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ON PARALLEL SURFACES OF WEINGARTEN TUBULAR SURFACE İN EUCLIDEAN 3-SPACE

SEZAİ KIZILTUĞ AND ÖMER TARAKCI

ABSTRACT. In this paper, we study the parallel surface of tubular surface in a Euclidean 3-space. Besides, we get parametric expression of the parallel surface of tubular surface in Euclidean 3-space satisfying a linear type and a quadric type with respect to the Gaussian curvature and the mean curvature.
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1. INTRODUCTION

In the study of the diffrential geometry of surfaces, it is common to determine some surfaces satisfying curvature conditions. An interesting curvature property to study for surfaces in a Euclidean 3-space is the one that requires the existence of a non-trivial functional relationship between the principal curvatures. The resulting surfaces are called Weingarten surfaces. In particular, a surface M in a Euclidean 3-space is called Weingarten surface if there is some(smooth) relation $U(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2, or equivalently, if there exists a non-trivial functional relation $\Phi(K, H) = 0$ with respect to its Gaussian curvature K and its mean curvature H. The existence of a non-trivial functional relation $\Phi(K, H) = 0$ on the surface M parametrized by $X(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely$\left|\frac{\partial(K,H)}{\partial(u,v)}\right| = 0$. Also, if a surface satisfies a linear equation with respect to K and H , that is, $aK + bH = c$ $(a, b, c) = (0.0.0)$, $a, b, c \in \mathbb{R}$ then it is said to be a linear Weingarten surface or LW-surface ([3]) First examples of LW-surfaces are the surface with constant mean curvature $(a = 0)$ and the surfaces with constant Gaussian curvature $(b = 0)$. Although these two kinds of surfaces have been extensively studied in the literature, the classification of LW-surface in the general case is almost completely open today. Besides several geometers ([2,3,7,11,12]) have studied Weingarten surfaces and LW-surface and obtained many interesting result. Recently N. G Kim and D. W Yoon ([5]) studied the ruled surfaces in a Euclidean 3-space satifying the quadric type with respect to the Gaussian curvature, the mean curvature and the second mean curvature.the second to the Gaussian curvature is the Gaussian curvature of non-degenernate second fundamental form of a surface. Also Y. Tunçer and D. W. Yoon and M. K. Karacan studied Weingarten and linear Weingarten type tubular surfaces in a Euclidean 3-space ([7]). In this paper, we will study the parallel surface of tubular surface in in a Euclidean 3-space satisfying the conditions:

$$a\overline{K} + b\overline{H} = c \;\; a, b \neq 0, \tag{1.1}$$

$$a\overline{K}^2 + b\overline{KH} + c\overline{H}^2 = d, \; b^2 - 4ac > 0, \tag{1.2}$$

where a, b, c, d are constants, and $\overline{K}$, $\overline{H}$ are the Gaussian curvature and the mean curvature of the parallel surface of tubular surface. If a surface satisfies the equation (1.2), the a surface is said to be $\overline{K}\,\overline{H}$-quadric.

2. Preliminaries

A surface $\overline{M}$ whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, let M be a surface in a Euclidean 3-space with unit normal vector N, for any constant $r \in \mathbb{R}$, let

$$\overline{M} = \left\{ f(p) = p + rN_p : \; p \in M \right\}.$$

Thus if p is on M, then $f(p) = p + rN_p$ defines a new surface $\overline{M}$. The map f is called the natural map on M into $\overline{M}$ and if f is univalent, then $\overline{M}$ is a parallel surface of M with unit normal vector N, $N_{f(p)} = N_p$ for all p in M.
Therefore the parametrization for $\overline{M}$ is given by

$$\overline{M}(u, v) = M(u, v) + \lambda N(u, v) \tag{2.1}$$

where λ is a constant scalar and N is the unit normal vector field on M. Let I, II, K, H be the first fundamental, the second fundamental form, the Gaussian curvature and the mean curvature of M, respectively, and let $\overline{I}$, $\overline{II}$, $\overline{K}$, $\overline{H}$ be the corresponding ones for $\overline{M}$. With the parametrization for a parallel surface, the following proposition holds.

