nabla}(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$ be the associated curvature operator. We define the *affine Jacobi operator* $J_{\mathcal{R}^{\nabla}}(X) : T_pM \longrightarrow T_pM$ with respect to a vector $X \in T_pM$ by

$$J_{\mathcal{R}^{\nabla}}(X)Y := \mathcal{R}^{\nabla}(Y,X)X.$$

Let Spect$\{J_{\mathcal{R}^{\nabla}}(X)\} \subset \mathbb{C}$ be the spectrum of $J_{\mathcal{R}^{\nabla}}(X)$; this is the set of roots of the characteristic polynomial $P_{\lambda}[J_{\mathcal{R}^{\nabla}}(X)]$. One says that an affine manifold (M, ∇) is *affine Osserman* if Spect$\{J_{\mathcal{R}^{\nabla}}(X)\} = \{0\}$ for any vector X; i.e the affine Jacobi operator is nilpotent.

Affine Osserman manifolds are well-understood in dimension two, due to the fact that an affine manifold is Osserman if and only if its Ricci tensor is skew-symmetric [3,7]. The situation is however involved more in higher dimensions where the skew-symmetric is a necessary (but not sufficient) condition for an affine manifold to be Osserman [4–6].

Affine Osserman manifolds are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Osserman metrics since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions (see [4,5,7]). Here it is worth to emphasize that some recent modifications of the usual Riemann extensions allowed some new applications [1, 2, 8]

Our paper is organized as follows. Section 1 introduces this topics. In section 2 we recall some basics definitions and results about affine manifolds. In section 3, we study the Osserman condition on a particular affine connection (cf. Theorem 3.2). Section 4 is devoted to the study the locally symmetric on the same affine connection (cf. Theorem 4.3). The purpose of the last section is to build examples of affine Osserman connections which are locally symmetric but not flat on a 3-dimensional affine manifold (cf. Theorem 5.1).

2. Affine manifolds

In this section, we give the necessary tools needed to reach our goal. Here are some basic definitions and results about affine manifolds taken from the book [9].

Let M be a 3-dimensional and ∇ a smooth affine connection. We choose a fixed coordinate domain $\mathcal{U}(u_1, u_2, u_3) \subset M$. In $\mathcal{U}$, the connection is given by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k,$$

where we denote $\partial_i = (\frac{\partial}{\partial_{u_i}})$ and the functions $\Gamma_{ij}^k (i, j, k = 1, 2, 3)$ are called the *Christoffel symbols* for the affine connection relative to the local coordinate system. We define a few tensors fields associated to a given affine connection ∇. The *torsion tensor field* T^∇, which is of type $(1, 2)$, is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The components of the torsion tensor T^∇ in local coordinates are

$$T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k.$$

If the torsion tensor of a given affine connection ∇ is 0, we say that ∇ is torsion-free.

The *curvature tensor field* $\mathcal{R}^\nabla$, which is of type $(1, 3)$, is defined by

$$\mathcal{R}^\nabla(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k, \partial_l)\partial_j = \sum_i R_{jkl}^i \partial_i$$

We shall assume that ∇ is torsion-free. If $\mathcal{R}^\nabla = 0$ on M, we say that ∇ is *flat affine connection*. It is know that ∇ is flat if and only if around point there exist a local coordinates system such that $\Gamma_{ij}^k = 0$ for all i, j and k.

We define the *Ricci tensor* Ric^∇, of type $(0, 2)$ by

$$Ric^\nabla(Y, Z) = \text{trace}\{X \mapsto \mathcal{R}^\nabla(X, Y)Z\}.$$

The components in local coordinates are given by

$$Ric^\nabla(\partial_j, \partial_k) = \sum_i R_{kij}^i.$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is, $Ric(Y, Z) = Ric(Z, Y)$. But this property is not true for an arbitrary affine connection with torsion-free.

