Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.2, Issue 1, pp.36-49

SOME STRUCTURES OF THE GROUP OF STRONG SYMPLECTIC HOMEOMORPHISMS

STÉPHANE TCHUIAGA

ABSTRACT. The group of strong symplectic homeomorphisms of a closed connected symplectic manifold was defined and studied in [5], [6] and [7]. In this paper, we introduce the notion of continuous symplectic flow, and we exhibit the C^0 analog of Hodge's decomposition theorem of symplectic isotopies [4]. We prove that any continuous symplectic flows can be decomposed as product of topological harmonic flow by continuous Hamiltonian flow. Finally, we describe some structures of strong symplectic homeomorphisms.

AMS Subject Classification: 53D05, 53D35.
Key Words and Phrases: Sympeomorphisms, generator of symplectic paths, symplectic homeomorphisms, C^0-symplectic topology.

1. INTRODUCTION

The notion of C^0- symplectic topology emerged from the work of Eliashberg and Gromov on the C^0- closure of the symplectic diffeomorphisms. In [14, 15] Oh and Müller defined the C^0- Hamiltonian topology and introduced the group $Hameo(M,\omega)$ of Hamiltonian homeomorphisms. More recently Banyaga [5, 6] defined the C^0- symplectic topology which generalizes the C^0- Hamiltonian topology, and defined the $L^{(1,\infty)}-$context of the group $SSympeo(M,\omega)$ of all ssympeomorphisms of a closed connected symplectic manifold (M,ω). This group generalizes the group $Hameo(M,\omega)$, of all Hamiltonian homeomorphisms. Our main results are the following :

Theorem 1.1. *Let (M,ω) be a closed connected symplectic manifold. The following equality holds*
$$\mathcal{P}SSympeo(M,\omega) = \mathcal{P}Harm(M,\omega) \circ \mathcal{P}Hameo(M,\omega).$$

Theorem 1.2. *Let (M,ω) be a closed connected symplectic manifold. Then,*

- *there exists an infinite number of non-differentiable elements of $SSympeo(M,\omega)$ whose Fathi's mass flow is non-trivial*
- *the inclusion $Hameo(M,\omega) \subset SSympeo(M,\omega)$ is strict.*

2. PRELIMINARIES

Let (M,ω) be a closed connected symplectic manifold. In the following we will mainly use notations from [5,6]. We denote by $C^\infty([0,1]\times M)$ the vector space of smooth time-dependent functions $H : [0,1]\times M \to \mathbb{R}$. By definition (h_t) is a symplectic isotopy if the map $(x,t) \mapsto h_t(x)$ is smooth with $h_t^*\omega = \omega$ for all t and $h_0(x) = x$ for all $x \in M$. We denote by $Iso(M,\omega)$ the set of all symplectic isotopies of M.

Fix a Riemannian metric g on M. Let $\mathbb{H}^1(M,\mathbb{R})$ be the first de Rham cohomology group. It is well known that $\mathbb{H}^1(M,\mathbb{R})$ is a finite dimensional vector space over $\mathbb{R}$ whose dimension is the first Betti number of M. We denote in this work the first Betti number of M by b_1, and by $harm^1(M,g)$ we denote the space of smooth harmonic $1-$forms on M. According to Hodge theory [17], the space $harm^1(M,g)$ is isomorphic to $\mathbb{H}^1(M,\mathbb{R})$. We will consider on the space $harm^1(M,g)$ the Euclidean norm $(|.|)$ defined as follows. Let $(h_i)_{1\leq i\leq b_1}$ be a basis of $harm^1(M,g)$. Let $\mathcal{H}$ in $harm^1(M,g)$ such that

$$\mathcal{H} = \Sigma_{i=1}^{b_1}\lambda_i h_i. \tag{2.1}$$

The norm of $\mathcal{H}$ is given by :

$$|\mathcal{H}| := \Sigma_{i=1}^{b_1}|\lambda_i|. \tag{2.2}$$

Denote by $\mathfrak{P}^\infty(harm^1(M,g))$ the space of smooth families of harmonic $1-$forms.

Definition 2.1. *A symplectic isotopy $\Psi = (\psi_t)$ is said to be harmonic if there exists $\mathcal{H} = (\mathcal{H}_t) \in \mathfrak{P}^\infty(harm^1(M,g))$ such that*

$$i_{\dot{\psi}_t}\omega = \mathcal{H}_t,$$

where

$$\dot{\psi}_t(x) = \frac{d}{dt}\psi_t(\psi_t^{-1}(x)),$$

for all t, and for all $x \in M$.

We call *harmonic diffeomorphism* any time-one evaluation map of a harmonic isotopy.
Let $C^\infty([0,1]\times M,\mathbb{R})$ be the vector space of smooth time-dependent functions from the space $[0,1]\times M$ onto the space $\mathbb{R}$.

Definition 2.2. *A symplectic isotopy $\Psi = (\psi_t)$ is said to be Hamiltonian if there exists $H \in C^\infty([0,1]\times M,\mathbb{R})$ such that*

$$i_{\dot{\psi}_t}\omega = dH_t \tag{2.3}$$

where

$$\dot{\psi}_t(x) = \frac{d\psi_t}{dt}((\psi_t)^{-1}(x)), \tag{2.4}$$

for all $x \in M$ and for all t.

Definition 2.3. *An element $U \in C^\infty([0,1]\times M,\mathbb{R})$ is normalized if*

$$\int_M U_t\omega^n = 0, \tag{2.5}$$

for all $t \in [0,1]$.

Denote by $\mathcal{N}([0,1]\times M\ ,\mathbb{R})$ the vector space of smooth time-dependent normalized functions. It is easy to show that the correspondence between the spaces $\mathcal{N}([0,1]\times M\ ,\mathbb{R})$ and $HIso(M,\omega)$ is bijective.

Definition 2.4. *The oscillation of any smooth function f is given by the following formula,*

$$osc(f) = \max_{x\in M} f(x) - \min_{x\in M} f(x). \tag{2.6}$$

We will also need the following well known result of [4]. Let (θ_t) be a smooth family of closed $1-$forms and let (ϕ_t) be an isotopy. Then

$$(\phi_t)^*\theta_t - \theta_t = d(\int_0^t (\theta_t(\dot{\phi}_s) \circ \phi_s)ds), \tag{2.7}$$

for all t.
Indeed, for a fixed t, we have $\frac{d}{ds}(\phi_s^*\theta_t) = \phi_s^*(\mathrm{L}_{\dot{\phi}_s}\theta_t)$ where L_X is the Lie derivative in the direction of the vector field X. Since the form θ_t is closed, deduce that $\frac{d}{ds}(\phi_s^*\theta_t) = \phi_s^*(di_{\dot{\phi}_s}\theta_t) = d(\theta_t(\dot{\phi}_s) \circ \phi_s)$. Integrating the above relation in the variable s between 0 et u one obtains :

$$\phi_u^*\theta_t - \theta_t = \int_0^u \frac{d}{ds}(\phi_s^*\theta_t)ds = d(\int_0^u (\theta_t(\dot{\phi}_s) \circ \phi_s)ds). \tag{2.8}$$

Take $t = u$ to obtain the desired result (see [4]).