Proposition 2.1. ([6]) Let $\overline{M}$ be a parallel surface of a surface M in a Euclidean 3-space. Then we have
(1) $\overline{I} = (1 - \lambda^2 K)I - 2\lambda(1 - \lambda H)II$,
(2) $\overline{II} = \lambda K I + (1 - 2\lambda H)II$,
(3) $\overline{K} = \frac{K}{1-2\lambda H+\lambda^2 K}$,
(4) $\overline{H} = \frac{H-\lambda K}{1-2\lambda H+\lambda^2 K}$.

From proposition (2.1), differentiating $\overline{K}$ and $\overline{H}$ with respect to u and v respectively, we get

$$(\overline{K})_u = \frac{K_u - 2\lambda K_u H + 2\lambda K H_u}{(1 - 2\lambda H + \lambda^2 K)^2}, \tag{2.2}$$

$$(\overline{K})_v = \frac{K_v - 2\lambda K_v H + 2\lambda K H_v}{(1 - 2\lambda H + \lambda^2 K)^2}, \tag{2.3}$$

$$(\overline{H})_u = \frac{H_u - \lambda^2 K H_u - \lambda K_u + \lambda^2 H K_u}{(1 - 2\lambda H + \lambda^2 K)^2}, \tag{2.4}$$

$$(\overline{H})_v = \frac{H_v - \lambda^2 K H_v - \lambda K_v + \lambda^2 H K_v}{(1 - 2\lambda H + \lambda^2 K)^2}. \tag{2.5}$$

Theorem 2.2. Let $\overline{M}$ be a parallel surface of a surface M in a Euclidean 3-space. If $\overline{M}$ is a Weingarten surface if and only if M is a Weingarten surface.

Proof. Which imply the jacobian function of the Gaussian curvature $\overline{K}$ and the mean curvature $\overline{H}$ of the parallel surface $\overline{M}$ is given

$$\Phi(\overline{K},\overline{H}) = \det\begin{bmatrix} \frac{K_u-2\lambda K_u H+2\lambda K H_u}{(1-2\lambda H+\lambda^2 K)^2} & \frac{K_v-2\lambda K_v H+2\lambda K H_v}{(1-2\lambda H+\lambda^2 K)^2} \\ \frac{H_u-\lambda^2 K H_u-\lambda K_u+\lambda^2 H K_u}{(1-2\lambda H+\lambda^2 K)^2} & \frac{H_v-\lambda^2 K H_v-\lambda K_v+\lambda^2 H K_v}{(1-2\lambda H+\lambda^2 K)^2} \end{bmatrix}. \tag{2.6}$$

From (2.6), we get

$$\Phi(\overline{K},\overline{H}) = \frac{K_u H_v - K_v H_u}{1-2\lambda H+\lambda^2 K}. \tag{2.7}$$

In a simple form, equation (2.7) becomes

$$\Phi(\overline{K},\overline{H}) = \Phi(K,H)\left(1-2\lambda H+\lambda^2 K\right).$$

This completes the proof. □

3. Tubular Surface In Euclidean 3-Space

Definition 3.1. Let $\alpha : [a,b] \to E^3$ be a unit-speed curve. A tubular surface of radius $r > 0$ about α is the surface with parametrization

$$M(u,v) = \alpha(u) + r\left(e_2(u)\cos v + e_3(u)\sin v\right) \tag{3.1}$$

$a \leq u \leq b$, where $e_2(u)$, $e_3(u)$ are the principal normal and binormal vectors of α, respectively ([1]). The curvature and the torsion of the curve α are denoted by κ,τ. Then Frenet formula of $\alpha(u)$ is defined by

$$e_1^{\prime}(u) = \kappa(u)\, e_2(u),\ e_2^{\prime}(u) = -\kappa(u)\, e_1(u) + \tau(u)\, e_3(u),\ e_3^{\prime}(u) = -\tau(u)\, e_2(u) \tag{3.2}$$