3. Affine Osserman connections on 3-dimensional manifolds

Let M a 3-dimensional manifold and ∇ a smooth torsion-free connection. We choose a fixed coordinates domain $\mathcal{U}(u_1, u_2, u_3) \subset M$. In $\mathcal{U}$, consider the connection is given by

$$\left\{ \begin{array}{lcl} \nabla_{\partial_1}\partial_1 & = & f_1(u_1, u_2, u_3)\partial_3; \\ \nabla_{\partial_2}\partial_2 & = & f_2(u_1, u_2, u_3)\partial_3; \\ \nabla_{\partial_3}\partial_3 & = & f_3(u_1, u_2, u_3)\partial_3. \end{array} \right. \tag{1}$$

where we denote $\partial_i = (\partial/\partial u_i)\,(i = 1, 2, 3)$. We denote the functions $f_1(u_1, u_2, u_3)$, $f_2(u_1, u_2, u_3)$, $f_3(u_1, u_2, u_3)$ by f_1, f_2, f_3 respectively, if there is no risk of confusion.

The components of the curvature operator of the connection (1) are given by

$$\begin{array}{rcl} \mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_1 & = & -\partial_2 f_1 \partial_3; \quad \mathcal{R}^{\nabla}(\partial_1, \partial_2)\partial_2 = \partial_1 f_2 \partial_3; \\ \mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_1 & = & (-\partial_3 f_1 - f_1 f_3)\partial_3; \quad \mathcal{R}^{\nabla}(\partial_1, \partial_3)\partial_3 = \partial_1 f_3 \partial_3, \\ \mathcal{R}^{\nabla}(\partial_2, \partial_3)\partial_2 & = & (-\partial_3 f_2 - f_2 f_3)\partial_3, \quad \mathcal{R}^{\nabla}(\partial_2, \partial_3)\partial_3 = \partial_2 f_3 \partial_3. \end{array}$$

The nonzero components of the Ricci tensor of the connection (1) are given by

$$\left\{ \begin{array}{rcl} Ric^{\nabla}(\partial_1, \partial_1) & = & \partial_3 f_1 + f_1 f_3; \quad Ric^{\nabla}(\partial_1, \partial_3) = -\partial_1 f_3; \\ Ric^{\nabla}(\partial_2, \partial_3) & = & -\partial_2 f_3; \quad Ric^{\nabla}(\partial_2, \partial_2) = \partial_3 f_2 + f_2 f_3. \end{array} \right. \tag{2}$$

Now, since the Ricci tensor of any affine Osserman connection is skew-symmetric, it follows that we have the following conditions for the connection to be Osserman

$$\partial_1 f_3 = 0, \quad \partial_2 f_3 = 0, \quad \partial_3 f_1 + f_1 f_3 = 0 \quad \text{and} \quad \partial_3 f_2 + f_2 f_3 = 0; \tag{3}$$

which implies that the connection is indeed Ricci flat, but not flat.

Let $X = \sum_1^3 \alpha_i \partial_i$ be a vector on M, then the affine Jacobi operator is given by

$$J_{\mathcal{R}^\nabla}(X)\partial_1 = (\alpha_2^2 \partial_1 f_2 - \alpha_1 \alpha_2 \partial_2 f_1)\partial_3 \quad \text{and} \quad J_{\mathcal{R}^\nabla}(X)\partial_2 = (\alpha_1^2 \partial_2 f_1 - \alpha_1 \alpha_2 \partial_1 f_2)\partial_3.$$

The matrix associated to $J_{\mathcal{R}^\nabla}(X)$ with respect to the basis $\{\partial_1, \partial_2, \partial_3\}$ is given by

$$(J_{\mathcal{R}^\nabla}(X)) = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & 0 \end{pmatrix},$$

with

$$\begin{array}{rcl} a_1 & = & \alpha_2^2 \partial_1 f_2 - \alpha_1 \alpha_2 \partial_2 f_1; \\ a_2 & = & \alpha_1^2 \partial_2 f_1 - \alpha_1 \alpha_2 \partial_1 f_2. \end{array}$$