3. Hodge decomposition of symplectic isotopies [5]

Fix a symplectic isotopy $\Phi = (\phi_t)$. The definition of symplectic isotopies implies that $i_{\dot{\phi}_t}\omega$ is a closed $1-$form for each t. To see this, for each t compute $L_{\dot{\phi}_t}\omega = d(i_{\dot{\phi}_t}\omega)$, $\phi_t^*(L_{\dot{\phi}_t}\omega) = \dfrac{d}{dt}(\phi_t^*(\omega)) = 0$, and derive that $di_{\dot{\phi}_t}\omega = 0$. Hodge's decomposition theorem of closed forms [17] implies that $i_{\dot{\phi}_t}\omega = dU_t + \mathcal{H}_t$ where $U_t \in C^\infty(M, \mathbb{R})$, and $\mathcal{H}_t \in harm^1(M, g)$ for all t. Let (ρ_t) be the harmonic isotopy such that $i_{\dot{\rho}_t}\omega = \mathcal{H}_t$ for all t. Consider the isotopy $\psi_t = \rho_t^{-1} \circ \phi_t$ for all t. From $\phi_t = \rho_t \circ \psi_t$ we get by differentiation $\dot{\phi}_t = \dot{\rho}_t + (\rho_t)_*\dot{\psi}_t$ which implies that :

$$\begin{aligned} i_{(\rho_t)_*\dot{\psi}_t}\omega &= i_{(\dot{\phi}_t - \dot{\rho}_t)}\omega \\ &= dU_t, \end{aligned}$$

for all t. We have :

$$\begin{aligned} i_{(\rho_t)_*\dot{\psi}_t}\omega &= (\rho_t^{-1})^*(i_{\dot{\psi}_t}(\rho_t^*\omega)) \\ &= (\rho_t^{-1})^*(i_{\dot{\psi}_t}\omega), \end{aligned}$$

for all t. Hence,

$$\begin{aligned} (\rho_t^{-1})^*(i_{\dot{\psi}_t}\omega) &= i_{(\rho_t)_*\dot{\psi}_t}\omega \\ &= dU_t, \end{aligned}$$

for all t. This shows that (ψ_t) is a Hamiltonian isotopy. Therefore we have just proved by the help of Hodge's decomposition theorem of closed forms [17] that any symplectic isotopy (ϕ_t) can be decomposed as a composition of a harmonic isotopy (ρ_t) by a Hamiltonian isotopy (ψ_t). The uniqueness in the latter decomposition of symplectic isotopies follows from the uniqueness of Hodge's decomposition of closed differential forms [17]. In [4] the above decomposition of symplectic isotopies has been called the *Hodge decomposition of symplectic isotopies*.

3.1. **A new description of symplectic isotopies.** Let $\Phi = (\phi_t)$ be a symplectic isotopy, and let $U^\Phi = (U_t^\Phi)$, $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)$ be Hodge decomposition of $i_{\dot{\phi}_t}\omega$, i.e $i_{\dot{\phi}_t}\omega = dU_t + \mathcal{H}_t$ for any t. Denote by U the function U^Φ normalized and by $\mathcal{H}$ the family of harmonic forms $\mathcal{H}^\Phi = (\mathcal{H}_t^\Phi)$. The map

$$\begin{aligned} Iso(M,\omega) &\rightarrow \mathcal{N}([0,1]\times M,\mathbb{R})\times\mathfrak{P}^\infty(harm^1(M,g)) \\ \Phi = (\phi_t) &\mapsto (U,\mathcal{H}), \end{aligned}$$

is a bijection. We denote by $\mathfrak{T}(M,\omega,g)$ the product space $\mathcal{N}([0,1]\times M,\mathbb{R})\times\mathfrak{P}^\infty(harm^1(M,g))$. It follows from the above statement that the correspondence between the sets $Iso(M,\omega)$ and $\mathfrak{T}(M,\omega,g)$ is bijective. We denote the latter correspondence by $\mathfrak{A} : Iso(M,\omega) \rightarrow \mathfrak{T}(M,\omega,g)$. We call any element therein $\mathfrak{T}(M,\omega,g)$ " smooth generator" of symplectic isotopy.
Let $\Phi_1 = (\phi_1^t)$ and $\Phi_2 = (\phi_2^t)$ be two elements of $Iso(M,\omega)$ such that $\mathfrak{A}(\Phi_i) = (U^i,\mathcal{H}^i)$ for $i = 1,2$. Consider the product $\phi_t = \phi_1^t\circ\phi_2^t$ for all t. From $\phi_t = \phi_1^t\circ\phi_2^t$ we get by differentiation $\dot{\phi}_t = \dot{\phi}_1^t + (\phi_1^t)_*\dot{\phi}_2^t$ which implies that :

$$\begin{aligned} i_{\dot{\phi}_t}\omega &= i_{\dot{\phi}_1^t}\omega + (\phi_1^{-t})^*(i_{\dot{\phi}_2^t}\omega) \\ &= (dU_t^1 + \mathcal{H}_t^1) + d(U_t^2\circ\phi_2^{-t} + \Delta_t(\mathcal{H}^2,\phi_1^{-1})) + \mathcal{H}_t^2, \\ &= d(U_t^1 + U_t^2\circ\phi_2^{-t} + \Delta_t(\mathcal{H}^2,\phi_1^{-1})) + \mathcal{H}_t^1 + \mathcal{H}_t^2, \end{aligned}$$

where $\Delta_t(\mathcal{H}^2,\phi_1^{-1}) := \int_0^t \mathcal{H}_t^2(\dot{\phi}_1^{-s})\circ\phi_1^{-s}ds$, and $\phi_1^{-t} := (\phi_1^t)^{-1}$ for all t. The above result suggests that when one decomposes the composition $\Phi_1\circ\Phi_2$ in the Hodge decomposition of symplectic isotopies, its harmonic part is generated by the sum $\mathcal{H}+\mathcal{K}$ and its Hamiltonian part is generated by the normalized function associated to the sum $U^1 + U^2\circ\Phi_1^{-1} + \Delta(\mathcal{H}^2,\Phi_1^{-1})$. By assumption, both functions U^1 and U^2 are already normalized. Hence, $\int_M U^2\omega^n = 0$ obviously implies that $\int_M U^2\circ\Phi_1^{-1}\omega^n = 0$. This implies that to normalize the sum $U^1 + U^2\circ\Phi_1^{-1} + \Delta(\mathcal{H}^2,\Phi_1^{-1})$ its suffices to normalize the function $\Delta(\mathcal{H}^2,\Phi_1^{-1})$. We denote by $\widetilde{\Delta}(\mathcal{H}^2,\Phi_1^{-1})$ the normalized function associated to $\Delta(\mathcal{H}^2,\Phi_1^{-1})$. Therefore, it follows from the above statement that the composition $\Phi_1\circ\Phi_2$ is generated by the element $(U^1 + U^2\circ\Phi_1^{-1} + \widetilde{\Delta}(\mathcal{H}^2,\Phi_1^{-1}),\mathcal{H}^1 + \mathcal{H}^2)$. Similarly, from $id = \phi_1^{-t}\circ\phi_1^t$ we get by differentiation $\dot{\phi}_1^{-t} = -(\phi_1^{-t})_*\dot{\phi}_1^t$ which implies that $i_{\dot{\phi}_1^{-t}}\omega = -(\phi_1^t)^*(i_{\dot{\phi}_1^t}\omega) = -d(U_t^1\circ\phi_1^t + \Delta_t(\mathcal{H}^1,\phi_1^t)) - \mathcal{H}_t^1$. It follows from the above that the isotopy Φ_1^{-1} is generated by $(-U^1\circ\Phi_1 - \widetilde{\Delta}(\mathcal{H}^1,\Phi_1),-\mathcal{H}^1)$.

