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POSITION VECTORS OF TIMELIKE GENERAL HELICES IN MINKOWSKI 3-SPACE

AHMAD T. ALI AND MELIH TURGUT

ABSTRACT. In this paper, position vector of a timelike general helix with respect to standard frame of Minkowski space E_1^3 is studied in terms of Frenet equations. First, we prove that position vector of every timelike space curve in Minkowski space E_1^3 satisfies a vector differential equation of fourth order. The general solution of mentioned vector differential equation has not yet been found. By special cases, we determine the parametric representation of the timelike general helices from the intrinsic equations (i.e. curvature and torsion are functions of arc-length). Moreover, we give some examples to illustrate how to find the position vector of timelike general helices from the intrinsic equations.

Mathematics Subject Classifications 2010: 53B40, 53C50.
Keywords: Minkowski 3-space; Frenet equations; General helix; Intrinsic equations.

1. INTRODUCTION

In the local differential geometry, we think of curves as a geometric set of points, or locus. Intuitively, we are thinking of a curve as the path traced out by a particle moving in E^3. So, investigating position vectors of the curves is a classical aim to determine behavior of the particle (or the curve, i.e.). Despite its long history, the theory of curve is still one of the most important interesting topics in a differential geometry and its is being study by many mathematicians until now, see for example T aim of these works is to obtain position vectors of the curves with respect to Frenet frame. And, in the classical differential geometry, it is well-known that determining position vector of an arbitrary curve according to standard frame is not easy. In a recent study, bin position vectors of spacelike $W-$curves according to standard frame of E_1^3 by means of vector differential equations. Also, Ali bined the position vectors of a spacelike general helices according to standard frame in E_1^3. Then, the authors also investigated position vector of a timelike slant helix in a similar way
urve of constant slope or general helix is defined by the property that the tangent lines make a constant angle with a fixed direction. A necessary and sufficient condition that a curve to be general helix is that ratio of curvature to torsion be constant. Indeed, a helix is a special case of the general helix. If both curvature and torsion are non-zero constants, it is called a helix or only a $W-$curve.
Helices arise in nanosprings, carbon nanotubes, $\alpha-$helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA, Ts fact was published for the first time by Watson and Crick in 1952 (see .ey constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases

guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds (for details, see) All helices ($W-$curves) in E_1^3 are completely classified by Walfare in F instance, the only planar spacelike degenerate helices are circles and hyperbolas. In t authors investigated position vectors of a timelike and a null helix ($W-$curve) with respect to Frenet frame.
In this work, we use vector differential equations established by means of Frenet equations in Minkowski space E_1^3 to determine position vectors of the timelike general helices according to standard frame of E_1^3. We obtain position vectors of timelike general helices with respect to standard frame of E_1^3. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

2. PRELIMINARIES

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E_1^3 are briefly presented (A more complete elementary treatment can be found in [1].)

The Minkowski 3-space E_1^3 is the real vector space R^3 provided with the standard flat Lorentzian metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since g is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(v, v) > 0$ or $v = 0$, timelike if $g(v, v) < 0$ and null if $g(v, v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors φ' are respectively spacelike, timelike or null (lightlike), for every $s \in I \subset R$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|g(a, a)|}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E_1^3$ it is said to be orthogonal if and only if $g(v, w) = 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along the curve φ in the space E_1^3. For an arbitrary curve φ with first and second curvature, κ and τ in the space E_1^3, the following Frenet formulae are given in [1]:

If φ is a timelike curve, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where

$$g(T, T) = -1, g(N, N) = g(B, B) = 1,$$
$$g(T, N) = g(T, B) = g(T, N) = g(N, B) = 0.$$

Recall that an arbitrary curve is called a $W-$curve if it has constant Frenet curvatures A, from the view of Differential Geometry, a helix is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ
cause the tangent vector T is a timelike vector for a timelike curve, the angle between

the vector T and any vector U depends on the kind of U. We state here the following definitions of the angle between two vectors in Minkowski space([5],[15]):

Definition 2.1. Let u be a spacelike vector and v a positive timelike vector in $\mathbb{E}_1^3$, then there is a unique non-negative real number ϕ such that

$$|g(u,v)| = \|u\|\|v\|\sinh[\phi].$$

The real number ϕ is called the Lorentzian timelike angle between u and v.