It follows from the matrix associated to $J_{\mathcal{R}^\nabla}(X)$, that its characteristic polynomial as written as follows:

$$P_\lambda[J_{\mathcal{R}^\nabla}(X)] = \lambda^3.$$

Lemma 3.1. *Let (M, ∇) be a 3-dimensional manifolds with a torsion free connection ∇ given system (1). The (M, ∇) is affine Osserman manifolds if and only if the functions f_1, f_2, f_3 satisfy the following PDE's*

$$\partial_1 f_3 = 0, \quad \partial_2 f_3 = 0, \quad \partial_3 f_1 + f_1 f_3 = 0 \quad \textit{and} \quad \partial_3 f_2 + f_2 f_3 = 0. \tag{4}$$

From (4) we have following:

Theorem 3.2. *Let M be a 3-dimensional manifold with torsion free connection (1). Then (M, ∇) is affine Osserman if and only if the components of the connection are given by*

$$f_3(u_1, u_1, u_3) = f(u_3), \quad \partial_3 f_1 + f_1 f_3(u_3) = 0 \quad \textit{and} \quad \partial_3 f_2 + f_2 f_3(u_3) = 0.$$

Example 3.3. *Using the connection (1) above, one can construct examples of affine Osserman connections which are Ricci flat but not flat.*

(1) *If we take* $f_1(u_1, u_2, u_3) = u_1u_2, f_2(u_1, u_2, u_3) = u_1 - u_2$ *and* $f_3 = 0$. *The nonvanishing components of the curvature tensor are given by* $\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 = -u_1\partial_3$ *and* $\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 = \partial_3$ *and it follows that this connection is a nonflat affine Osserman connection.*

(2) *If we take* $f_1(u_1, u_2, u_3) = e^{u_2 - \frac{1}{2}u_3^2}, f_2(u_1, u_2, u_3) = e^{u_1 - \frac{1}{2}u_3^2}$ *and* $f_3 = u_3$. *The nonvanishing components of the curvature tensor are* $\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 = e^{(u_2 - \frac{1}{2}u_3^2)}\partial_3$ *and* $\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 = e^{(u_1 - \frac{1}{2}u_3^2)}\partial_3$. *It follows that this connection is a nonflat affine Osserman connection.*

4. Affine locally symmetric connections

Let M be a manifold with an affine connection ∇. A smooth diffeomorphism f of M is called an *affine transformation* of M if f preserves the affine connection, i.e., $(f^*\nabla)_X(Y) = \nabla_X Y$ for all $X, Y \in \mathfrak{X}(M)$. If f is a affine transformation, then the curvature R^∇ and the torsion T^∇ are invariants with respect to f, i.e., $f^*\mathcal{R}^\nabla = \mathcal{R}^\nabla$ and $f^*T^\nabla = T^\nabla$.

Definition 4.1. *Let* M *be a manifold with an affine connection* ∇. *A symmetry at* $p \in M$ *is an affine transformation* s_p *defined on some neighborhood* $U \subset M, p \in U$ *such that:*

- $s_p(p) = p$
- $T_p s_p = -id_{T_pM}$.

A manifold M with an affine connection is called *affine locally symmetric space* if there is some symmetry in each $p \in M$. There is the well known description of an affine locally symmetric given by its curvature and torsion.

Proposition 4.2. *The manifold* M *with an affine connection* ∇ *is affine locally symmetric if and only if*

$$T^\nabla = 0 \quad and \quad \nabla\mathcal{R}^\nabla = 0. \tag{5}$$

Writing this formula in local coordinates, we find that any locally symmetric affine connections must satisfy eighteen equations.

Theorem 4.3. *The connection* ∇ *defined by (1) is locally symmetric if and only if the functions* f_1, f_2, f_3 *are solutions of the following:*