For short, in the rest of this work, exceptionally if mention is made to the contrary we will denote any symplectic isotopy by $\phi_{(U,\mathcal{H})}$ to mean that its image by the map $\mathfrak{A}$ is $(U,\mathcal{H})$. In particular, any symplectic isotopy of the form $\phi_{(0,\mathcal{H})}$ is considered as a harmonic isotopy and any symplectic isotopy of the form $\phi_{(U,0)}$ is considered as a Hamiltonian isotopy. We endow the space $\mathfrak{T}(M,\omega,g)$ with the composition law $\bowtie$ defined by :

$$(U,\mathcal{H})\bowtie(U',\mathcal{H}') = (U + U'\circ\phi_{(U,\mathcal{H})}^{-1} + \widetilde{\Delta}(\mathcal{H}',\phi_{(U,\mathcal{H})}^{-1}),\mathcal{H}+\mathcal{H}') \tag{3.1}$$

The inverse of $(U,\mathcal{H})$ denoted $\overline{(U,\mathcal{H})}$ is given by

$$\overline{(U,\mathcal{H})} = (-U\circ\phi_{(U,\mathcal{H})} - \widetilde{\Delta}(\mathcal{H},\phi_{(U,\mathcal{H})}),-\mathcal{H}) \tag{3.2}$$

One can check that $(\mathfrak{T}(M,\omega,g),\bowtie)$ is a group (see [7] for more detail).

Remark 3.1. *Hodge decomposition of any symplectic isotopy generated by an element $(U,\mathcal{H}) \in \mathfrak{T}(M,\omega,g)$ can be written as :*

$$(U,\mathcal{H}) = (0,\mathcal{H}) \bowtie (U\circ\phi_{(0,\mathcal{H})},0).$$

We define two metrics D^2 and D^1 on the space $\mathfrak{T}(M,\omega,g)$ as follows.

$$D^1((U,\mathcal{H}),(V,\mathcal{K})) = \frac{D_0((U,\mathcal{H}),(V,\mathcal{K})) + D_0(\overline{(U,\mathcal{H})},\overline{(V,\mathcal{K})})}{2} \tag{3.3}$$

$$D^2((U,\mathcal{H}),(V,\mathcal{K})) = \frac{D_0^\infty((U,\mathcal{H}),(V,\mathcal{K})) + D_0^\infty(\overline{(U,\mathcal{H})},\overline{(V,\mathcal{K})})}{2} \tag{3.4}$$

where

$$D_0((U,\mathcal{H}),(V,\mathcal{K})) = \int_0^1 osc(U_t - V_t) + |\mathcal{H}_t - \mathcal{K}_t|dt,$$
$$D_0^\infty((U,\mathcal{H}),(V,\mathcal{K})) = \max_{t\in[0,1]}(osc(U_t - V_t) + |\mathcal{H}_t - \mathcal{K}_t|).$$

Let $Homeo(M)$ be the group of all homeomorphisms of M endowed with the C^0-topology. This is the metric topology induced by the following distance:

$$d_0(f,g) = \max(d_{C^0}(f,g), d_{C^0}(f^{-1},g^{-1})) \tag{3.5}$$

where $d_{C^0}(f,g) = \sup_{x\in M} d(g(x),f(x))$ and d is a distance on M induced by the Riemannian metric g. The space $\mathcal{P}Homeo(M)$ of continuous paths $\varrho : [0,1] \to Homeo(M)$ such that $\varrho(0) = id$ is endowed with the C^0-topology induced by the following metric

$$\bar{d}(\lambda,\mu) = \max_{t\in[0,1]} d_0(\lambda(t),\mu(t)). \tag{3.6}$$

Lemma 3.1. *Let $((U^n,\mathcal{H}^n))_n$ and $((V^n,\mathcal{K}^n))_n$ be two D^2-Cauchy sequences. If the sequences $(\phi_{(U^n,\mathcal{H}^n)})_n$ and $(\phi_{(V^n,\mathcal{K}^n)})_n$ are $\bar{d}-$Cauchy. Then,*

(1) *$((U^n,\mathcal{H}^n)\bowtie(V^n,\mathcal{K}^n))_n$ is D^2-Cauchy,*
(2) *$\overline{(U^n,\mathcal{H}^n)}$ is D^2-Cauchy.*

Proof of Lemma 3.1. Fist of all, for any integer n, we compute :

$$(U^n,\mathcal{H}^n)\bowtie(V^n,\mathcal{K}^n) = (U^n + V^n\circ\phi^{-1}_{(U^n,\mathcal{H}^n)} + \Delta(\mathcal{K}^n,\phi^{-1}_{(U^n,\mathcal{H}^n)}),\mathcal{K}^n + \mathcal{H}^n),$$

$$\overline{(U^n,\mathcal{H}^n)} = (-U^n\circ\phi_{(U^n,\mathcal{H}^n)} - \Delta(\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)}),-\mathcal{H}^n),$$
$$\overline{(V^n,\mathcal{K}^n)} = (-V^n\circ\phi_{(V^n,\mathcal{K}^n)} - \Delta(\mathcal{K}^n,\phi_{(V^n,\mathcal{K}^n)}),-\mathcal{K}^n).$$

Define :

$$\Pi_1^n =: -U^n\circ\phi_{(U^n,\mathcal{H}^n)} - \Delta(\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)}),$$
$$\Pi_2^n =: -V^n\circ\phi_{(V^n,\mathcal{K}^n)} - \Delta(\mathcal{K}^n,\phi_{(V^n,\mathcal{K}^n)}).$$

Therefore, we have

$$\overline{(V^n,\mathcal{K}^n)}\bowtie\overline{(U^n,\mathcal{H}^n)} = (\Pi_2^n + \Pi_1^n\circ\phi_{(V^n,\mathcal{K}^n)} + \Delta(-\mathcal{H}^n,\phi_{(V^n,\mathcal{K}^n)}),-\mathcal{K}^n - \mathcal{H}^n).$$

To check that the quantity $D_0^\infty((U^n,\mathcal{H}^n)\bowtie(V^n,\mathcal{K}^n),(U^m,\mathcal{H}^m)\bowtie(V^m,\mathcal{K}^m))$ tends to zero when n and m go at infinite, it suffices to prove that each of the quantities $\max_t(osc(\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^m,\phi^{-1}_{(U^m,\mathcal{H}^m)})))$ and

$\max_t(osc(V_t^n \circ \phi^{-1}_{(U^n,\mathcal{H}^n)} - V_t^m \circ \phi^{-1}_{(U^m,\mathcal{H}^m)}))$ tend to zero when n and m go at infinite.
To prove the above statements, fist of all, we fix an very large integer n_0, and pass to subsequences so that each of the quantities $\bar{d}(\phi_{(U^{n_0},\mathcal{H}^{n_0})}, \phi_{(U^n,\mathcal{H}^n)})$ and $\bar{d}(\phi_{(V^{n_0},\mathcal{K}^{n_0})}, \phi_{(V^n,\mathcal{K}^n)})$ becomes arbitrarily small for $n \geq 1$. The above considerations are always possible since both sequences $(\phi_{(U^n,\mathcal{H}^n)})_n$ and $(\phi_{(V^n,\mathcal{K}^n)})_n$ are $\bar{d}-$Cauchy.