Definition 2.2. Let u and v be positive (negative) timelike vectors in $\mathbb{E}_1^3$, then there is a unique non-negative real number ϕ such that

$$g(u,v) = \|u\|\|v\|\cosh[\phi].$$

The real number ϕ is called the Lorentzian timelike angle between u and v.

3. Main Results

Now, we give the two following lemmas using the definitions 2.1 and 2.2. The following propositions are *new characterizations* for timelike general helices in $\mathbb{E}_1^3$:

Lemma 3.1. Let $\psi : I \to \mathbb{E}_1^3$ be a timelike curve that is parameterized by arclength with intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$. The curve ψ is a general helix (its tangent vectors make a constant Lorentzian timelike angle $\phi = \pm\text{arcsinh}[n]$, with a fixed spacelike straight line in the space) if and only if $\left|\frac{\tau(s)}{\kappa(s)}\right| = \left|\tanh[\phi]\right| < 1$.

Proof. ($\Rightarrow$) Let $\mathbf{d}$ be the unitary fixed spacelike vector makes a constant timelike angle, $\phi = \pm\text{arcsinh}[n]$, with the tangent vector T. Therefore

$$g(\mathbf{d}, T) = n. \tag{3.1}$$

Differentiating the equation (3.1) with respect to the variable s and using the Frenet equations (2.1), we get

$$\kappa\, g(\mathbf{d}, N) = 0. \tag{3.2}$$

The curvature $\kappa(s)$ do not equal to zero, therefore the vector $\mathbf{d}$ is orthogonal to N takes the form:

$$\mathbf{d} = -n\,T + \lambda\,B. \tag{3.3}$$

Because the vector $\mathbf{d}$ is a unitary spacelike vector, we can get $\lambda = \pm\sqrt{1+n^2}$.
If we differentiate the equation (3.3), we obtain $\frac{\tau(s)}{\kappa(s)} = \pm\frac{n}{\sqrt{1+n^2}} = \pm\tanh[\phi]$, the desired result.
($\Leftarrow$) Suppose that ψ is a timelike curve and $\tau(s) = \pm\tanh[\phi]\kappa(s)$. Let us consider a spacelike vector

$$\mathbf{d} = \sinh[\phi]\,T \pm \cosh[\phi]\,B.$$

We will prove that the vector $\mathbf{d}$ is a constant vector. Indeed, applying Frenet formula (2.1), we have

$$\mathbf{d}'(s) = \Big(\sinh[\phi]\kappa(s) \mp \cosh[\phi]\tau(s)\Big)N = 0.$$

Therefore, the vector $\mathbf{d}$ is constant. This concludes the proof of lemma (3.1). $\square$

Lemma 3.2. Let $\psi : I \to \mathrm{E}_1^3$ be a timelike curve that is parameterized by arclength with intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$. The curve ψ is a general helix (its tangent vectors make a constant Lorentzian timelike angle $\phi = \pm\text{arccosh}[n]$, with a fixed timelike straight line in the space) if and only if $\Big|\frac{\tau(s)}{\kappa(s)}\Big| = \Big|\coth[\phi]\Big| > 1$.

The proof of the lemma (3.2) is similar as the proof of the lemma (3.1).
In this section, first, we adapt important theorems in the classical differential geometry of the curves to timelike curves of Minkowski 3-space.

Lipschutz [1] stated and proved the following two important theorem in Euclidean space E^3. Here we state the same theorems in Minkowski space E_1^3 but without proof.

