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EQUAL-AREA PROJECTION: SPHEROID TO SPHERE TO PLANE

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ABSTRACT. An accurate method for the equal-area projection from spheroid to plane is proposed. The points of interest are first projected from the spheroid to an equivalent sphere, that is, to a sphere of equal area. Then, the points of the equivalent sphere are projected onto the surface of a circumscribing cylinder (cylindrical equal-area projection) following the standard mapping procedure of mapping the sphere to the plane.

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1. INTRODUCTION

In the current practice, the equal-area projection of ellipsoid of revolution to plane takes advantage of the standard integrals of both Lambert conformal conic mapping and equal-area conic mapping. The final step are the standard mapping procedures of mapping the sphere to the plane. Further details can be found in Grafarent and Krumm ([3]). An alternative approach is to reduce the spheroid to (an equivalent) sphere by keeping the longitude constant whilst the latitude on the equivalent sphere must be determined (please see [1]). It is important to be mentioned that both procedures are rather complicated, whilst, moreover, Adams ([1]) implies that the second procedure may result to equations that cannot be solved. In the present paper, a new method for the equal-area projection from spheroid to plane is proposed. The method in question, which is more simple than the existing ones, makes use of the concept of equivalent sphere keeping the latitude constant while the longitude is determined. The derived coordinates are given in the Cartesian system.

2. EQUAL-AREA PROJECTION: SPHEROID TO SPHERE TO PLANE

The method, in essence, is based on a concept initially proposed by Archimedes of Syracuse (287 BC - 212 BC) that, the projection of a sphere to a lateral surface of a circumscribing cylinder is area-preserving. Since, the same does not directly stand for spheroids, a modification is needed, where, the area of interest must first be projected onto an equivalent sphere (that is, to a sphere of surface area equal to the surface area of spheroid) and then to a cylinder that circumscribes the sphere. As known, a spheroid may be generated by the revolution of an ellipse about either axis. If the axis of revolution is the minor axis, the surface is called an oblate spheroid (disk-like solid), and if the major axis, a prolate spheroid ([2]). The Cartesian equation of ellipse (in a $y-z$ plane) and of spheroid are given by Eq. 1 and Eq. 2 respectively.

$$\frac{y^2}{r^2} + \frac{z^2}{R^2} = 1 \tag{1}$$

$$\frac{x^2+y^2}{r^2} + \frac{z^2}{R^2} = 1 \tag{2}$$

and, as the method involves projection on an equivalent (as for the surface area) sphere, the equation of the sphere in question is also given:

$$x^2 + y^2 + z^2 = r_{eq}^2 \tag{3}$$

As the equivalent sphere is assumed to have surface area A_{sph} equal to the surface area of spheroid A_{sphd}, the radius of the equivalent sphere can be found using the area of the spheroid:

$$r_{eq} = \sqrt{\frac{A_{sph}}{4\pi}} = \sqrt{\frac{A_{sphd}}{4\pi}} \tag{4}$$

The area A_{sphd} can be treated as surface of revolution in Euclidean space using the equation of ellipse given in Eq. 1 ([2]). The function $y = f(z) = r\sqrt{1-z^2/R^2}$ (see Eq. 1) has a continuous derivative between $z = -R$ and $z = R$ and therefore, the area A_{sphd} can be obtained by revolving the ellipse about the $z-$ axis by 360^o ([4]):

$$\begin{aligned} A_{sphd} &= 2\pi \int_{-R}^{R} f(z)\sqrt{1+[f'(z)]^2}dz \\ &= 2\pi \int_{-R}^{R} r\sqrt{1-\frac{z^2}{R^2}}\sqrt{1+\left[\frac{d}{dz}\left(r\sqrt{1-\frac{z^2}{R^2}}\right)\right]^2}dz \end{aligned} \tag{5}$$

Now, it is assumed that the equivalent sphere is circumscribed by a cylindrical surface. It is reminded that, the projection of a sphere to a lateral surface of a circumscribing cylinder is area-preserving. Therefore, any slice of the sphere cut by two $z-$planes, e.g. $z = z_\alpha$ and $z = z_b$ ($z_b > z_\alpha$), will have surface area $A_{sph,i}$ equal to the surface area of the circumscribing cylinder of the sphere between the same planes $A_{cyl,i}$ (it refers exclusively to the curved surfaces):

$$A_{sph,i} = A_{cyl,i} = 2\pi r_{eq}\left(z_b - z_\alpha\right) \tag{6}$$

Taking $z_\alpha = 0$ and, since, $A_{sph,i} = A_{sphd,i}$, the previous equation can be rewritten as

$$A_{sphd,i} = 2\pi\, r_{eq} z_{eq,i} \tag{7}$$

where, $A_{sph,i}$ is the area of spheroid surface between $z = 0$ and $z = z_i$ as given by Eq. 8 and $z_{eq,i}$ is the equivalent coordinate of z_i (see Figure 1).

$$A_{sphd,i} = 2\pi \int_{0}^{z_i} r\sqrt{1-\frac{z^2}{R^2}}\sqrt{1+\left[\frac{d}{dz}(r\sqrt{1-\frac{z^2}{R^2}})\right]^2}dz \tag{8}$$