- Stage (a) : Consider $S_1 = \max_t(osc(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})})))$, and compute :

$$\begin{aligned} S_1 \leq\ & \max_t(osc(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}))) \\ +\ & \max_t(osc(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}) - \Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}))) \\ +\ & \max_t(osc(\Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}) - \Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^n,\mathcal{H}^n)}))) \\ +\ & \max_t(osc(\Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})}))). \end{aligned}$$

Therefore, one checks from [7] (Lemma 3.4) that,

$$\max_t(osc(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}) - \Delta_t(\mathcal{K}^{n+1}, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}))) \to 0, n \to \infty.$$

On the another hand, let r_0 be the injectivity radius of the manifold M (r_0 is positive since M is a compact Riemannian manifold). By assumption we have $\bar{d}(\phi_{(U^n,\mathcal{H}^n)}, \phi_{(U^{n_0},\mathcal{H}^{n_0})}) \leq r_0$ and $\bar{d}(\phi_{(V^{n_0},\mathcal{K}^{n_0})}, \phi_{(V^n,\mathcal{K}^n)}) \leq r_0$ for all n. Put,

$$\phi_n^t =: \phi^{-1}_{(U^n,\mathcal{H}^n)},$$

$$\phi_{n_0}^t =: \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})},$$

for each t, and for each n. Derive from formula 2.7 that for each t, and for each n we have :

$$(\phi_n^t)^*(\mathcal{K}_t^n) - (\phi_{n_0}^t)^*(\mathcal{K}_t^n) = d(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})})).$$

Fix a point $\blacktriangle$ on M, for all $x \in M$, pick any curve ζ_x from $\blacktriangle$ to x and consider the function

$$\mu_{t,n}(x) = \int_{\zeta_x} (\phi_n^t)^*(\mathcal{K}_t^n) - (\phi_{n_0}^t)^*(\mathcal{K}_t^n),$$

for all fixed $t \in [0,1]$. The function $\mu_{t,n}$ does not depend on the choice of the curve ζ from $\blacktriangle$ to x. It is easy to check that

$$osc(\mu_{t,n}) = os(\Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^n,\mathcal{H}^n)}) - \Delta_t(\mathcal{K}^n, \phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})})).$$

Let x_0 be any point of M that realizes the supremum of the function $x \mapsto |\mu_t(x)|$, or equivalently

$$\sup_x |\mu_{t,n}(x)| = |\int_{\zeta_{x_0}} (\phi_n^t)^*(\mathcal{K}_t^n) - (\phi_{n_0}^t)^*(\mathcal{K}_t^n)|.$$

Compute,

$$\sup_x |u_{t,n}(x)| = |\int_{\zeta_{x_0}} (\phi_n^t)^*(\mathcal{K}_t^n) - (\phi_{n_0}^t)^*(\mathcal{K}_t^n)| = |\int_{\phi_n^t \circ \zeta_{x_0}} \mathcal{K}_t^n - \int_{\phi_{n_0}^t \circ \zeta_{x_0}} \mathcal{K}_t^n|.$$

Since by assumption the inequality $\bar{d}(\phi_{(U^n,\mathcal{H}^n)},\phi_{(U^{n_0},\mathcal{H}^{n_0})}) \leq r_0$ holds true, we derive that $\sup_x d(\phi_n^t(x),\phi_{n_0}^t(x)) \leq r_0$ for all t. It follows that there is a homotopy $F:[0,1]\times M \to M$ between the isotopies (ϕ_n^t) and $(\phi_{n_0}^t)$. That is, $F(0,y)=\phi_n^t(y)$ and $F(1,y)=\phi_{n_0}^t(y)$ for all $y\in M$. We may define $F(s,y)$ to be the unique minimizing geodesic χ_z joining $\phi_n^t(z)$ to $\phi_{n_0}^t(z)$ for all $z\in M$ (see Theorem 12.9 of [8] for more details). Consider the following set,

$$\boxdot := \{F(s,\zeta_{x_0}(u)), 0\leq s,u\leq 1\}.$$

Since the form $\mathcal{K}_t^n$ is closed, one derives by the help of Stokes' theorem that $\int_{\partial\boxdot}\mathcal{K}_t^n=0$, this is equivalent to

$$\int_{\phi_n^t\circ\zeta_{x_0}}\mathcal{K}_t^n-\int_{\phi_{n_0}^t\circ\zeta_{x_0}}\mathcal{K}_t^n=\int_{\chi_{x_0}}\mathcal{K}_t^n-\int_{\chi_\blacktriangle}\mathcal{K}_t^n,$$

since the boundary of the set $\boxdot$ is exactly $Im(\phi_n^t\circ\zeta_{x_0})\cup Im(\phi_{n_0}^t\circ\zeta_{x_0})\cup Im(\chi_{x_0})\cup Im(\chi_\blacktriangle)$, see [6] for more details. The integral over the geodesic χ_{x_0} is bounded by the quantity $\sup_s|\mathcal{K}_t^n(\dot\chi_{x_0}(s))|.d(\phi_{n_0}^t(x_0),\phi_n^t(x_0))$, because the speed of any minimizing geodesic is bounded from the above by the distance between its end points. That is,

$$|\int_{\chi_{x_0}}\mathcal{K}_t^n|\leq\sup_s|\mathcal{K}_t^n(\dot\chi_{x_0}(s))|.\bar{d}(\phi_{(U^n,\mathcal{H}^n)},\phi_{(U^{n_0},\mathcal{H}^{n_0})}), \tag{3.7}$$

Analogously for the integral over the geodesic $\chi_\blacktriangle$ i.e

$$|\int_{\chi_\blacktriangle}\mathcal{K}_t^n|\leq\sup_s|\mathcal{K}_t^n(\dot\chi_\blacktriangle(s))|.\bar{d}(\phi_{(U^n,\mathcal{H}^n)},\phi_{(U^{n_0},\mathcal{H}^{n_0})}) \tag{3.8}$$

for each integer n, and for each t. Relations 3.7 and 3.8 imply that the quantity $\max_t(osc(\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^n,\mathcal{H}^n)})-\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})})))$ is always bounded from above by $2.B_n\bar{d}(\phi_{(U^n,\mathcal{H}^n)},\phi_{(U^{n_0},\mathcal{H}^{n_0})})$ where $B_n=:\{\sup_{t,s}|(\mathcal{K}_t^n)(\dot\chi_{x_0}(s))|+\sup_{t,s}|(\mathcal{K}_t^n)(\dot\chi_\blacktriangle(s))|\}$ is bounded for each n. That is, the quantity $\max_t(osc(\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^n,\mathcal{H}^n)})-\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})})))$ tends to zero when n goes at infinite. Similarly, one derives that each of the quantities $\max_t(osc(\Delta_t(\mathcal{K}^{n+1},\phi^{-1}_{(U^n,\mathcal{H}^n)})-\Delta_t(\mathcal{K}^{1+n},\phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})})))$ and $\max_t(osc(\Delta_t(\mathcal{K}^{n+1},\phi^{-1}_{(U^n,\mathcal{H}^n)})-\Delta_t(\mathcal{K}^{1+n},\phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})})))$ tends to zero when n goes at infinite. Therefore,

$$\max_t(osc(\Delta_t(\mathcal{K}^n,\phi^{-1}_{(U^n,\mathcal{H}^n)})-\Delta_t(\mathcal{K}^m,\phi^{-1}_{(U^m,\mathcal{H}^m)})))\to 0, n,m\to\infty.$$

- Stage (b) : We have,

$$\max_t(osc(V_t^n\circ\phi^{-1}_{(U^n,\mathcal{H}^n)}-V_t^{n+1}\circ\phi^{-1}_{(U^{n_0},\mathcal{H}^{n_0})}))$$
$$\leq\max_t(osc(V_t^n\circ\phi^{-1}_{(U^n,\mathcal{H}^n)}-V_t^n\circ\phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})}))$$
$$+\max_t(osc(V_t^{n+1}\circ\phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})}-V_t^n\circ\phi^{-1}_{(U^{n+1},\mathcal{H}^{n+1})})).$$

Using simultaneously the uniform continuity of the function $z\mapsto V_t^n(z)$ with the fact that the sequence $\phi^{-1}_{(U^n,\mathcal{H}^n)}$ is $\bar{d}-$Cauchy, we conclude that the hand right side of the above estimate tends to zero when n goes at infinite.