Theorem 3.3. A curve is defined uniquely by its curvature and torsion as function of a natural parameters.

The equations

$$\kappa = \kappa(s), \quad \tau = \tau(s)$$

which give the curvature and torsion of a curve as functions of s are called the natural or intrinsic equations of a curve, for they completely define the curve.
We observe that the Frenet equations form a system of three vector differential equations of the first order in T, N and B. It is reasonable to ask, therefore, given arbitrary continuous functions κ and τ, whether or not there exist solutions T, N, B of the Frenet equations, and hence, since $\psi' = T$, a curve

$$\psi = \int T ds + C$$

which the prescribed curvature and torsion. The answer is in the affirmative and is given by

Theorem 3.4. **(Fundamental existence and uniqueness theorem for space curve).** Let $\kappa(s)$, $\tau(s)$ be arbitrary continuous function on $a \leq s \leq b$. Then there exists, except for position in space, one and only one timelike curve C for which $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and s is a natural parameter along C.

The problem of the determination of parametric representation of the position vector of an arbitrary space curve according to the intrinsic equations is still open in the Euclidean space E^3 and in the Minkowski space E_1^3 [1, 1]. This problem is not easy to solve in general case. We solved this problem in the case of the timelike general helix ($\frac{\tau}{\kappa}$ is constant) in Minkowski space E_1^3.

In the light of above statements, first, we give:

Theorem 3.5. Let $\psi = \psi(s)$ be a timelike unit speed curve. Then, position ψ satisfies a vector differential forth order as follows

$$\frac{d}{ds}\Big[\frac{1}{\tau}\frac{d}{ds}\Big(\frac{1}{\kappa}\frac{d^2\psi}{ds^2}\Big)\Big] + \Big(\frac{\tau}{\kappa} - \frac{\kappa}{\tau}\Big)\frac{d^2\psi}{ds^2} - \frac{d}{ds}\Big(\frac{\kappa}{\tau}\Big)\frac{d\psi}{ds} = 0. \tag{3.4}$$

Proof. Let $\psi = \psi(s)$ be an unit speed timelike curve with non-vanishing curvature and torsion. If we substitute first equation of (2.1) to second equation of (2.1), we have

$$B = \frac{d}{ds}\left(\frac{1}{\kappa}\frac{dT}{ds}\right) - \frac{\kappa}{\tau}T. \tag{3.5}$$

Differentiating of (3.5) and using in third equation of (2.1), we write

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\frac{dT}{ds}\right)\right] + \left(\frac{\tau}{\kappa} - \frac{\kappa}{\tau}\right)\frac{dT}{ds} - \frac{d}{ds}\left(\frac{\kappa}{\tau}\right)T = 0. \tag{3.6}$$

Denoting $\frac{d\psi}{ds} = T$, we have the following vector differential equation of fourth order

$$\frac{d}{ds}\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\frac{d^2\psi}{ds^2}\right)\right] + \left(\frac{\tau}{\kappa} - \frac{\kappa}{\tau}\right)\frac{d^2\psi}{ds^2} - \frac{d}{ds}\left(\frac{\kappa}{\tau}\right)\frac{d\psi}{ds} = 0. \tag{3.7}$$

□

If we put $\tau(s) = \kappa(s)\, f(s)$, the equation (3.7) takes the following simple form

$$\frac{d}{d\theta}\Big(\frac{1}{f}\frac{d^2T}{d\theta^2}\Big) + \Big(\frac{f^2-1}{f}\Big)\frac{dT}{d\theta} + \frac{1}{f^2}\frac{df}{d\theta}T = 0, \quad f = f(\theta), \quad \theta = \int \kappa(s)ds. \tag{3.8}$$

By means of solution of the above equation, position vector of an arbitrary timelike curve can be determined. However, the general solution of it has not been found. So, we investigate special cases. Now we give two theorems correspondence to the two lemmas above as follows:

