ON PARALLEL SURFACES OF WEINGARTEN TUBULAR SURFACE IN EUCLIDEAN 3-SPACE

SEZAİ KIZILTUĞ AND ÖMER TARAKCI

ABSTRACT. In this paper, we study the parallel surface of tubular surface in a Euclidean 3-space. Besides, we get parametric expression of the parallel surface of tubular surface in Euclidean 3-space satisfying a linear type and a quadric type with respect to the Gaussian curvature and the mean curvature.

AMS Subject Classification: 53A05.

Key Words and Phrases: Parallel surfaces, Weingarten property, Euclidean space.

1. INTRODUCTION

In the study of the differential geometry of surfaces, it is common to determine some surfaces satisfying curvature conditions. An interesting curvature property to study for surfaces in a Euclidean 3-space is the one that requires the existence of a non-trivial functional relationship between the principal curvatures. The resulting surfaces are called Weingarten surfaces. In particular, a surface $M$ in a Euclidean 3-space is called Weingarten surface if there is some (smooth) relation $U(\kappa_1, \kappa_2) = 0$ between its two principal curvatures $\kappa_1$ and $\kappa_2$, or equivalently, if there exists a non-trivial functional relation $\Phi(K, H) = 0$ with respect to its Gaussian curvature $K$ and its mean curvature $H$. The existence of a non-trivial functional relation $\Phi(K, H) = 0$ on the surface $M$ parametrized by $X(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0$. Also, if a surface satisfies a linear equation with respect to $K$ and $H$, that is, $aK + bH = c \ (a, b, c) = (0, 0, 0), a, b, c \in \mathbb{R}$ then it is said to be a linear Weingarten surface or LW-surface ([3]). First examples of LW-surfaces are the surface with constant mean curvature ($a = 0$) and the surfaces with constant Gaussian curvature ($b = 0$). Although these two kinds of surfaces have been extensively studied in the literature, the classification of LW-surface in the general case is almost completely open today. Besides several geometers ([2,3,7,11,12]) have studied Weingarten surfaces and LW-surface and obtained many interesting result. Recently N. G Kim and D. W Yoon ([5]) studied the ruled surfaces in a Euclidean 3-space satisfying the quadric type with respect to the Gaussian curvature, the mean curvature and the second mean curvature. the second to the Gaussian curvature is the Gaussian curvature of non-degenerate second fundamental form of a surface. Also Y. Tunçer and D. W. Yoon and M. K. Karacan studied Weingarten and linear Weingarten type tubular surfaces in a Euclidean 3-space ([7]). In this paper, we will study the parallel surface of tubular surface in in a Euclidean 3-space satisfying the conditions:


\[ aK + bH = c \quad a, b \neq 0, \quad (1.1) \]

\[ aK^2 + bKH + cH^2 = d, \quad b^2 - 4ac > 0, \quad (1.2) \]

where \( a, b, c, d \) are constants, and \( K, H \) are the Gaussian curvature and the mean curvature of the parallel surface of tubular surface. If a surface satisfies the equation (1.2), the surface is said to be \( KH \)-quadric.

2. Preliminaries

A surface \( M \) whose points are at a constant distance along the normal from another surface \( M \) is said to be parallel to \( M \). So, let \( M \) be a surface in a Euclidean 3-space with unit normal vector \( N \), for any constant \( r \in \mathbb{R} \), let

\[ \overline{M} = \{ f(p) = p + rN_p : p \in M \}. \]

Thus if \( p \) is on \( M \), then \( f(p) = p + rN_p \) defines a new surface \( \overline{M} \). The map \( f \) is called the natural map on \( M \) into \( \overline{M} \) and if \( f \) is univalent, then \( \overline{M} \) is a parallel surface of \( M \) with unit normal vector \( N, N_f(p) = N_p \) for all \( p \) in \( M \).

Therefore the parametrization for \( \overline{M} \) is given by

\[ \overline{M}(u, v) = M(u, v) + \lambda N(u, v) \quad (2.1) \]

where \( \lambda \) is a constant scalar and \( N \) is the unit normal vector field on \( M \). Let \( I, II, K, H \) be the first fundamental, the second fundamental form, the Gaussian curvature and the mean curvature of \( M \), respectively, and let \( \overline{I}, \overline{II}, \overline{K}, \overline{H} \) be the corresponding ones for \( \overline{M} \).