We have proved that

$$D_0^\infty((V^n,\mathcal{K}^n)\bowtie(U^n,\mathcal{H}^n),(V^{n+1},\mathcal{K}^{n+1})\bowtie(U^{n+1},\mathcal{H}^{n+1}))\to 0, n\to\infty.$$

Its remains to check that

$$D_0^\infty(\overline{(V^n,\mathcal{K}^n)}\bowtie\overline{(U^n,\mathcal{H}^n)},\overline{(V^{n+1},\mathcal{K}^{n+1})}\bowtie\overline{(U^{n+1},\mathcal{H}^{n+1})})\to 0, n\to\infty,$$

to do that, it suffices to check that each of the following identities tends to zero when n goes at infinite :

$$\max_t(osc(\Delta_t(-\mathcal{K}^n,\phi_{(V^n,\mathcal{K}^n)})-\Delta_t(-\mathcal{K}^{n+1},\phi_{(V^{n+1},\mathcal{K}^{n+1})}))) \tag{3.9}$$

$$\max_t(osc(\Delta_t(-\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)})\circ\phi_{(V^n,\mathcal{K}^n)}-\Delta_t(-\mathcal{H}^{n+1},\phi_{(U^{n+1},\mathcal{H}^{n+1})})\circ\phi_{(V^{n+1},\mathcal{K}^{n+1})})) \tag{3.10}$$

$$\max_t(osc(\Delta_t(-\mathcal{H}^n,\phi_{(V^n,\mathcal{K}^n)})-\Delta_t(-\mathcal{H}^{n+1},\phi_{(V^{n+1},\mathcal{K}^{n+1})}))). \tag{3.11}$$

Remark that the identities (3.9) and (3.11) are obvious, since one obtains their proofs by using the same arguments as in the stage (a). For identity (3.10), observe that

$$\max_t(osc(\Delta_t(-\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)})\circ\phi_{(V^n,\mathcal{K}^n)}-\Delta_t(-\mathcal{H}^{n+1},\phi_{(U^{n+1},\mathcal{H}^{n+1})})\circ\phi_{(V^{n+1},\mathcal{K}^{n+1})}))$$
$$\leq\max_t(osc(\Delta_t(-\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)})\circ\phi_{(V^n,\mathcal{K}^n)}-\Delta_t(-\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)})\circ\phi_{(V^{n_0},\mathcal{K}^{n_0})}))$$
$$+$$
$$\max_t(osc(\Delta_t(-\mathcal{H}^n,\phi_{(U^n,\mathcal{H}^n)})\circ\phi_{(V^{n_0},\mathcal{K}^{n_0})}-\Delta_t(-\mathcal{H}^{n+1},\phi_{(U^{n+1},\mathcal{H}^{n+1})})\circ\phi_{(V^{n_0},\mathcal{K}^{n_0})}))$$
$$+$$
$$\max_t(osc(\Delta_t(-\mathcal{H}^{n+1},\phi_{(U^{n+1},\mathcal{H}^{n+1})})\circ\phi_{(V^{n_0},\mathcal{K}^{n_0})}-\Delta_t(-\mathcal{H}^{n+1},\phi_{(U^{n+1},\mathcal{H}^{n+1})})\circ\phi_{(V^{n+1},\mathcal{K}^{n+1})})).$$

As a consequence of the arguments stated in stage (a), it falls out that the hand right side of the above estimates tend to zero when n goes at infinite. One uses similar arguments to prove that the sequence $\overline{(U^n,\mathcal{H}^n)}$ is D^2-Cauchy. This achieves our proof.□

The $L^{(1,\infty)}-$version of Lemma 3.1 was proved in [6]. Indeed, given two Cauchy sequences $((U^n,\mathcal{H}^n))_n$ and $((V^n,\mathcal{K}^n))_n$ in the metric D^1, it was proved in [6] that if the sequences of paths $\phi_{(U^n,\mathcal{H}^n)}$ and $\phi_{(V^n,\mathcal{K}^n)}$ are Cauchy in the metric $\bar{d}$, then the sequences $((U^n,\mathcal{H}^n)\bowtie(V^n,\mathcal{K}^n))_n$ and $(\overline{(U^n,\mathcal{H}^n)})_n$ are D^1-Cauchy.

Lemma 3.2. *Assume that $((0,\mathcal{H}^n))_n$ is D^2-Cauchy. Then, $(\phi_{(0,\mathcal{H}^n)})_n$ is $\bar{d}-$Cauchy.*

Proof. The space $harm^1(M,g)$ of smooth harmonic $1-$forms of any compact manifold is a complete vector space, hence by the compactness of $[0,1]$, the set of continuous maps from $[0,1]$ onto $harm^1(M,g)$ is a complete vector space for the following norm $\|(\mathcal{H}_t)_t\|_\infty =: \sup_t|\mathcal{H}_t|$. Since $((0,\mathcal{H}^n))_n$ is D^2-Cauchy, then $(\mathcal{H}^n)$ converges uniformly to $\mathcal{K}=(\mathcal{K}_t)$, and $\mathcal{K}$ being a continuous family of smooth harmonic $1-$forms. Consider the isomorphism ϖ induced by the symplectic form, defined from the tangent bundle onto the cotangent bundle (i.e $\varpi: TM\to T^*M$) and define $X_n^t =: \varpi^{-1}(\mathcal{H}_t^n)$, $X^t =: \varpi^{-1}(\mathcal{K}_t)$ for all n, for all t. By assumption, the sequence of smooth $1-$ parameter family of smooth vector fields $(X_n^t)_n$ converges uniformly to a continuous $1-$ parameter family of smooth vector fields (X^t). Hence, due to R. Abraham and J. Robbin [1], the sequence of flows generated by the sequence of vector fields (X_n^t) converges uniformly to a continuous family $\Phi: t\mapsto\phi^t$ of smooth diffeomorphisms. That is, $\bar{d}(\phi_{(0,\mathcal{H}^n)},\Phi)\to 0, n\to\infty$.□

4. Symplectic homeomorphisms

According to Oh and Müller ([15]), the automorphism group of the C^0 symplectic topology is the closure of the group $Symp(M,\omega)$ in the group $Homeo(M)$ of homeomorphism of M endowed with the C^0 topology. That group, denoted $Sympeo(M,\omega)$ has been called group of symplectic homeomorphisms :
$Sympeo(M,\omega) = \overline{Symp(M,\omega)} \subset Homeo(M)$. This definition has been motivated by the following celebrated rigidity theorem of Eliashberg [10] and Gromov [12]. Let $Sympeo_0(M,\omega)$ denotes the identity component in the group $Sympeo(M,\omega)$ endowed with the subspace topology. Proposition 2.1.4 of Müller's thesis shows that any symplectic homeomorphism preserves the Liouville measure μ_0 induced by the volume form $\frac{1}{n!}\omega^n$, or equivalently $Sympeo_0(M,\omega) \subset Homeo_0(M,\mu_0)$.