Theorem 3.6. The position vector ψ of a timelike general helix whose tangent vector makes a constant Lorentzian timelike angle, with a fixed spacelike straight line in the space, is computed in the natural representation form:

$$\psi(s) = \sqrt{1+n^2}\int \Big(\cosh\Big[\sqrt{1-m^2}\int \kappa(s)ds\Big], \sinh\Big[\sqrt{1-m^2}\int \kappa(s)ds\Big], m\Big)ds, \tag{3.9}$$

or in the parametric form

$$\psi(\theta) = \int \frac{\sqrt{1+n^2}}{\kappa(\theta)}\Big(\cosh[\sqrt{1-m^2}\,\theta], \sinh[\sqrt{1-m^2}\,\theta], m\Big)d\theta, \tag{3.10}$$

where $\theta = \int \kappa(s)ds$, $m = \frac{n}{\sqrt{1+n^2}}$, $n = \sinh[\phi]$ and ϕ is the timelike angle between the fixed spacelike straight line $\mathbf{e}_3$ (axis of a timelike general helix) and the tangent vector of the curve.

Proof. If $\psi(\theta)$ is a timelike general helix whose tangent vector T makes a timelike angle $\phi = \pm\text{arcsinh}[n]$ with a straight spacelike line U, then we can write $f(\theta) = \tanh[\phi] = m$, where $\theta = \int \kappa(s)ds$ and $m = \frac{n}{\sqrt{1+n^2}}$. Therefore the equation (3.8) becomes

$$T'''(\theta) - (1-m^2)T'(\theta) = 0. \tag{3.11}$$

If we write the tangent vector as the following:

$$T = T_1(\theta)\mathbf{e}_1 + T_2(\theta)\mathbf{e}_2 + T_3(\theta)\mathbf{e}_3. \tag{3.12}$$

Now, the curve ψ is a timelike general helix, i.e. the tangent vector T makes a constant timelike angle, ϕ, with the constant spacelike vector called the axis of the general helix. So, with out loss of generality, we take the axis of a general helix is parallel to the spacelike vector $\mathbf{e}_3$. Then

$$T_3 = g(T, \mathbf{e}_3) = n. \tag{3.13}$$

On other hand the tangent vector T is a unit timelike vector, so the following condition is satisfied

$$T_1^2(\theta) - T_2^2(\theta) = 1 + n^2. \tag{3.14}$$

The general solution of equation (3.14) can be written in the following form:

$$T_1 = \sqrt{1+n^2}\,\cosh[t(\theta)], \quad T_2 = \sqrt{1+n^2}\,\sinh[t(\theta)], \tag{3.15}$$

where t is an arbitrary function of θ. Every component of the vector T is satisfied the equation (3.11). So, substituting the components $T_1(\theta)$ and $T_2(\theta)$ to the equation (3.11), we have the following differential equations of the function $t(\theta)$

$$3t't''\cosh[t] - \Big[(1-m^2)t' - t'^3 - t'''\Big]\sinh[t] = 0, \tag{3.16}$$

$$3t't''\sinh[t] - \Big[(1-m^2)t' - t'^3 - t'''\Big]\cosh[t] = 0. \tag{3.17}$$

It is easy to prove that the above two equations lead to the following two equations:

$$t'\,t'' = 0, \tag{3.18}$$

$$(1-m^2)t' - t'^3 - t''' = 0. \tag{3.19}$$

Because, the parameter t is a variable (not constant), then $t' \neq 0$, so that the general solution of the equation (3.18) is

$$t(\theta) = c_2 + c_1\,\theta, \tag{3.20}$$

where c_1 and c_2 are constants of integration. The constant c_2 can disappear if we change the parameter $t \to t + c_2$. Substituting the solution (3.20) in the equation (3.19), we obtain the following condition:

$$c_1 = \sqrt{1+m^2}.$$

Now, the tangent vector takes the following form:

$$T(\theta) = \sqrt{1+n^2}\Big(\cosh[\sqrt{1-m^2}\,\theta], \sinh[\sqrt{1-m^2}\,\theta], m\Big). \tag{3.21}$$