Besides, the gaussian curvature, mean curvature and the unit surface normal of $M(u,v)$ tubular surface in a Euclidean 3- space are given by (see [1]), respectively

$$K = \frac{-\kappa\cos v}{r\sigma}, \tag{3.3}$$

$$H = \frac{1-2r\kappa\cos v}{2r\sigma}, \tag{3.4}$$

$$N = e_2(u)\cos v + e_3(u)\sin v \tag{3.5}$$

where $\sigma = 1 - r\kappa\cos v$.

Theorem 3.2. Suppose that M is linear Weingarten type tubular surface in Euclidean 3-space. Then M is a tubular surface around a circle or a helix ([7]).

4. Main Results-The Parallel Tubular Surface İn E^3

In this section, we study the parallel surface $\overline{M}(u,v)$ of a tubular surface in a Euclidean 3-space which satisfies a linear weingarten equation (1.1) and a quadric equation (1.2) with respect to the Gaussian curvature $\overline{K}$ and The mean curvature $\overline{H}$ of the parallel surface $\overline{M}(u,v)$.

Theorem 4.1. Let $\overline{M}(u,v)$ be a parallel surface of tubular surface $M(u,v)$ in a Euclidean 3-space. If $\overline{M}(u,v)$ is a linear weingarten surface satisfing $a\overline{K}+b\overline{H}=c \;\; a,b\neq 0, c\in\mathbb{R}$. Then $\overline{M}(u,v)$ is parametrized by

$$\overline{M}(u,v)=\left(\alpha\left(u\right)+r\left(e_2\left(u\right)\cos v+e_3\left(u\right)\sin v\right)-\frac{br+2a}{b}\left(e_2\left(u\right)\cos v+e_3\left(u\right)\sin v\right)\right)$$

Proof. Let $\overline{M}(u,v)$ be a parallel surface of tubular surface $M(u,v)$ in E^3. Then, by the definition of a parallel surface, the parametrization for $\overline{M}(u,v)$ is given by

$$\overline{M}(u,v)=M(u,v)+\lambda N(u,v). \tag{4.1}$$

Suppose that parallel surface $\overline{M}(u,v)$ in E^3 is a linear weingarten surface. Then by (1.1) we have

$$a\overline{K}+b\overline{H}=c. \tag{4.2}$$

Differentiating $\overline{K}$ and $\overline{H}$ with respect to v, we get

$$a\overline{K}_v+b\overline{H}_v=0. \tag{4.3}$$

From (2.3) and (2.5), we get

$$a\left(K_v-2\lambda K_vH+2\lambda KH_v\right)+b\left(H_v-\lambda^2KH_v-\lambda K_v+\lambda^2HK_v\right)=0. \tag{4.4}$$

Differentiating K given by (3.3) and H given by (3.4) with respect to v, we obtain

$$K_v=\frac{\kappa\sin v}{r\sigma^2},\, H_v=\frac{\kappa\sin v}{2\sigma^2} \tag{4.5}$$

where $\sigma=1-r\kappa\cos v$. By using (4.5), Eq. (4.4) becomes

$$a\left[\frac{\kappa\sin v}{r\sigma^2}-2\lambda\left(\frac{\kappa\sin v}{r\sigma^2}\right)\left(\frac{1-2r\kappa\cos v}{2r\sigma}\right)+2\lambda\left(\frac{-\kappa\cos v}{r\sigma}\right)\left(\frac{\kappa\sin v}{2\sigma^2}\right)\right]$$

$$+b\left[\frac{\kappa\sin v}{2\sigma^2}-\lambda^2\left(\frac{-\kappa\cos v}{r\sigma}\right)\left(\frac{\kappa\sin v}{2\sigma^2}\right)-\lambda\frac{\kappa\sin v}{r\sigma^2}+\lambda^2\left(\frac{1-2r\kappa\cos v}{2r\sigma}\right)\left(\frac{\kappa\sin v}{r\sigma^2}\right)\right]=0.$$