Definition 4.1. *(Oh-Müller [14, 15]) A homeomorphism h is called Hamiltonian homeomorphisms in the $L^{(1,\infty)}$ context (resp. in the L^∞ context) if*

(1) *there exists $\lambda \in \mathcal{P}(Homeo(M), id)$ with $\lambda(1) = h$,*
(2) *there exists a Cauchy sequence $(U^n, 0)$ for the metric D^1 (resp. D^2) such that $\phi_{(U^n,0)}$ converges $\bar{d}$ to λ.*

Denote by $Hameo(M,\omega)$ the set of Hamiltonian homeomorphisms of all closed connected symplectic manifold (M,ω) [14].

Definition 4.2. *(Banyaga, [5, 6]) A homeomorphism h is called strong symplectic homeomorphism in the $L^{(1,\infty)}-$context if there exists a D^1-Cauchy sequence $((U^i,\mathcal{H}^i))_i \subseteq \mathfrak{T}(M,\omega,g)$, generating a sequence of symplectic isotopies $\phi_{(U^i,\mathcal{H}^i)}$ such that $\bar{d}(\phi_{(U^i,\mathcal{H}^i)}, \phi_{(U^j,\mathcal{H}^j)}) \to 0, i, j \to \infty$ and $\phi^1_{(U^i,\mathcal{H}^i)} \xrightarrow{C^0} h$.*

Definition 4.3. *(Banyaga, [5, 6]) A homeomorphism h is called strong symplectic homeomorphism in the L^∞-context if there exists a D^2-Cauchy sequence $((V^i,\mathcal{K}^i))_i \subseteq \mathfrak{T}(M,\omega,g)$, generating a sequence of symplectic isotopies $\phi_{(V^i,\mathcal{K}^i)}$ such that $\bar{d}(\psi_{(V^i,\mathcal{K}^i)}, \psi_{(V^j,\mathcal{K}^j)}) \to 0, i, j \to \infty$ and $\psi^1_{(V^i,\mathcal{K}^i)} \xrightarrow{C^0} h$.*

In [5, 6], the author denoted by $SSympeo(M,\omega)^{(1,\infty)}$ the group of strong symplectic homeomorphisms in the $L^{(1,\infty)}-$context and denoted by $SSympeo(M,\omega)^\infty$ the group of strong symplectic homeomorphisms in the $L^\infty-$context. The following theorem is the main result of [7].

Theorem 4.1. *(Banyaga-Tchuiaga [7]) For any closed connected symplectic manifold (M,ω), the following equality holds*

$$SSympeo(M,\omega)^\infty = SSympeo(M,\omega)^{(1,\infty)}.$$

In regard of theorem 4.1, for short, we will denote both sets of strong symplectic homeomorphisms by the same notation $SSympeo(M,\omega)$. As we will see later, theorem 4.1 will play an important role in the description of the structure of strong symplectic homeomorphisms. The group $SSympeo(M,\omega)$ is the topological analogue of the identity component of symplectic diffeomorphisms' group endows with the C^∞ compact-open topology.

5. Continuous symplectic flows

It is well known that on any compact symplectic manifold, any symplectic isotopy can be decomposed in a unique way as product of a harmonic isotopy by a Hamiltonian isotopy [4,5,7]. Then, combining Lemma 3.2 with Hodge's decomposition theorem of symplectic isotopies, it seems natural that one can exhibit the C^0 analog of Hodge's decomposition theorem of symplectic isotopies (without uniqueness). That is the goal of this section.

Definition 5.1. *(Continuous symplectic flow)*
A continuous map $\xi : [0,1] \to Sympeo_0(M,\omega)$ with $\xi(0) = id$ will be called continuous symplectic flow in the $L^\infty-$context (or $L^\infty-$ssympeotopy) if there exists a sequence $((F_n, \lambda_n))_n \subset \mathfrak{T}(M,\omega,g)$, generating a sequence of symplectic isotopies $\phi_{(F_n,\lambda_n)}$ such that :

(1) $\bar{d}(\phi_{(F_n,\lambda_n)}, \xi) \to 0, n \to \infty$,
(2) $D^2((F_n,\lambda_n),(F_m,\lambda_m)) \to 0, n, m \to \infty$.

We denote by $\mathcal{P}SSympeo(M,\omega)$, the space of $L^\infty-$ssympeotopies of a closed connected symplectic manifold (M,ω). Obviously, the notion of strong symplectic isotopies or continuous symplectic flows generalizes the notion of symplectic isotopies as well as the notion of continuous Hamiltonian flows introduced in [9,16].

Proposition 5.1. *$\mathcal{P}SSympeo(M,\omega)$ is a group for paths composition.*

The proof of Proposition 5.1 is a direct consequence of the proof of main theorem [6]. That consequence is given in the present work by Lemma 3.1.

Remark 5.1. *$\mathcal{P}SSympeo(M,\omega)$ contains $\mathcal{P}Hameo(M,\omega)$ as a subgroup, and if the manifold is simply connected then $\mathcal{P}Hameo(M,\omega) = \mathcal{P}SSympeo(M,\omega)$.*

Definition 5.2. *([9,15,16])*
A continuous symplectic flow ν is called continuous Hamiltonian flow if there exists a D^2 Cauchy sequence $((U_n,0))_n \subset \mathfrak{T}(M,\omega,g)$ which generates a sequence of Hamiltonian isotopies $(\phi_{(U_n,0)})_n$ that converges $\bar{d}$ to ν.

Let $\mathcal{P}Hameo(M,\omega)$ denotes the space of continuous Hamiltonian flows. The notion of continuous Hamiltonian flows or Hameotopies was studied in [9,15,16]. This is the topological analogue of Hamiltonian isotopies group.

Definition 5.3. *A continuous symplectic flow μ is called topological harmonic flows or harmeotopy if there exists a D^2-Cauchy sequence $((0,\mathcal{H}^n))_n \subset \mathfrak{T}(M,\omega,g)$ which generates a sequence of harmonic isotopies $\phi_{(0,\mathcal{H}^n)}$ that converges $\bar{d}$ to μ.*

We denote by $\mathcal{P}Harm(M,\omega)$ the space of topological harmonic flows. The set of topological harmonic flows is the topological analogue of space of harmonic isotopies [4,5]. We will see later that this set has a remarkable contribution in the description of some structure of the group of strong symplectic homeomorphisms. One can check that the space $\mathcal{P}Hameo(M,\omega)$ is a subgroup of $\mathcal{P}SSympeo(M,\omega)$, while the space $\mathcal{P}Harm(M,\omega)$ is not a subgroup of $\mathcal{P}SSympeo(M,\omega)$. A justification of the latter statements lies in the fact that in smooth case, the set of Hamiltonian isotopies is closed under path composition in the group of symplectic isotopies, while the set of harmonic isotopies is not.
The set $\mathcal{P}Harm(M,\omega)$ generalizes the well known set of harmonic isotopies [4].