With the parametrization for a parallel surface, the following proposition holds.

**Proposition 2.1.** ([6]) Let \( M \) be a parallel surface of a surface \( M \) in a Euclidean 3-space. Then we have

1. \( \overline{I} = (1 - \lambda^2 K)I - 2\lambda(1 - \lambda H)II \)
2. \( \overline{II} = \lambda KI + (1 - 2\lambda H)II \)
3. \( \overline{K} = \frac{K}{1 - 2\lambda H - \lambda^2 K} \)
4. \( \overline{H} = \frac{H - \lambda K}{1 - 2\lambda H + \lambda^2 K} \)

From proposition (2.1), differentiating \( \overline{K} \) and \( \overline{H} \) with respect to \( u \) and \( v \) respectively, we get

\[ (\overline{K})_u = \frac{K_u - 2\lambda K_u H + 2\lambda K H_u}{(1 - 2\lambda H + \lambda^2 K)^2}, \quad (2.2) \]

\[ (\overline{K})_v = \frac{K_v - 2\lambda K_v H + 2\lambda K H_v}{(1 - 2\lambda H + \lambda^2 K)^2}, \quad (2.3) \]

\[ (\overline{H})_u = \frac{H_u - \lambda^2 K H_u - \lambda K_u + \lambda^2 H K_u}{(1 - 2\lambda H + \lambda^2 K)^2}, \quad (2.4) \]

\[ (\overline{H})_v = \frac{H_v - \lambda^2 K H_v - \lambda K_v + \lambda^2 H K_v}{(1 - 2\lambda H + \lambda^2 K)^2}. \quad (2.5) \]
Theorem 2.2. Let $\overline{M}$ be a parallel surface of a surface $M$ in a Euclidean 3-space. If $\overline{M}$ is a Weingarten surface if and only if $M$ is a Weingarten surface.

Proof. Which imply the jacobian function of the Gaussian curvature $K$ and the mean curvature $H$ of the parallel surface $M$ is given

$$\Phi(K, H) = \det \left[ \begin{array}{cc}
\frac{K - 2\lambda K H + 2\lambda^2 H_k}{(1-2\lambda H + \lambda^2 K)^2} & \frac{K - 2\lambda K H + 2\lambda^2 H_k}{(1-2\lambda H + \lambda^2 K)^2} \\
\frac{H - 2\lambda K H - \lambda K + \lambda^2 K H}{(1-2\lambda H + \lambda^2 K)^2} & \frac{H - 2\lambda K H - \lambda K + \lambda^2 K H}{(1-2\lambda H + \lambda^2 K)^2}
\end{array} \right]. \quad (2.6)$$

From (2.6), we get

$$\Phi(K, H) = \frac{K_u H_v - K_v H_u}{1 - 2\lambda H + \lambda^2 K}. \quad (2.7)$$

In a simple form, equation (2.7) becomes

$$\Phi(K, H) = \Phi(K, H) \left(1 - 2\lambda H + \lambda^2 K\right).$$

This completes the proof. □

3. Tubular Surface In Euclidean 3-Space

Definition 3.1. Let $\alpha : [a, b] \rightarrow E^3$ be a unit-speed curve. A tubular surface of radius $r > 0$ about $\alpha$ is the surface with parametrization

$$M(u, v) = \alpha(u) + r \left( e_2(u) \cos v + e_3(u) \sin v \right) \quad (3.1)$$

$a \leq u \leq b$, where $e_2(u)$, $e_3(u)$ are the principal normal and binormal vectors of $\alpha$, respectively ([1]). The curvature and the torsion of the curve $\alpha$ are denoted by $\kappa, \tau$. Then Frenet formula of $\alpha(u)$ is defined by

$$e_1'(u) = \kappa(u) e_2(u), \quad e_2'(u) = -\kappa(u) e_1(u) + \tau(u) e_3(u), \quad e_3'(u) = -\tau(u) e_2(u) \quad (3.2)$$

Besides, the gaussian curvature, mean curvature and the unit surface normal of $M(u, v)$ tubular surface in a Euclidean 3-space are given by (see [1]), respectively

$$K = \frac{-\kappa \cos v}{r\sigma}, \quad (3.3)$$

$$H = \frac{1 - 2r\kappa \cos v}{2r\sigma}, \quad (3.4)$$

$$N = e_2(u) \cos v + e_3(u) \sin v \quad (3.5)$$

where $\sigma = 1 - r\kappa \cos v$.