In a simple form, above equation becomes

$$\left[b\left(\lambda-r\right)^2-2a\left(\lambda-r\right)\right]\left(\sin v\right)-\left[b\left(\lambda-r\right)^2-2a\left(\lambda-r\right)\right]\left(\kappa r\sin v\cos v\right)=0.$$

According to Theorem (3.2), $\kappa=0$. Thus, we have

$$\left[b\left(\lambda-r\right)^2-2a\left(\lambda-r\right)\right]\left(\sin v\right)=0$$

According to the definition of the linear independent of vectors, we find that

$$b\left(\lambda-r\right)^2-2a\left(\lambda-r\right)=0,$$

from which

$$\lambda=\frac{br+2a}{b}.$$

This completes the proof. □

Theorem 4.2. Let $\overline{M}(u,v)$ be a parallel surface of tubular surface $M(u,v)$ in a Euclidean 3-space and let a, b, c, d are constants such that $b^2-4ac>0$ $c\neq 0$. If $\overline{M}(u,v)$ is a $\overline{K}$ $\overline{H}$-quadric surface satisfying $a\overline{K}^2+b\overline{KH}+c\overline{H}^2=d$. Then $\overline{M}(u,v)$ is parametrized by

$$\overline{M}(u,v)=\left(\alpha\left(u\right)+r\left(e_2\left(u\right)\cos v+e_3\left(u\right)\sin v\right)-\frac{cr+b}{c}\left(e_2\left(u\right)\cos v+e_3\left(u\right)\sin v\right)\right)$$

Proof. The parametrization for $\overline{M}(u,v)$ is given by

$$\overline{M}(u,v)=M(u,v)+\lambda N(u,v). \tag{4.6}$$

Suppose that a parallel surface $\overline{M}(u,v)$ in E^3 is $\overline{K}$ $\overline{H}$-quadric. Then, by (1.2) we have

$$a\overline{K}^2+b\overline{KH}+c\overline{H}^2=d. \tag{4.7}$$

Differentiating $\overline{K}$ and $\overline{H}$ with respect to v, we get

$$2a\overline{KK}_v+b\overline{K}_v\overline{H}+b\overline{KH}_v+2c\overline{H}_v\overline{H}=0. \tag{4.8}$$

Then by using (2.3) and (2.5), we get

$$\begin{aligned}&\left[\left(H_v-\lambda^2KH_v-\lambda K_v+\lambda^2HK_v\right)\left(2c\left(H-\lambda K\right)+bK\right)\right]\\&+\left[\left(K_v-2\lambda K_vH+2\lambda KH_v\right)\left(2aK+b\left(H-\lambda K\right)\right)\right]=0.\end{aligned} \tag{4.9}$$

Differentiating K given by (3.3) and H given by (3.4) with respect to v, we obtain

$$K_v=\frac{\kappa\sin v}{r\sigma^2},\ H_v=\frac{\kappa\sin v}{2\sigma^2} \tag{4.10}$$

where $\sigma=1-r\kappa\cos v$. By using (4.10), Eq. (4.9) becomes

$$\begin{aligned}&\left[\left(\frac{\left(\lambda^2-2\lambda r+r^2\right)\left(\kappa\sin v-\kappa^2r\sin v\cos v\right)}{2r^2\sigma^3}\right)\left(\frac{\kappa\cos v\left(2\lambda c-2cr-b\right)+c}{r\sigma}\right)\right]\\&+\left[\left(\frac{\left(\lambda-r\right)\left(\kappa^2r\sin v\cos v-\kappa\sin v\right)}{r^2\sigma^3}\right)\left(\frac{\kappa\cos v\left(\lambda b-2br-4a\right)+b}{2r\sigma}\right)\right]=0.\end{aligned}$$