5.1. **Proof of Theorem 1.1.** Let $\lambda \in \mathcal{P}SSympeo(M,\omega)$. Our job here is to find $\nu \in \mathcal{P}Harm(M,\omega)$ and $\mu \in \mathcal{P}Hameo(M,\omega)$ such that $\lambda = \nu \circ \mu$. By definition of λ, there exists a sequence $((U^n, \mathcal{H}^n))_n \subset \mathfrak{T}(M,\omega,g)$ such that $\bar{d}(\phi_{(U^n,\mathcal{H}^n)}, \lambda) \to 0, n \to \infty$ and $D^2((U^n,\mathcal{H}^n),(U^m,\mathcal{H}^m)) \to 0, n, m \to \infty$. It falls from the definition of group isomorphism $\mathfrak{A}$ that Hodge decomposition of symplectic isotopies induces a decomposition on the space $\mathfrak{T}(M,\omega,g)$ such that for each fixed integer n, one can decompose $(U^n,\mathcal{H}^n)$ as follows : $(U^n,\mathcal{H}^n) = (0,\mathcal{H}^n) \bowtie (U^n \circ \phi_{(0,\mathcal{H}^n)}, 0)$. Lemma 3.2 implies that the sequence $\phi_{(0,\mathcal{H}^n)}$ is $\bar{d}$ converging because by assumption, the sequence $(0,\mathcal{H}^n)$ is D^2-Cauchy. Hence, we derive from the above that the sequence $\phi_{(U^n \circ \phi_{(0,\mathcal{H}^n)},0)}$ is $\bar{d}$ converging as the composition of $\bar{d}$ converging sequences. To achieve our proof, it remains to prove that the sequence $(U^n \circ \phi_{(0,\mathcal{H}^n)}, 0)$ is D^2-Cauchy i.e $\max_t(osc(U^n_t \circ \phi^t_{(0,\mathcal{H}^n)} - U^m_t \circ \phi^t_{(0,\mathcal{H}^m)})) \to 0, n, m \to \infty$. To do that, remark that for each t we have

$$osc(U^n_t \circ \phi^t_{(0,\mathcal{H}^n)} - U^m_t \circ \phi^t_{(0,\mathcal{H}^m)}) \leq osc(U^n_t \circ \phi^t_{(0,\mathcal{H}^n)} - U^n_t \circ \phi^t_{(0,\mathcal{H}^m)})$$
$$+osc(U^n_t \circ \phi_{(0,\mathcal{H}^m)} - U^m_t \circ \phi_{(0,\mathcal{H}^m)}).$$

By assumption,

$$osc(U^n_t \circ \phi_{(0,\mathcal{H}^m)} - U^m_t \circ \phi_{(0,\mathcal{H}^m)}) = \max_t(osc(U^n_t - U^m_t)) \to 0, n, m \to \infty,$$

while the uniform continuity of the mapping $x \mapsto U^n_t(x)$ combined with the fact that the sequence $\phi_{(0,\mathcal{H}^n)}$ is $\bar{d}-$Cauchy imply that $osc(U^n_t \circ \phi^t_{(0,\mathcal{H}^n)} - U^n_t \circ \phi^t_{(0,\mathcal{H}^m)})$ can be considered arbitrarily small as we want when both integers n, m tend to infinite. That is, $\max_t(osc(U^n_t \circ \phi^t_{(0,\mathcal{H}^n)} - U^m_t \circ \phi^t_{(0,\mathcal{H}^m)})) \to 0, n, m \to \infty$. Therefore, we get $\mathcal{P}SSympeo(M,\omega) \subset \mathcal{P}Har(M,\omega) \circ \mathcal{P}Hameo(M,\omega)$. It is easy to see that $\mathcal{P}Harm(M,\omega) \circ \mathcal{P}Hameo(M,\omega) \subset \mathcal{P}SSympeo(M,\omega)$. This achieves our proof.□

Remark 5.2. *Theorem 1.1 and Lemma 3.2 show some advantages of the L^∞ symplectic topology over the $L^{(1,\infty)}$ symplectic topology.*

Consider the surjective group morphism

$$ev_1 : \mathcal{P}SSympeo(M,\omega) \to SSympeo(M,\omega),$$
$$\xi \mapsto \xi(1),$$

and define $harmeo(M,\omega) =: ev_1(\mathcal{P}Harm(M,\omega))$.

Corollary 5.1. *Let (M,ω) be a closed connected symplectic manifold. Then*

$$SSympeo(M,\omega) = harmeo(M,\omega) \circ Hameo(M,\omega).$$

The proof of Corollary 5.1 is an immediate consequence of Theorem 1.1 because the time-one evaluation map is a group morphism from the space $\mathcal{P}SSympeo(M,\omega)$ onto the space $SSympeo(M,\omega)$.

Lemma 5.1. *Let (M,ω) be a closed connected symplectic manifold. Let $\nu \in \mathcal{P}Harm(M,\omega)$. For each t, $\nu(t)$ is a symplectic diffeomorphism.*

Lemma 5.1 states that a long of any topological harmonic flow, one can find only the symplectomorphisms. In regard of the above facts, the following questions make sense.

Questions:

- Is $\mathcal{P}SSympeo(M,\omega)$ strictly contains $\mathcal{P}Hameo(M,\omega)$ as a subgroup?
- Is $\mathcal{P}Hameo(M,\omega)$ normal to $\mathcal{P}SSympeo(M,\omega)$?
- Is $\mathcal{P}Harm(M,\omega)$ strictly contains the space of all harmonic isotopies of (M,ω) ?
- Is $\mathcal{P}Hameo(M,\omega) \cap \mathcal{P}Harm(M,\omega) = \{Id\}$?

Proof of Lemma5.1: By definition of ν, there exists a sequence $((0,\mathcal{H}^n))_n \subset \mathfrak{T}(M,\omega,g)$ such that : $\bar{d}(\phi_{(0,\mathcal{H}^n)},\nu) \to 0, n \to \infty$ and $D^2((0,\mathcal{H}^n),(0,\mathcal{H}^m)) \to 0, n,m \to \infty$. Then, the proof of Lemma 3.2 implies that the sequence of harmonic flows $\phi_{(0,\mathcal{H}^n)}$ converges uniformly to a continuous family of smooth diffeomorphisms that we denote it here by (ϕ_t). Hence, the second Hausdorff axiom implies that $\nu(t) = \phi_t$ for all t. On the another hand, since ν is a continuous family of smooth diffeomorphisms which is a C^0 limit of a sequence of smooth families of symplectic diffeomorphisms, we derive from the celebrated rigidity theorem of Eliashberg [10] and Gromov [12] that $\phi_t \in Symp(M,\omega)$ for all t. $\square$

5.2. **Proof of theorem 1.2.** The proof of Theorem 1.2 is a direct consequence of Corollary 5.1. It suffices to pick any non-trivial harmonic diffeomorphism ρ and any non-differentiable Hamiltonian homeomorphism ν. Then, Corollary 5.1 implies that the product $\rho\circ\nu$ lies inside the group $SSympeo(M,\omega)^{\infty}$. The product $\rho\circ\nu$ is continuous and non-differentiable. Hence, Fathi's mass flow of such product reduces to Fathi's mass flow of ρ because in [15], Oh and Müller showed that the group $Hameo(M,\omega)$ is contained in the kernel of Fathi's mass flow. The latter statement implies that Fathi's mass flow of the product $\rho\circ\nu$ reduces to Fathi's mass flow of ρ. In [3], Banyaga showed that the flux of ρ is non-trivial i.e the Fathi mass flow of ρ is non-trivial by duality since it is showed in [11] that the flux for volume-preserving diffeomorphisms is the Poincaré dual of Fathi's mass flow. This obviously implies that the inclusion of $Hameo(M,\omega)$ in $SSympeo(M,\omega)$ is strict.$\square$