If we integrate the equation (3.21), we get the two equations (3.9) and (3.10), which it completes the proof. □

Theorem 3.7. The position vector ψ of a timelike general helix whose tangent vector makes a constant Lorentzian timelike angle, with a fixed timelike straight line in the space, is computed in the natural representation form:

$$\psi(s) = \sqrt{n^2-1}\int\Big(m, \cos\Big[\sqrt{m^2-1}\int\kappa(s)ds\Big], \sin\Big[\sqrt{m^2-1}\int\kappa(s)ds\Big]\Big)ds, \tag{3.22}$$

or in the parametric form

$$\psi(\theta) = \int\frac{\sqrt{n^2-1}}{\kappa(\theta)}\Big(m, \cos[\sqrt{m^2-1}\,\theta], \sin[\sqrt{m^2-1}\,\theta]\Big)d\theta, \tag{3.23}$$

where $\theta = \int \kappa(s)ds$, $m = \frac{n}{\sqrt{n^2-1}}$, $n = \cosh[\phi]$ and ϕ is the timelike angle between the fixed timelike straight line $-\mathbf{e}_1$ (axis of a timelike general helix) and the tangent vector of the curve.

According to lemma 3.2 the prove of the above theorems 3.7 is similar as the proof of the theorem 3.6.

4. Examples

In this section, we take several choices for the curvature κ and torsion τ, and next, we apply theorems 3.6 and 3.7.

Example 4.1. The case of a timelike general helix with

$$\kappa = \kappa(s), \quad \tau = 0, \tag{4.1}$$

which is the intrinsic equations of a timelike plane curve. There are two cases correspondence to two theorems 3.6 and 3.7.

Case 1: If ψ is a timelike plane curve, then the tangent vectors make a constant Lorentzian timelike angle $\phi = \text{arctanh}[n]$ with a fixed spacelike straight line $\mathbf{e}_3$. According to lemma 3.1, we have $\tanh[\phi] = \frac{\tau(s)}{\kappa(s)} = 0$ which leads to $n = \sinh[\phi] = 0$ and $m = \tanh[\phi] = 0$. Substituting values ($n = m = 0$) in the equations (3.9) and (3.10) we have the explicit natural and parametric representation of such curve as follows:

$$\psi(s) = \int \Big(\cosh\Big[\int \kappa(s)ds\Big], \sinh\Big[\int \kappa(s)ds\Big], 0\Big) ds, \tag{4.2}$$

$$\psi(\theta) = \int \frac{1}{\kappa(\theta)} \Big(\cosh[\theta], \sinh[\theta], 0\Big) d\theta, \tag{4.3}$$

where $\theta = \int \kappa(s)ds$. Now, we give the parametric representation of a special example of a plane curve.

(1): The position vector ψ of a timelike plane curve ($\kappa(s) = \frac{a}{a^2-s^2}$), whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed spacelike straight line in the space, takes the form:

$$\psi(s) = a\Big(2\arctan\Big[\tanh[\frac{\theta}{2}]\Big], -\text{sech}[\theta], 0\Big), \tag{4.4}$$

where $s = a\tanh[\theta]$ and $\kappa(\theta) = \frac{\cosh^2[\theta]}{a}$. One can see a special example of such curve when $a = 2$ in the lift hand side of the figure 1.

(2): The position vector ψ of a timelike plane curve ($\kappa(s) = \frac{a}{s}$), whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed spacelike straight line in the space, takes the form:

$$\psi(s) = e^{\theta/a}\Big(a\sinh[\theta] - \cosh[\theta], a\cosh[\theta] - \sinh[\theta], 0\Big), \tag{4.5}$$

where $s = e^{\theta/a}$ and $\kappa(\theta) = a e^{-\theta/a}$. One can see a special example of such curve when $a = 2$ in the right hand side of the figure 1.