$$\begin{aligned}
\partial_1\partial_2 f_1 = 0,\ \partial_1^2 f_2 = 0,\ \partial_1^2 f_3 = 0,\ \partial_1\partial_3 f_1 + f_1\partial_1 f_3 + f_3\partial_1 f_1 = 0,\\
\partial_1\partial_2 f_3 = 0,\ \partial_1\partial_3 f_2 + f_2\partial_1 f_3 + f_3\partial_1 f_2 = 0,\ \partial_2^2 f_1 = 0,\ \partial_2\partial_1 f_2 = 0,\\
\partial_2\partial_3 f_1 + f_1\partial_2 f_3 + f_3\partial_2 f_1 = 0,\ \partial_2\partial_1 f_3 = 0,\ \partial_2\partial_3 f_2 + f_2\partial_2 f_3 + f_3\partial_2 f_2 = 0,\\
\partial_2^2 f_3 = 0,\ \partial_3^2 f_1 + f_1\partial_3 f_3 + 2f_3\partial_3 f_1 + f_1 f_3^2 = 0,\ \partial_3\partial_1 f_3 + f_3\partial_1 f_3 = 0,\\
\partial_3\partial_2 f_1 + f_3\partial_2 f_1 = 0,\ \partial_3\partial_1 f_2 + f_3\partial_1 f_2 = 0,\ \partial_3\partial_2 f_3 + f_3\partial_2 f_3 = 0,\\
\partial_3^2 f_2 + f_2\partial_3 f_3 + 2f_3\partial_3 f_2 + f_2 f_3^2 = 0.
\end{aligned}$$

Proof. Let $X_k = \alpha_i^k\partial_i$, $k = 1, 2, 3, 4$, $i = 1, 2, 3$. The condition

$$\nabla_{X_1}\mathcal{R}^\nabla(X_2, X_3)X_4 = 0$$

leads to

$$\nabla_{\alpha_i^1\partial_i}\mathcal{R}^\nabla(\alpha_j^2\partial_j, \alpha_k^3\partial_k)\alpha_l^4\partial_l = 0, \quad i, j, k, l = 1, 2, 3.$$

Equivalently,

$$\nabla_{\alpha_1^1\partial_1}\mathcal{R}^\nabla(\alpha_j^2\partial_j,\alpha_k^3\partial_k)\alpha_l^4\partial_l+\nabla_{\alpha_2^1\partial_2}\mathcal{R}^\nabla(\alpha_j^2\partial_j,\alpha_k^3\partial_k)\alpha_l^4\partial_l+\nabla_{\alpha_3^1\partial_3}\mathcal{R}^\nabla(\alpha_j^2\partial_j,\alpha_k^3\partial_k)\alpha_l^4\partial_l=0.$$

Straightforward calculation gives

$$\begin{aligned}
\nabla_{\partial_1}\mathcal{R}^\nabla(\partial_2,\partial_1)\partial_1 &= \partial_1\partial_2 f_1\partial_3;\ \nabla_{\partial_1}\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2=\partial_1^2 f_2\partial_3;\ \nabla_{\partial_1}\mathcal{R}^\nabla(\partial_1,\partial_3)\partial_3=\partial_1^2 f_3\partial_3;\\
\nabla_{\partial_1}\mathcal{R}^\nabla(\partial_3,\partial_1)\partial_1 &= (\partial_1\partial_3 f_1+f_1\partial_1 f_3+f_3\partial_1 f_1)\partial_3;\ \nabla_{\partial_1}\mathcal{R}^\nabla(\partial_2,\partial_3)\partial_3=\partial_1\partial_2 f_3\partial_3;\\
\nabla_{\partial_1}\mathcal{R}^\nabla(\partial_3,\partial_2)\partial_2 &= (\partial_1\partial_3 f_2+f_2\partial_1 f_3+f_3\partial_1 f_2)\partial_3;\ \nabla_{\partial_2}\mathcal{R}^\nabla(\partial_2,\partial_1)\partial_1=\partial_2^2 f_1\partial_3;\\
\nabla_{\partial_2}\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2 &= \partial_2\partial_1 f_2\partial_3;\ \nabla_{\partial_2}\mathcal{R}^\nabla(\partial_3,\partial_1)\partial_1=(\partial_2\partial_3 f_1+f_1\partial_2 f_3+f_3\partial_2 f_1)\partial_3;\\
\nabla_{\partial_2}\mathcal{R}^\nabla(\partial_1,\partial_3)\partial_3 &= \partial_2\partial_1 f_3\partial_3;\ \nabla_{\partial_2}\mathcal{R}^\nabla(\partial_3,\partial_2)\partial_2=(\partial_2\partial_3 f_2+f_2\partial_2 f_3+f_3\partial_2 f_2)\partial_3;\\
\nabla_{\partial_2}\mathcal{R}^\nabla(\partial_2,\partial_3)\partial_3 &= \partial_2^2 f_3\partial_3;\ \nabla_{\partial_3}\mathcal{R}^\nabla(\partial_3,\partial_1)\partial_1=(\partial_3^2 f_1+f_1\partial_3 f_3+2f_3\partial_3 f_1+f_1 f_3^2)\partial_3;\\
\nabla_{\partial_3}\mathcal{R}^\nabla(\partial_1,\partial_3)\partial_3 &= (\partial_3\partial_1 f_3+f_3\partial_1 f_3)\partial_3;\ \nabla_{\partial_3}\mathcal{R}^\nabla(\partial_2,\partial_1)\partial_1=(\partial_3\partial_2 f_1+f_3\partial_2 f_1)\partial_3;\\
\nabla_{\partial_3}\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2 &= (\partial_3\partial_1 f_2+f_3\partial_1 f_2)\partial_3;\ \nabla_{\partial_3}\mathcal{R}^\nabla(\partial_2,\partial_3)\partial_3=(\partial_3\partial_2 f_3+f_3\partial_2 f_3)\partial_3;\\
\nabla_{\partial_3}\mathcal{R}^\nabla(\partial_3,\partial_2)\partial_2 &= (\partial_3^2 f_2+f_2\partial_3 f_3+2f_3\partial_3 f_2+f_2 f_3^2)\partial_3;
\end{aligned}$$