Theorem 1.2 agrees with a result found by Müller, which states that when $\mathbb{H}^1(M,\mathbb{R}) \neq \{0\}$, then $Hameo(M,\omega)$ is a proper subgroup in the identity component of $Sympeo(M,\omega)$, see Theorem 2.5.3 of Müller thesis for more details. From the above, we derive the following inclusions :

$$Ham(M,\omega) \subset Hameo(M,\omega) \subsetneq SSympeo(M,\omega) \subset Sympeo_0(M,\omega).$$

5.3. **Examples of continuous ssympeomorphism with non-trivial Fathi's mass flow on $\mathbb{T}^2$.** In this section, we construct a non-trivial ssympeomorphism on any closed connected symplectic manifold (M,ω) which is continuous, not differentiable and admits a non-trivial Fathi's mass flow. In particular, any harmonic diffeomorphism defines a smooth ssympeomorphism whose Fathi's mass flow is non-trivial [2,3,17]. We recall that according to Müller, considering any $2n$ dimensional symplectic manifold (M,ω), by choosing $\mathbb{D}^2(\epsilon)\times\ldots\mathbb{D}^2(\epsilon) = D^{2n}(\epsilon)$ inside the domain of some Darboux chart in M one can construct a Hamiltonian homeomorphism on M which is not differentiable. Assume this done, and denote by ψ such Hamiltonian homeomorphism on (M,ω). When $\mathbb{H}^1(M,\mathbb{R}) \neq 0$, it follows from Hodge's theory that there exists at least a non-trivial harmonic $1-$form $\mathcal{H}$ on M. The flow $\phi_{(0,\mathcal{H})}$ generated by $(0,\mathcal{H})$ is a harmonic isotopy which is different from the constant path Id, or equivalently there exists $t_0 \in]0,1]$ such that $\phi^{t_0}_{(0,\mathcal{H})} \neq id$. We then derive from Corollary 5.1 that $\sigma = \phi^{t_0}_{(0,\mathcal{H})}\circ\psi \in SSympeo(M,\omega)$. Constructively, σ is continuous and not differentiable. Furthermore, σ is not an element of $Hameo(M,\omega)$ because its Fathi's mass flow reduces to Fathi's mass flow of $\phi^{t_0}_{(0,\mathcal{H})}$, and the Fathi mass flow of $\phi^{t_0}_{(0,\mathcal{H})}$ is non-trivial because Banyaga showed that the symplectic flux of $\phi^{t_0}_{(0,\mathcal{H})}$ is non-trivial [3]. For instance, for $M = \mathbb{T}^2$ where $\mathbb{T}^2$ is the two dimensional torus. The identification $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ provides the space $\mathbb{T}^2$ with a natural symplectic structure obtained by projecting the symplectic form w_0 through the map π from $\mathbb{R}^2$ to T^2 through the canonical projection $\pi : \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2$. We denote by $\tilde{\omega}_0$ the symplectic form of T^2. According to Müller the closed connected symplectic manifold $(\mathbb{T}^2, \tilde{\omega}_0)$ admits at least a Hamiltonian homeomorphism which is not differentiable. Let us denote by ψ such Hamiltonian homeomorphism. Fix $u = (a, b) \in \mathbb{T}^2$ such that $u \neq (0, 0)$ and consider the translation

$$R_u : \mathbb{T}^2 \to \mathbb{T}^2,$$

$$(x, y) \mapsto (x + a, y + b).$$

We have $R_u \circ \psi \in SSympeo(\mathbb{T}^2, \tilde{\omega}_0)$ and $R_u \circ \psi$ is continuous but not differentiable. Moreover, the Fathi mass flow of $R_u \circ \psi$ is non-trivial.

Acknowledgments: I would like to thank Augustin Banyaga for advices and improvements of the preliminary drafts of this paper. I thanks the Germany Office of University Exchanges (DAAD) for financial support of my research at the Institute of Mathematics and Physical Sciences.

References

[1] R. Abraham and J. Robbin, : *Transversal mappings and flow,* W. A. Benjamin. Inc., New York-Amsterdam, **39**(1967).

[2] A. Banyaga, : *Sur la structure de difféomorphismes qui préservent une forme symplectique,* comment. Math. Helv. **53** (1978) pp 174 − 2227.

[3] A. Banyaga, : *The Structure of classical diffeomorphisms groups,* Mathmatics and its applications vol 400. Kluwer Academic Publisher's Goup, Dordrescht, The Netherlands (1997).

[4] A. Banyaga, : *A Hofer-like metric on the group of symplectic diffeomorphisms,* Contemp. Math. Amer. Math. Soc. RI. Vol 512 (2010) pp 1 − 24.

[5] A. Banyaga, : *On the group of strong symplectic homeomorphisms,* Cubo, A Mathematical Journal. Vol 12 **3**(2010) pp 49 − 69.

[6] A. Banyaga, : *Erratum To : On the group of strong symplectic homeomorphisms,* Cubo. A Mathematical Journal. Vol 14 **3**(2012) pp 57 − 59.

[7] A. Banyaga and S. Tchuiaga, : *The group of strong symplectic homeomorphisms in L^∞-norm,* Accepted in Advance in Geometry. To appear.

[8] T. Brocker and K. Janich, : *Introduction to differential topology,* Cambridge University Press (1982).

[9] L. Buhovsky and S. Seyfaddini, : *Uniqueness of generating Hamiltonian for continuous Hamiltonian flows,* Arxiv (2010).

[10] Y. Eliashberg, : *A theorem on the structure of wave fronts and its application in symplectic topology,* Funct. Anal. and Its Applications(1987)21 227 − 232.

[11] A. Fathi, : *Structure of the group of homeomorphisms preserving a good measure on a compact manifold,* Ann. Scient. Ec. Norm. Sup., **13**(1980) 45 − 93.

[12] M. Gromov, : *Partial differential relations,* Ergebnisse der Mathematik, Springer (1986).

[13] F. Lalonde and D. McDuff, : *The geometry of symplectic energy,* Ann. of Math. **141**(1995) 711 − 727.

[14] S. Müller, : *The group of Hamiltonian homeomorphism in the L^∞-norm,* T. Korean Math. Soc. **45**(2008), 1769-1784.

[15] Y-G. Oh and S. Müller, : *The group of hamiltonian homeomorphisms and C^0-symplectic topology,* J. Symp. Geometry **5**(2007) 167 − 225.

[16] C, Viterbo, : *On the uniqueness of generating Hamiltonian for continuous limits of Hamiltonian flows,* Internat. Math. Res. Notices, vol (2006), Article ID 34028, 9 pages. Erratum, ibid Vol 2006 article ID 38784, 4 pages.

[17] F. Warner, : *Foundation of differentiable manifolds and Lie groups,* Scotl., Foresman and Co., London, (1971).

INSTITUT DE MATHÉMATIQUES,
ET DE SCIENCES PHYSIQUES
IMSP/PORTO-NOVO.
E-mail address: tchuiagas@gmail.com, tchuiaga.kameni@imsp-uac.org