Case 2: If ψ is a timelike plane curve and the tangent vectors make a constant Lorentzian timelike angle $\phi = \text{arccosh}[n]$ with a fixed timelike straight line $-\mathbf{e}_1$. According to lemma 3.2, we have $\coth[\phi] = \frac{\tau(s)}{\kappa(s)} = 0$ which is contradiction $(\coth[\phi] \neq 0)$. Therefore, we can write the following lemma.

Lemma 4.2. There are no timelike plane curves whose tangent vector makes a constant Lorentzian timelike angle with a fixed timelike straight line.

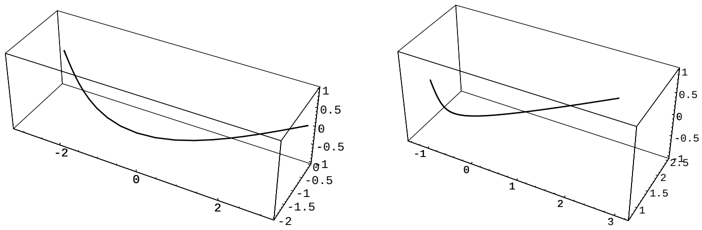

FIGURE 1. Some timelike plane curves

Example 4.3. The case of a timelike general helix with

$$\kappa = constant, \quad \tau = constant, \tag{4.6}$$

which is the intrinsic equations of a timelike W-curve or helix. There are two cases correspondence to two theorems 3.6 and 3.7.

Case 1: The position vector ψ of a timelike W-curve whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed spacelike straight line in the space, is computed in the parametric representation form:

$$\psi(s) = \frac{\kappa}{\kappa^2 - \tau^2}\Big(\sinh[\xi], \cosh[\xi], \frac{\tau}{\kappa}\,\xi\Big), \tag{4.7}$$

where $\xi = \sqrt{\kappa^2 - \tau^2}\,s$ and $\phi = \operatorname{arctanh}[\frac{\tau}{\kappa}]$. One can see a special example of such curve when $\kappa = 3$ and $\tau = 2$ in the left hand side of the figure 2.

Case 2: The position vector ψ of a timelike W-curve whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed timelike straight line in the space, is computed in the parametric representation form:

$$\psi(s) = \frac{\kappa}{\tau^2 - \kappa^2}\Big(\frac{\tau}{\kappa}\,\xi, \sin[\xi], -\cos[\xi],\Big), \tag{4.8}$$

where $\xi = \sqrt{\tau^2 - \kappa^2}\,s$ and $\phi = \operatorname{arccoth}[\frac{\tau}{\kappa}]$. One can see a special example of such curve when $\kappa = \frac{1}{3}$ and $\tau = \frac{1}{2}$ in the right hand side of the figure 2.

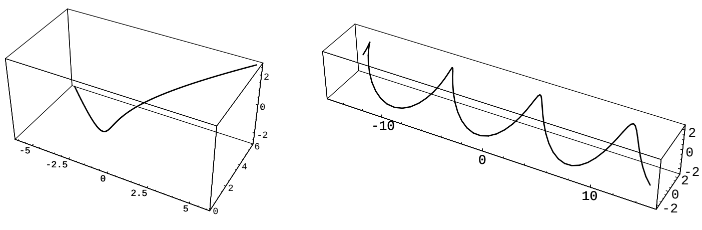

FIGURE 2. Some timelike W-curves.

Example 4.4. The case of a timelike general helix with

$$\kappa = \frac{h}{s}, \quad \tau = \frac{r}{s}, \tag{4.9}$$

where h and r are arbitrary constants. There are two cases correspondence to two theorems 3.6 and 3.7.