The proof is complete. □

5. Affine Osserman locally symmetric connections

In this section, we will give some examples of affine Osserman connections on 3-dimensional manifolds which are locally symmetric.

Theorem 5.1. *The connection ∇ defined by (1) is affine Osserman locally symmetric if and only if the functions f_1, f_2, f_3 are solutions of the following:*

$$\begin{aligned}
\partial_1\partial_2 f_1=0,\ \partial_2\partial_1 f_2=0,\ \partial_1^2 f_2=0,\ \partial_2^2 f_1=0,\\
\partial_3\partial_2 f_1+f_3\partial_2 f_1=0,\ \partial_3\partial_1 f_2+f_3\partial_1 f_2=0,\\
\partial_3^2 f_1+2f_3\partial_3 f_1+f_1 f_3^2=0,\ \partial_3^2 f_2+2f_3\partial_3 f_2+f_2 f_3^2=0.
\end{aligned}$$

Example 5.2. *Using the Theorem (3.2) and the Theorem (5.1), we can construct examples of affine Osserman locally symmetric connections which are not flat.*

(1) *If we take $f_1(u_1,u_2,u_3)=u_2, f_2(u_1,u_2,u_3)=0$ and $f_3=0$. It is easy to see that $\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_1=\partial_3$ and $\nabla\mathcal{R}^\nabla=0$. It follows that this connection is a nonflat affine Osserman locally symmetric connection.*
(2) *If we take $f_1(u_1,u_2,u_3)=0, f_2(u_1,u_2,u_3)=u_1$ and $f_3=0$. It is easy to see that $\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2=\partial_3$ and $\nabla\mathcal{R}^\nabla=0$. It follows that this connection is a nonflat affine Osserman locally symmetric connection.*
(3) *If we take $f_1(u_1,u_2,u_3)=u_2, f_2(u_1,u_2,u_3)=u_1$ and $f_3=0$. It is easy to see that $\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_1=\partial_3$ and $\mathcal{R}^\nabla(\partial_1,\partial_2)\partial_2=\partial_3$ and $\nabla\mathcal{R}^\nabla=0$. It follows that this connection is a nonflat affine Osserman locally symmetric connection.*

Remark 5.3. *Note that in dimension 2, the affine Osserman connections locally symmetric are flat (cf. [[7], Theorem 5).*

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