In a simple form, above equation becomes

$$\begin{aligned}&-\kappa^2\sin v\cos v\left[\left(\lambda-r\right)^2\left(2\lambda c-2cr-b\right)+\left(\lambda-r\right)\left(\lambda b-2br-4a\right)\right]\\&+\kappa\sin v\cos v\left[\left(\lambda-r\right)^2\left(2\lambda c-3cr-b\right)+\left(\lambda-r\right)\left(\lambda b-3br-4a\right)+\sin v\left[c\left(\lambda-r\right)^2+b\left(\lambda-r\right)\right]\right]=0.\end{aligned}$$

According to Theorem (3.2), $\kappa=0$. Thus we have

$$\sin v\left[c\left(\lambda-r\right)^2-b\left(\lambda-r\right)\right]=0.$$

Coefficient obtained from the last equation must be zero. So, we get

$$\lambda=\frac{cr+b}{c}.$$

This completes the proof. □

REFERENCES

[1] J. S. Ro and D. W. Yoon, *Tubes of Weingarten Types in a Euclidean 3- spaces*, Journal of the Chungcheong Mathematical Society, vol.22, no.3,(2009), pp.359-366.

[2] F. Dillen and W. Sodsiri, *Ruled surfaces of Weingarten type in Minkowski 3-space*, J.Geom. 83 (2005), pp.10-21.

[3] J. A. Galvez, A. Martinez and F. Milan, *Linear Weingarten surfaces in Euclidean 3-space*, Monatsh. Math., 138 (2003), pp.133-144.

[4] Y. H. Kim and D. W. Yoon, *On non-developable ruled surfaces in Lorentz-Minkowski 3-spaces*, Taiwanes J. Math. 11, (2007), pp.197-214.

[5] N. G. Kim and D. W. Yoon, *Mean curvature of non-degenerate second fundamental form of ruled surfaces*, Honam Math. J., 28 (2006), pp.549-558.

[6] A. Pressley, *Elementary differential geometry*, Springer, (2002).

[7] Y. Tunçer, D. W. Yoon and M. K. Karacan, *Weingarten and Linear Weingarten Type Tubular Surfaces in* E^3, Mathematical Problems in Engineering, Vol.10 **(11)**, (2011).

[8] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, NewYork, (1983).

[9] H. S. Abdelaziz, M. Khalifa Saad and S. Kızıltuğ,*On Parallel Surfaces of Weingarten Type in Minkowski 3-Space*, International Mathematical Forum, Vol. 7, (2012), no. 46, pp.2293 - 2302.

[10] S. Kızıltuğ, S. Kaya, Ö. Tarakcı,*Tube Surfaces with Type-2 Bishop Frame of Weingarten Types*, Int. Journal of Math. Analysis, Vol. 7, (2013), no. 1, pp.9 - 18.

[11] S. Kızıltuğ, Y.Yaylı, *Timelike tubes with Darboux frame in Minkowski 3-space*, International Journal of Physical Sciences, Vol. 8(1), (2013), pp.31-36.

[12] S. Kızıltuğ, A. Çakmak, S. Kaya,*Timelike Tubes Around A Spacelike Curve With Darboux Frame Of Weingarten Types in* E_1^3 , International Journal of Physical and Mathematical Sciences, Vol.4, No.1 (2013), pp.9-17.

ATATÜRK UNIVERSITY,
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM- TURKEY.
E-mail address: skiziltug24@hotmail.com

ATATÜRK UNIVERSITY,
FACULTY OF ARTS AND SCIENCES
DEPARTMENT OF MATHEMATICS
ERZURUM- TURKEY.
E-mail address: tarakci@atauni.edu.tr