Case 1: The position vector $\psi = (\psi_1, \psi_2, \psi_3)$ of a timelike general helix whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed spacelike straight line in the space, is computed in the parametric representation form:

$$\begin{cases} \psi_1(\theta) = \frac{h\,e^{\theta/h}}{h^2-r^2-1}\left(\sinh\left[\frac{\sqrt{h^2-r^2}}{h}\,\theta\right] - \frac{1}{\sqrt{h^2-r^2}}\cosh\left[\frac{\sqrt{h^2-r^2}}{h}\,\theta\right]\right), \\ \psi_2(\theta) = \frac{h\,e^{\theta/h}}{h^2-r^2-1}\left(\cosh\left[\frac{\sqrt{h^2-r^2}}{h}\,\theta\right] - \frac{1}{\sqrt{h^2-r^2}}\sinh\left[\frac{\sqrt{h^2-r^2}}{h}\,\theta\right]\right), \\ \psi_3(\theta) = \frac{r\,e^{\theta/h}}{\sqrt{h^2-r^2}}, \end{cases} \tag{4.10}$$

where $s = e^{\theta/h}$ and $\phi = \operatorname{arctanh}[\frac{r}{h}]$. One can see a special example of such curve when $h = 3$ and $r = 2$ in the left hand side of figure 3.

Case 2: The position vector ψ of a timelike general helix whose tangent vector makes a constant Lorentzian timelike angle, ϕ, with a fixed timelike straight line in the space, is computed in the parametric representation form:

$$\begin{cases} \psi_3(\theta) = \frac{r\,e^{\theta/h}}{\sqrt{r^2-h^2}}, \\ \psi_1(\theta) = \frac{h\,e^{\theta/h}}{1-h^2+r^2}\left(\frac{1}{\sqrt{r^2-h^2}}\cos\left[\frac{\sqrt{r^2-h^2}}{h}\,\theta\right] + \sin\left[\frac{\sqrt{r^2-h^2}}{h}\,\theta\right]\right), \\ \psi_2(\theta) = \frac{h\,e^{\theta/h}}{1-h^2+r^2}\left(\frac{1}{\sqrt{r^2-h^2}}\sin\left[\frac{\sqrt{r^2-h^2}}{h}\,\theta\right] - \cos\left[\frac{\sqrt{r^2-h^2}}{h}\,\theta\right]\right), \end{cases} \tag{4.11}$$

where $s = e^{\theta/h}$ and $\phi = \operatorname{arccoth}[\frac{r}{h}]$. One can see a special example of such curve when $h = 20$ and $r = 30$ in the right hand side of the figure 3.

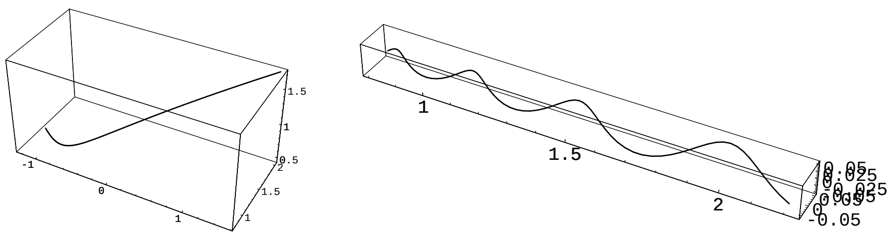

FIGURE 3. Some timelike general helices.

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AL-AZHAR UNIVERSITY,
FACULTY OF SCIENCE, MATHEMATICS DEPARTMENT,
NASR CITY, 11884, CAIRO, EGYPT.
KING ABDUL AZIZ UNIVERSITY,
FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS,
PO BOX 80203, JEDDAH, 21589, SAUDI ARABIA.
E-mail address: `atali71@yahoo.com`

DEPARTMENT OF ELEMENTARY EDUCATION,
FACULTY OF EDUCATION,
ESKISEHIR OSMANGAZI UNIVERSITY,
ESKISEHIR, TURKEY.
E-mail address: `mturgut@ogu.edu.tr`