Solving Eq. 7 as for $z_{eq,i}$, we, finally, have:

$$z_{eq,i} = \frac{A_{sphd,i}}{2\pi r_{eq}} \tag{9}$$

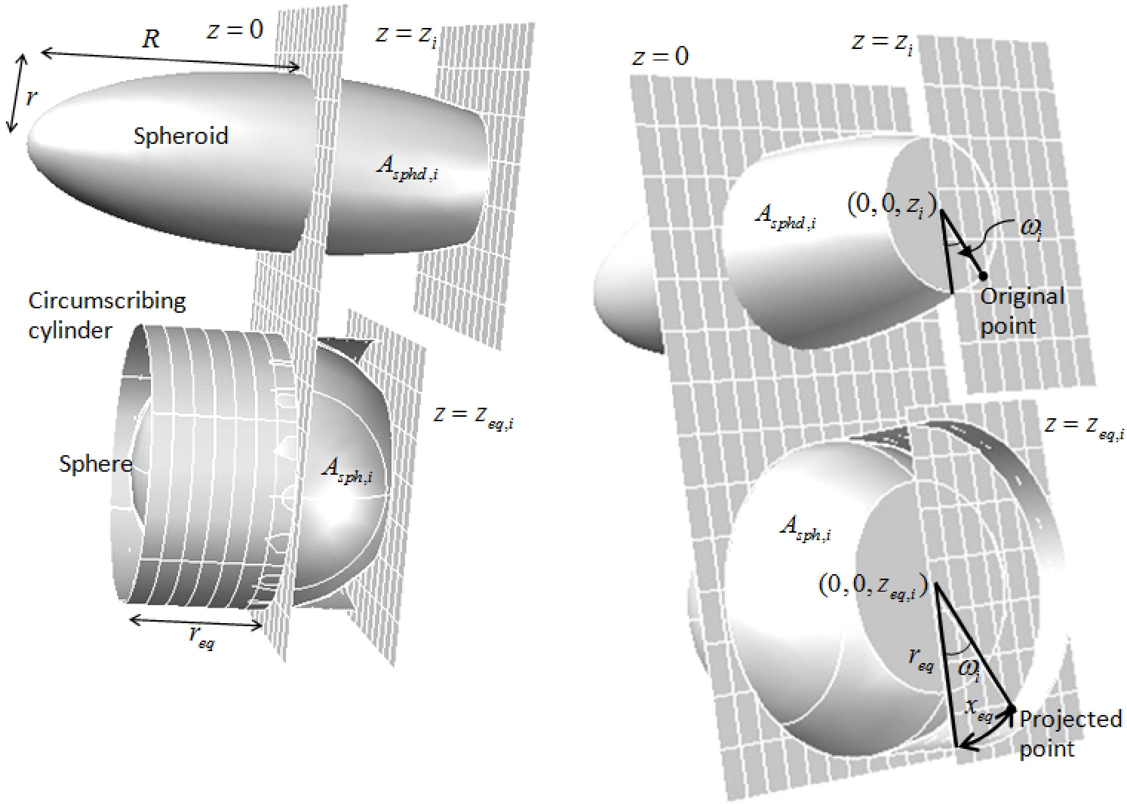

FIGURE 1. Geometric elements of the problem.

which, in essence, is the transformation of the coordinate z_i of a given point on the spheroid to an equivalent coordinate $z_{eq,i}$ on the projection plane. The projection plane is nothing more than the cylindrical surface, unrolled. As regards to the second coordinate on the projection plane of the point in question (either $x-$ or $y-$coordinate), rationally thinking, the point keeps its original radial direction on the circle where it belongs:

$$x^2 + y^2 = r^2 \left(1 - \frac{z_i^2}{R^2}\right) \tag{10}$$

The above means that, the projected point on the equivalent circle (see Eq. 11) will have the same radial direction.

$$x^2 + y^2 = r_{eq}^2 - z_{eq,i}^2 \tag{11}$$

Therefore, the second coordinate on the plane is given by the product of angular distance ω_i multiplied by the radius of the cylinder, r_{eq}:

$$x_{eq} = \omega_i r_{eq} \tag{12}$$

3. SUMMARY

This paper presents an accurate method for the equal-area projection from spheroid to equivalent sphere (as for the surface area) to plane. The procedure makes use of the concept of surface of revolution in Euclidean space and that, the projection of a sphere to a lateral surface of a circumscribing cylinder is area-preserving. The derived coordinates are given in the Cartesian system. The proposed method could be used in the equal-area map projection.

REFERENCES

[1] O.S. Adams, *General theory of equivalent projections*, US Government Printing Office, Washington 1945.
[2] P. Frost and J. Wolstenholme, *A Treatise on Solid Geometry*, McMillan and Co., Cambridge 1863.
[3] E.W. Grafarend and F.W. Krumm, *Map projections: Cartographic information systems*, Springer, Berlin 2006.
[4] R.R. Singh and M. Bhatt, *Engineering mathematics: A tutorial approach*, McGraw-Hill, New Delhi 2010.

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