Theorem 3.2. Suppose that $M$ is linear Weingarten type tubular surface in Euclidean 3-space. Then $M$ is a tubular surface around a circle or a helix ([7]).
4. MAIN RESULTS-THE PARALLEL TUBULAR SURFACE IN $E^3$

In this section, we study the parallel surface $\overline{M}(u,v)$ of a tubular surface in a Euclidean 3-space which satisfies a linear weingarten equation (1.1) and a quadric equation (1.2) with respect to the Gaussian curvature $\overline{K}$ and The mean curvature $\overline{H}$ of the parallel surface $\overline{M}(u,v)$.

**Theorem 4.1.** Let $\overline{M}(u,v)$ be a parallel surface of tubular surface $M(u,v)$ in a Euclidean 3-space. If $\overline{M}(u,v)$ is a linear weingarten surface satisfying $a\overline{K} + b\overline{H} = c$, $a, b \neq 0$, $c \in \mathbb{R}$. Then $\overline{M}(u,v)$ is parametrized by

$$\overline{M}(u,v) = (\alpha(u) + r(e_2(u)\cos v + e_3(u)\sin v) - \frac{br + 2a}{b}(e_2(u)\cos v + e_3(u)\sin v))$$

**Proof.** Let $\overline{M}(u,v)$ be a parallel surface of tubular surface $M(u,v)$ in $E^3$. Then, by the definition of a parallel surface, the parametrization for $\overline{M}(u,v)$ is given by

$$\overline{M}(u,v) = M(u,v) + \lambda N(u,v).$$

Suppose that parallel surface $\overline{M}(u,v)$ in $E^3$ is a linear weingarten surface. Then by (1.1) we have

$$a\overline{K} + b\overline{H} = c.$$  (4.2)

Differentiating $\overline{K}$ and $\overline{H}$ with respect to $v$, we get

$$a\overline{K}_v + b\overline{H}_v = 0.$$  (4.3)

From (2.3) and (2.5), we get

$$a(K_v - 2\lambda K_v H + 2\lambda K H_v) + b(H_v - \lambda^2 K H_v - \lambda K_v + \lambda^2 K H_v) = 0.$$  (4.4)

Differentiating $K$ given by (3.3) and $H$ given by (3.4) with respect to $v$, we obtain

$$K_v = \frac{\kappa \sin v}{r \sigma^2}, \quad H_v = \frac{\kappa \sin v}{2r \sigma}.$$  (4.5)

where $\sigma = 1 - r \kappa \cos v$. By using (4.5), Eq. (4.4) becomes

$$a \left[ \frac{\kappa \sin v}{r \sigma^2} - 2\lambda \left( \frac{\kappa \sin v}{r \sigma^2} \right) \left( \frac{1 - 2r \kappa \cos v}{2r \sigma} \right) + 2\lambda \left( -\frac{\kappa \cos v}{r \sigma} \right) \left( \frac{\kappa \sin v}{2r \sigma^2} \right) \right]$$

$$+ b \left[ \frac{\kappa \sin v}{2r \sigma^2} - \lambda^2 \left( \frac{\kappa \cos v}{r \sigma} \right) \left( \frac{\kappa \sin v}{2r \sigma^2} \right) - \lambda \frac{\kappa \sin v}{r \sigma^2} + \lambda^2 \left( 1 - \frac{2r \kappa \cos v}{2r \sigma} \right) \left( \frac{\kappa \sin v}{r \sigma^2} \right) \right] = 0.$$

In a simple form, above equation becomes

$$\left[ b (\lambda - r)^2 - 2a (\lambda - r) \right] (\sin v) - \left[ b (\lambda - r)^2 - 2a (\lambda - r) \right] (\kappa r \sin v \cos v) = 0.$$  

According to Theorem (3.2), $\kappa = 0$. Thus, we have

$$\left[ b (\lambda - r)^2 - 2a (\lambda - r) \right] (\sin v) = 0$$  

According to the definition of the linear independent of vectors, we find that

$$b (\lambda - r)^2 - 2a (\lambda - r) = 0,$$

from which

$$\lambda = \frac{br + 2a}{b}.$$  

This completes the proof. □

33
Theorem 4.2. Let \( \overline{M}(u, v) \) be a parallel surface of tubular surface \( M(u, v) \) in a Euclidean 3-space and let \( a, b, c, d \) be constants such that \( b^2 - 4ac > 0 \) \( c \neq 0 \). If \( \overline{M}(u, v) \) is a \( K \Pi \)-quadric surface satisfying \( aK^2 + b\Pi + c\Pi^2 = d \). Then \( \overline{M}(u, v) \) is parametrized by

\[
\overline{M}(u, v) = \left( \alpha(u) + r \left( e_2(u) \cos v + e_3(u) \sin v \right) - \frac{cr + b}{c} (e_2(u) \cos v + e_3(u) \sin v) \right)
\]

Proof. The parametrization for \( \overline{M}(u, v) \) is given by

\[
\overline{M}(u, v) = M(u, v) + \lambda N(u, v).
\] (4.6)

Suppose that a parallel surface \( \overline{M}(u, v) \) in \( E^3 \) is \( K\Pi \)-quadric. Then, by (1.2) we have

\[
aK^2 + b\Pi + c\Pi^2 = d.
\] (4.7)

Differentiating \( K \) and \( \Pi \) with respect to \( v \), we get

\[
2aK_vK_v + bK_v\Pi_v + b\Pi_v\Pi_v + 2c\Pi_v\Pi_v = 0.
\] (4.8)

Then by using (2.3) and (2.5), we get

\[
\left[ (H_v - \lambda^2KH_v - \lambda K_v + \lambda^2HK_v) (2c (H - \lambda K) + bK) \right] + \left[ (K_v - 2\lambda K_vH + 2\lambda KH_v) (2aK + b (H - \lambda K)) \right] = 0.
\] (4.9)

Differentiating \( K \) given by (3.3) and \( H \) given by (3.4) with respect to \( v \), we obtain

\[
K_v = \frac{\kappa \sin v}{r \sigma^2}, H_v = \frac{\kappa \sin v}{2 \sigma^2}
\] (4.10)

where \( \sigma = 1 - r \kappa \cos v \). By using (4.10), Eq. (4.9) becomes

\[
\left[ \left( \frac{\lambda^2 - 2\lambda r + r^2}{2r^2 \sigma^3} \left( \kappa \sin v - \kappa^2 r \sin v \cos v \right) \right) \left( \frac{\kappa \cos v (2\lambda c - 2cr - b) + c}{r \sigma} \right) \right] + \left[ \left( \frac{(\lambda - r) (\kappa^2 r \sin v \cos v - \kappa \sin v)}{r^2 \sigma^3} \right) \left( \frac{\kappa \cos v (\lambda b - 2br - 4a) + b}{2r \sigma} \right) \right] = 0.
\]

In a simple form, above equation becomes

\[
-\kappa^2 \sin v \cos v \left[ (\lambda - r)^2 (2\lambda c - 2cr - b) + (\lambda - r) (\lambda b - 2br - 4a) \right] + \kappa \sin v \cos v \left[ (\lambda - r)^2 (2\lambda c - 3cr - b) + (\lambda - r) (\lambda b - 3br - 4a) + \sin v \left( c (\lambda - r)^2 + b (\lambda - r) \right) \right] = 0.
\]

According to Theorem (3.2), \( \kappa = 0 \). Thus we have

\[
\sin v \left[ c (\lambda - r)^2 - b (\lambda - r) \right] = 0.
\]

Coefficient obtained from the last equation must be zero. So, we get

\[
\lambda = \frac{cr + b}{c}.
\]

This completes the proof. 

34
On Parallel Surfaces of Weingarten Tubular Surface In Euclidean 3-Space

REFERENCES


ATATÜRK UNIVERSITY,
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM- TURKEY.
E-mail address: skiziltug24@hotmail.com

ATATÜRK UNIVERSITY,
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM- TURKEY.
E-mail address: tarakci@atauni.edu.tr